Solitonic and extended periodic solutions of the Quasi 1D Gross-Pitavaeskii equation with a piecewise-constant nonlinearity

Thomas Mooney

September 5, 2008

ABSTRACT

This dissertation is concerned with the study of solitary wave structures in Quasi 1D Bose Einstein condensates(BEC). Here we study the exist and stability properties of bright and dark matter wave solitons in the presence of a piecewise constant nonlinearity. This work largely is an duplication of results presented in [2] as well as following to a limited degree some of the extensions suggested there in. The calculations performed in this dissertation, in the context of Hamiltonian perturbation theory and the numerical simulations which accompany them are my own independent calculations although the results are presented in [2]. We also make some suggestions for methods which extend the work given in [2].

0.1 Acknowlegments

I would like to thank my supervisor Dr Mason Porter for supervising me these past four months. I would also like to sincerely thank the EPSRC for kindly providing me an complete financial help so that I could take up my place here at Oxford. I would finally like to thank Dr A.S. Rodrigues at the University of Porto for his helpful advice via correspondence.

CONTENTS

1. INTRODUCTION AND FORMULATION OF THE MODEL

In 1925 Satyendra Nath Bose discovered bose statistics (with application to photons) which determine the distribution of the number of indistinguishable bosons (This is a fundamental type of particle, the other being a fermion. Any number of bosons can occupy a given state at the same time where as only one fermion at can occupy a given state at a time) in each state over a given energy range in thermal equilibrium [13]. Bose communicated his work to Albert Einstein who applied the statistics to atoms and found that if the atoms were bosons too then at a critical temperature T_c near absolute zero ($0K$ where K is the unit of Kelvin's) all the atoms collapsed into the same ground state (hence the atoms have the same ground state energy).This was the discovery of a new state of matter which is called a Bose-Einstein Condensate (BEC).

Seventy years later in 1995 Eric Cornell and Carl Wieman from the University of Colorado in Boulder produced one of the first BECs using a gas of rubidium atoms [8] at the NIST/JILA lab. Also in that same year BECs were produced with sodium atoms and Lithium atoms [8].

The theoretical study of BECs takes place within the framework of Quantum field theory. The model which is formulated in this framework using the mean field approximation introduced by Bogoliubov in 1947 is the Gross-Pitaevskii (GP) equation [8]. The GP equation is a variant of the nonlinear Schrodinger equation were the nonlinearity term is introduced by taking into account the interatomic interactions in the formulation of the model. The strength of the nonlinearity is governed by the s-wave scattering length (this is the length scale in which the interaction potential looks spherically symmetric to leading order [11]) and the interactions can be either attractive or repulsive depending on whether the s-wave scatter length is negative or positive [2]. These types of interactions lead to the formation of solitary wave structures (which I will call solitons, but technically they are not the same)in the BEC which in the regime with attractive interactions, create bright matter wave solitons and repulsive interactions create dark matter wave solitons(I simply refer to matter wave solitons as solitons in the following discussions) respectively.

These interactions which cause the nonlinear dynamics of the BEC can be controlled and manipulated with experimental tools such as the confining magnetic trap of the BEC or optical Feshbach resonances [2]. Both temporally and spatially (collisionally inhomogeneous) dependent methods of manipulating the interactions have been employed and it is the later which was studied in [2] for the particular case of a piece-wise constant nonlinearity (See [2] for a description of the possible experimental setup for this type of nonlinearity). There they gave the bright and dark soliton profiles initially for a collisionally homogenous BEC(constant nonlinearity) and then introduced the piecewise constant term as a perturbation of the original integrable equation. They then determined the stability/instability of the respective perturbed solutions within the frame work of Hamiltonian perturbation theory and then confirmed their finds with linear stability analysis and direct numerical simulations. They also gave a method to semi analytically stitch a bright solition with the view of obtaining analytical results as opposed to just numerical simulations.

The outline of this dissertation is as follows. First we shall formulate the full 3D model following the details given in [8]. Then we shall reduce the dimension of the full 3D model to a Quasi 1D limit following the reasoning given in [3].Then we introduce the piecewise constant nonlinearity and find the bright and dark matter wave soliton solutions of which only the results and not the details are given in [2]. Then we shall give the stitching method for bright solitons from [2] and the suggest a corresponding method for dark solitons. Next we reproduce the stability analysis(Hamiltonian perturbation theory and suggest a way to solve the BdG equations in Matlab) results given in [2]. Follow the derivation of the extended periodic solutions to the Quasi 1D GP equation as in [1]. Finally, we suggest a method for the stitching of attractive elliptic function solutions and then present the conclusions and possible extensions

1.1 Formulation of the model

A BEC contains millions of particles [1] and this results in a formidable many-body problem in which tens of thousands of interactions take place for each particle. This is an intractable problem to solve due to the number of particles present if we used linear quantum mechanics. In order to overcome this issue, we consider all the interactions that a single particle experiences and replace these with an effective interaction. This assumption of the validity in replacing the many interaction experienced by a single particle with an effective interaction makes the analysis tractable. The fundamental equation in the study of BECs which we need to solve parallels the role of the Schrodinger equation in linear quantum mechanics it is called the Gross-Pitaveski(GP) equation. The GP equation governs the evolution of the macroscopic wave function $\varphi(\mathbf{r},t)$[1](in this case the amplitude of the macroscopic wave function gives particle density at a point in space, instead of a probability density of being a particular point in space as in quantum mechanics)which allows us to determine all of the dynamical quantities of interested for a BEC such as the momentum distribution, energy distribution etc as well as the stability/instability of nonlinear wave structures induced in the BEC.

BECs can be treated as a quantum field [1] so an appropriate framework in which we will work is that of Quantum field theory (QFT). This is an efficient choice because QFT can easily handle systems with an infinite number of particles and so is a general framework suitable for larger sizes of BECs as well as ours.

When deriving the evolution equation for a dynamical system such as a BEC we start by writing down the Hamiltonian of the system (which represents the energy contained in the BEC considered in this dissertation) which is given in equation (1.1)

$$\hat{H} = \int d\mathbf{r}\hat{\phi}(\mathbf{r},t)^{'}\hat{H}_0\hat{\phi}(\mathbf{r},t) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}^{'}\hat{\phi}(\mathbf{r},t)^{'}\hat{\phi}(\mathbf{r}^{'},t)^{'}V(\mathbf{r}-\mathbf{r}^{'})\hat{\phi}(\mathbf{r}^{'},t))\hat{\phi}(\mathbf{r},t) \tag{1.1}$$

where $H_0 = (\hbar^2/2m)\nabla^2 + V_{ext}(\mathbf{r},t)$ is the single particle Hamiltonian(with external trapping potential $V_{ext}(\mathbf{r},t)$ and $V(\mathbf{r}-\mathbf{r}^{'})$ is the single particle interatomic potential. We can interpret equation (1.1) as the 'sum' of the Hamiltonians and interactions of all the particles in the BEC. Equation (1.1) is not a ordinary sum because we are treating the collection of particles of the BEC as a field instead of the aggregate of single particles.

If we now assume that the temperature of the gas is well below the critical temperature T_c, a significant fraction of the particles will be in the same ground state

[1] $\mathbf{r}$ is a three dimensional spatial variable and t is time

i.e. the BEC constitutes a significant volume of the condensed gas and there will be a negligible volume of particles still excited and forming an enveloping gas cloud. In 1947 Bogoliubov used this idea of a central core of particles making up the BEC with a negligible surrounding gas cloud in the construction of a zeroth order mean field approximation of a BEC by making the assumption that the bosonic field operator $\hat{\phi}(\mathbf{r}, t)$ (which creates a state with wave function $\phi(\mathbf{r}, t)$ out of the vacuum [11]) in this regime could be written as in (1.2)

$$\hat{\phi}(\mathbf{r}, t) = \left\langle \hat{\phi}(\mathbf{r}, t) \right\rangle + \hat{\phi}(\mathbf{r}, t)' \tag{1.2}$$

where the first term on the right of equation (1.2) is the mean field approximation which for convenience we will just write as $\phi(\mathbf{r}, t)$ and can be thought of as the mean number of particles in a state with wave function $\phi(\mathbf{r}, t)$(The second term on the right in equation (1.2) is the bosnic field operator which creates the excited states in the enveloping gas cloud around the BEC).The interpretation of the macroscopic wave function(I will refer to it simply as the wave function of the BEC) is different from that of linear quantum mechanics. Here $|\phi(\mathbf{r}, t))|^2$ should be interpreted as the particle density at $(\mathbf{r}, t)$ as opposed the probability of finding a particle there at time t. Now we consider an important constraint which follows from this is the conservation of particle number in the BEC. This is given below in (1.3) and has been normalized for convenience [1]

$$\int_{\Omega} d\mathbf{r} \, |\phi(\mathbf{r}, t))|^2 = 1 \tag{1.3}$$

Where Ω is the region of space occupied by the BEC. The next step is to consider the evolution equation for (1.2) in the Heisenberg picture. This evolution equation is given in equation (1.4)

$$i\hbar \frac{\partial \hat{\phi}(\mathbf{r}, t)}{\partial t} = [\hat{\phi}(\mathbf{r}, t), \hat{H}] \tag{1.4}$$

Where $[*, *]$ is the commutator which has the property that for two field operators $\widehat{A}, \widehat{B}$ $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$ and $\hbar = h/2\pi$ where h is Planck's constant. If we then substitute (1.1) into (1.4) and simplify we arrive at (1.5) [8]

$$i\hbar \frac{\partial \phi(\hat{\mathbf{r}}, t)}{\partial t} = \left[\hat{H}_0 + \int d\mathbf{r}' \phi(\hat{\mathbf{r}}, t)' V(\mathbf{r} - \mathbf{r}') \phi(\hat{\mathbf{r}'}, t)) \right] \phi(\hat{\mathbf{r}}, t) \tag{1.5}$$

As the BEC is dilute and ultra cold we can reasonably assume that the majority of interactions will be two bodied with very low energy. Therefore we can use a delta function $V(\mathbf{r}-\mathbf{r}^{'}) = g\delta(\mathbf{r}-\mathbf{r}^{'})$ to represent the interaction potential in a two body collision where $\mathbf{r}$ is the position vector of one of the particles involved in the collision and $\mathbf{r}^{'}$ is the position vector of the other particle. If $\mathbf{r} \neq \mathbf{r}^{'}$ then $V = 0$ and there is no interaction where as if $\mathbf{r} = \mathbf{r}^{'}$ then $V = g$ i.e. g represents the strength of the interaction. The constant g is given by $g = 4\pi\hbar^2 a/m$ where a is the s-wave scattering length, and m is the mass of the particles. The sign of a and hence the sign of the constant g determines whether an interaction is attractive ($g < 0$) or repulsive ($g > 0$) and hence determines the nonlinear dynamics of a BEC. Using the delta function interaction potential and the mean field approximation (1.2) we get to zeroth order the following model for a BEC called the 3D Gross-Pitaveskii(GP) equation

$$i\hbar\frac{\partial\phi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{r},t) + V_{ext}(\mathbf{r})\phi(\mathbf{r},t) + g\left|\phi(\mathbf{r},t)\right|^2\phi(\mathbf{r},t) \tag{1.6}$$

1.2 Quasi 1D Model

The external trapping potential $V_{ext}(\mathbf{r},t)$ is used to confine and manipulate the behavior of a BEC(which could be an electric or magnetic field whose geometry can be manipulated to change the dynamics of a BEC). Here we consider the first type of trap used on BECs in 1995 which were the magnetic traps and the external potential has the form [3].

$$V_{ext}(\mathbf{r},t) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{1.7}$$

This is a harmonic oscillator type potential and the three harmonic confining frequencies ω_x, ω_y and ω_z are in general different and can be varied to set the length scales and hence the physical size of a BEC. These experimentally fixed length scales are called characteristic length scales and are given by $a_i = \sqrt{\hbar^2/m\omega_i}$ where $i = x, y, z$[3]. Another important length scale involved with BECs is the healing length ξ. This is the distance over which the density of a BEC grows from 0 to ρ.If we are in a regime in which the confining frequencies are such that $\omega_x = \omega_y = \omega_r >> \omega_z$(r is the radial distance in the xy plane from the origin)and the transverse oscillator length is such that $a_{x,y} < \xi$, then the density of the BEC in the transverse directions is bounded by ρ and is much smaller than density variations in the remaining z direction. We therefore have a tightly constrained condensate along the z axis which is a 'cigar' shaped BEC. This is not a true 1D regime since the density variation in the transverse directions is not zero. We therefore refer to this model as a Quasi 1D model [3].

To demonstrate how we obtain the quasi 1d model, we decompose the 3D wave function as follows[3] (using polar variable r as instead of x, y for simplicity)

$$\phi(\mathbf{r}, t) = \psi(z, t)\varphi(r, t) \tag{1.8}$$

The second factor in equation (1.8) is given by $\varphi(r, t) = \varphi_0(r)e^{i\mu t}$ which is the steady transverse wave function where μ is the chemical potential(this is defined to be the rate of change of the Gibbs function with respect to the change in the number of moles of a particular constituent[12]) of the BEC. The steady transverse wave function $\varphi(r, t)$ also satisfies the auxiliary 2D harmonic oscillator problem

$$\frac{\hbar^2}{2m}\nabla_r^2\varphi_0 - \frac{1}{2}m\omega_r^2 r^2\varphi_0 + \mu\varphi_0 = 0 \tag{1.9}$$

where $\nabla_r^2 = (1/r)\partial/\partial r(r\partial/\partial r)$.The steady transverse wave function $\varphi(r, t)$ is always in the ground state since the regime is quasi 1D, so the ground state solution of (1.8) is $\varphi(r, t) = \pi^{-1/2}a_r^{-1}e^{-r^2/2a_r^2+i\mu t}$[3]. Now we substitute equation (1.8) into (1.6) to get equation (1.10)

$$i\hbar\varphi\frac{\partial\psi}{\partial t} + i\hbar\psi\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} - \frac{\hbar^2}{2m}\psi\nabla_r^2\varphi + V_{ext}\psi\varphi + g\left|\psi\right|^2\left|\varphi\right|^2\psi\varphi \tag{1.10}$$

Note that in equation(1.10) we have split the Laplacian up as $\nabla^2 = \partial^2/\partial z^2 + \nabla_r^2$.If we use the ground state wave function of equation (1.9) in equation (1.10) and then multiply by its complex conjugate $\varphi(\mathbf{r}, t)^*$(this cancels factors of $e^{i\mu t}$ in each term) we can integrate with respect to r from 0 to ∞ to obtain the Quasi 1D model given by equation (1.11)[3]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} + V_{ext}\psi + g_{1D}\left|\psi\right|^2\psi \tag{1.11}$$

where $g_{1D} = g/2\pi a_r^2 = 2a\hbar\omega_r$(remember that $g = 4\pi\hbar^2 a/m$) and $V_{ext}(z) = (1/2m)\omega_z^2 z^2$. This reduction is also valid for many other types of potentials (such as elliptic function potentials [4]) and leads to the same quasi 1D model. We shall assume here from this point onwards that we have constant external potential and in such a case we can then take $V_{ext}(z) = 0$.We make this assumption in order to be able to study solutions of the Quasi 1D model in the presence of a piecewise constant nonlinearity without any other mechanisms influencing the dynamics such as a nonzero potential(we can introduce a potential after the analysis of the regime without it).

1.3 NonDimensionalisation

Now we generalize the collisionally homogenous s-wave scattering length which we now denote by a_0 so that it can be spatially dependent i.e. $a = a(z)$. As a result, the nonlinearity coefficient in equation (1.11) becomes spatially dependent (we also drop the subscript $1D$ on g_{1D} and just write it as g). Now we nondimensionalise the Quasi 1D model by rescaling z, t and the wave function ψ such that $z = a_r\tilde{z}$,$t = \omega_r^{-1}\tilde{t}$ and $\psi = \sqrt{2\,|a_0|}\tilde{\psi}$. Then equation (1.11) reduces to the form (1.12) after dropping tildes

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial z^2} + g\,|\psi|^2\,\psi \tag{1.12}$$

Where the nonlinearity coefficient (which we have assumed can be spatially dependent) is given by $g(z) = a(z)/a_0$. If $g < 0$ then we have an attractive BEC and if $g > 0$ we have a repulsive BEC. Note that as a result of this nondimensionalisation, if we have a collisionally homogenous attractive/repulsive BEC which s-wave scattering length $\mp a_0$, the nonlinearity coefficient becomes $g(z) = \mp a_0/a_0 = \mp 1$.

2. INTEGRABILITY AND HAMILTONIAN STRUCTURE OF THE QUASI 1D MODEL

For constant g equation (1.12) is integrable which means that it can be completely integrated twice to yield a closed form solution. Equation (1.12) is equivalent to the variational problem of extremising the integral in (2.1) which is the Hamiltonian for a collisonally homogeneous BEC. This indicates that equation (1.12) with constant g has a Hamiltonian structure [2].

$$H = \frac{1}{2}\int_{-\infty}^{\infty}(|\partial_z\psi|^2 + g\,|\psi|^2)dz \tag{2.1}$$

The new contribution made in [2] is the introduction of a spatially piecewise constant $g(z)$. The nonlinearity coefficient g in this case has the form $g(z) = g_0 + \Delta g(z)$ (we show the form of $\Delta g(z)$ in fig(2.1)) given in equation (2.2)

$$\Delta g(z) = \Delta g_0 \sum_{n=-\infty}^{n=\infty}(\Theta(z - [nL + L_1]) - \Theta(z - [n+1]\,L)) \tag{2.2}$$

where g_0 is the homogeneous nonlinearity coefficient, $\Delta g_0 = g_1 - g_0$ is the in homogeneity strength, $L1$ and L are the spatial periods(which for numerical simulations we take to be $L1 = 2$ and $L = 4$) which determines which value for the nonlinearity coefficient g is used in each region, and Θ is the Heaviside side step function. For convenience in later discussions I also state in equation (2.3) the 1D GP equation with a piecewise constant nonlinearity given by $g(z) = g_0 + \Delta g(z)$.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial z^2} + g(z)\,|\psi|^2\,\psi \tag{2.3}$$

The introduction of this piecewise constant $g(z)$ destroys the integrability of equation (1.12). However since the result perturbations induced by equation (2.2) are also equivalent to the variational problem associated with (2.4) for bright solitons and

(2.5) for dark solitons

$$H_1 = \int_{-\infty}^{\infty} \frac{\Delta g(z)}{2\Delta g_0} |\psi|^4 \, dz \tag{2.4}$$

$$H_1 = \int_{-\infty}^{\infty} \frac{\Delta g(z)}{4\Delta g_0} (|\eta|^4 - |\psi|^4) dz \tag{2.5}$$

Where $\epsilon = \Delta g_0$,the perturbations are also Hamiltonian and so we can find conditions under which localized bright(these are associated with attractive BECs with nonlinearity coefficient g=-1) and dark soliton (these are associated with repulsive BECs with nonlinearity coefficient g=+1) solutions of the unperturbed equation (1.12) persist under the induced perturbation caused by (2.2) [2].

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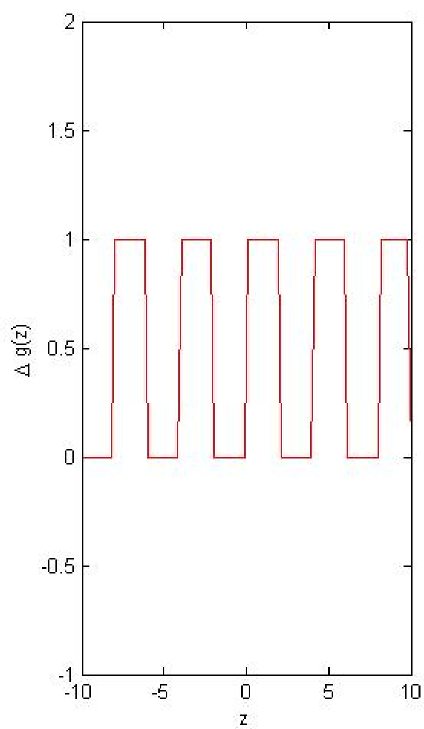

Fig. 2.1: The piecewise constant nonlinearity

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3. PHASE PLANE ANALYSIS AND SOLITON SOLUTIONS

We now construct the phase plane of the Quasi 1-D model with stationary solutions given by $\psi(z,t) = f(z)e^{i\mu t}$. We are interested in stationary solutions because when these are linearly stable they provide structures (solitary waves) that would have potential applications, such as in the construction of atom lasers [2]. Substituting the stationary solution ansatz $\psi(z,t) = f(z)e^{i\mu t}$ into equation (1.12)and cancelling the exponentials we arrive at equation (3.1)(when we also include boundary conditions with equation (3.1) we have a standing wave equation and solving this is equivalent to determining the stationary solutions of the corresponding time dependent equation (1.12))

$$\frac{d^2 f}{dz^2} = 2\mu f + g f^3 \tag{3.1}$$

If we set $p = \frac{df}{dz}$ then equation (3.1) reduces to the first order system

$$\frac{d}{dz}\begin{pmatrix} f \\ p \end{pmatrix} = \begin{pmatrix} p \\ 2\mu f + 2g f^3 \end{pmatrix} \tag{3.2}$$

The critical points of this system are $(0,0)$ and $(0, \pm\sqrt{\mu/g})$. The linearized systems can be straightforwardly found by using the substitutions $f = \eta$, $p = \nu$ for the critical point $(0,0)$ and $f = \eta \pm \sqrt{\mu/g}$, $p = \nu$ for the critical point $(0, \pm\sqrt{\mu/g})$ then we keep only the leading order terms. We are then left with a 2×2 stability matrix system $Ax = b$ and in each case we proceed to find the eigenvalues of A. Depending on their type (i.e. real and distinct, pure imaginary) the critical points can have a number of different stability properties i.e. they could be saddles, centres, nodes etc. I carried out this procedure and I show the phase portrait for negative nonlinearity only in fig (3.1)

As we can see there are both periodic and aperiodic solutions. The aperiodic solutions are both spatially stable and unstable and as we shall see these correspond to localized soliton solutions. Then there are the periodic solutions which are not and Settings/Thomas Mooney/My Documents/MATLAB/sech portrait.jpg

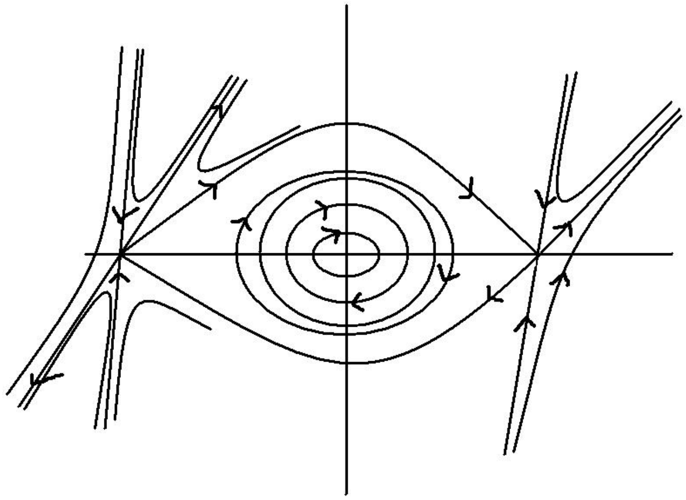

Fig. 3.1: The phase portrait for solutions of the Quasi 1D GP equation with negative nonlinearity

localized i.e. extended spatially, and these as we shall see are expressible as Jacobi elliptic functions. We shall derive these solutions systematically. What determine's each of them are the initial conditions and boundary conditions because different initial conditions put us on one of the trajectories shown in the phase portraits.

3.1 Soliton Solutions

In the following subsections we derive the bright and dark soliton solutions for attractive ($g < 0$) and repulsive BECs ($g > 0$) as well as the corresponding stationary solutions to equation (1.12). I am reproducing the solutions given in [2] although the details are not given there and I am performing the calculations myself.

3.1.1 Bright Soliton Solutions

For collisionally homogenous attractive BECs the nonlinearity coefficent may be taken as $g = -1$.The equation we then need to solve is

$$\frac{d^2 f}{dz^2} = 2\mu f - 2f^3 \tag{3.3}$$

With associated boundary conditions given by $f(z) \to 0$ as $z \to \pm\infty$. Multiply (3.3) by $\frac{df}{dz}$ and then integrating gives eqn (3.4)

$$\left(\frac{df}{dz}\right)^2 = -(\eta f)^2 - \frac{1}{2}f^4 + f_0 \tag{3.4}$$

Where $\eta^2 = -2\mu$ and f_0 is a constant of integration. The boundary conditions imply the additional property that $\frac{df(z)}{dz} \to 0$ as $z \to \pm\infty$ and so by applying this condition and the boundary conditions we see that $f_0 = 0$. Then if we rearrange the resulting equation and take the integral we obtain equation (3.5)

$$\int dz = \int \frac{df}{f\sqrt{\eta^2 - f^2}} \tag{3.5}$$

To transform the integrand of equation (3.5) to a form that is readily integrable we can use the substitution $f = \eta \mathrm{sech} x$. Equation (3.5) then leads to equation (3.6)

$$z - z_0 = -\frac{x}{\eta} \tag{3.6}$$

Although we have used the boundary conditions already to determine f_0, is localized and so the soliton is automatically zero at infinity. Therefore we may leave the second constant of integration unknown as it is arbitrary. This is due to this property of being localized (This second constant of integration z_0 can play the role of the centre of the bright soliton). Finally, inverting the transformation $f = \eta \mathrm{sech} x$ leads us to the bright soliton solution in equation (3.7)

$$f(z) = \eta \mathrm{sech} \eta(z - z_0) \tag{3.7}$$

which satisfies the boundary conditions and equation (3.3).In addition a last step we attach back the time dependence factor $e^{i\mu t}$ and the time dependent solution to equation(1.12) is equation (3.8)

$$\psi(z,t) = \eta \mathrm{sech}\left[\eta(\mathrm{z} - \mathrm{z}_0)\right] \mathrm{e}^{\mathrm{i}\mu\mathrm{t}} \tag{3.8}$$

3.1.2 Dark Soliton Solutions

Repulsive BECs are associated with a positive nonlinearity coefficient which can be set to $g = +1$.The equation we need to solve in this regime is equation (3.9)

$$\frac{d^2 f}{dz^2} = 2\mu f + 2f^3 \tag{3.9}$$

With associated boundary conditions given by $f(z) \to \pm\eta$ as $z \to \pm\infty$ (again there is the implication that $df(z)/dz \to 0$ as $z \to \pm\infty$) and the transformation used to evaluate the resulting integral is $f = \eta \tanh x$. Here $\eta^2 = \mu$. The resulting dark soliton solution to equation (3.9) and which satisfies the boundary conditions is given in equation

$$f(z) = \eta \tanh \eta (z - z_0) \tag{3.10}$$

Again we can find the stationary solution to equation (1.12) by attaching the factor $e^{i\mu t}$ to equation (3.10) which leads to equation (3.11)

$$\psi(z,t) = \eta \tanh(\eta\, [z - z_0])e^{i\mu t} \tag{3.11}$$

4. STABILITY ANALYSIS OF SOLITON SOLUTIONS

4.1 Stability of perturbed bright solitons solutions via Hamiltonian perturbation theory

The stationary bright soliton solution of equation (1.12) was found in section (3.1). It will remain a solution of perturbed equation (2.3) if it is an extremum/critical point of the perturbation Hamiltonian H_1 given in equation (2.4)[2]. The perturbed solution will be an extremum/critical point of (2.4) if it is located at the centre of a region with nonlinearity coefficient g_0 or g_1 where $g_1(g_1 < 0)$ (we haven't fixed g_1 yet but we will do so for numerical simulations)[2]. The numerical solutions to equation (2.3) are given in the next section in figures () and (). Now that we know the conditions under which we the soliton solution of equation (1.12) survives, the next step is to find the stationary solutions to equation (2.3). We have to find the solutions to equation(2.3) numerically as it is not integrable due to the introduction of the aforementioned perturbation given in (2.4)[2].

Now we consider whether numerical solutions to equation (2.4) are stable or unstable and what type of instability (i.e. exponential growth, oscillatory instability etc) is present if they are unstable. If we consider equation (0.12) with constant nonlinearity and apply the following transformations to z and ψ(which are called translational and gauge transformations)

$$z \to z^{'} + a \tag{4.1}$$

$$\psi \to e^{i\phi}\psi^{'} \tag{4.2}$$

Where a is a constant, we find that it is invariant under both these transformations (i.e. these are symmetries of equation (1.12) with constant nonlinearity), that is

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial z^2} + g\left|\psi\right|^2\psi \to i\frac{\partial \psi^{'}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi^{'}}{\partial z^{'2}} + g^{'}\left|\psi^{'}\right|^2\psi^{'} \tag{4.3}$$

When we introduce the perturbation given by equation (2.2) we break the translational symmetry (as (2.2) is not translationally invariant for all translations along the z axis). The gauge symmetry corresponding to the invariance of equation (1.12) under the transformation (4.2) is preserved even after the introduction of the perturbation (2.2) and so this will not affect the stability/instability of the numerical solutions as all the eigenvalues related to this symmetry remain zero [2].Instability can only arise in our numerical solutions to equation (2.3) if we break the symmetry corresponding to (4.2)[2]. After the breaking of this symmetry, instability will arise if there are translational eigenvalues() with nonzero imaginary part i.e. $im() \neq 0$. These translational eigenvalues are related to the eigenfrequencies of equation (2.3) through the relation $\omega^2 = -\lambda^2$[2].We will investigate the stability of the perturbed solitons by locating the eigenfrequencies (find the eigenfrequencies indicates if there are unstable eigenmodes in the Fourier decomposition of the soliton) which we will display in the spectral plane(the spectral plane is where the eigenfrequencies of a PDE/system of PDEs are located and spectral planes are used in the stability analysis of such equations)of equation(2.3) with [2].However, if all the eigenfrequencies for equation(2.3) have zero imaginary part then the numerical solution will be stable and we expect that all the eigenfrequencies to lie on the real line in the spectral plane[2].

We take advantage of the fact that equation (2.3) is Hamiltonian so that we can use the framework of Hamiltonian perturbation theory to determine the eigenfrequencies and hence the stability/instability of the perturbed soliton solutions.

To start with ω is determined in the case of attractive BECs from equation (4.4)[2]

$$\det(\epsilon \mathrm{M} - \omega^2 \mathrm{D}) = 0 \tag{4.4}$$

The matrices M and D have the following structure

$$M = \begin{pmatrix} \frac{\partial}{\partial z_0}\left\langle \frac{\delta H_1}{\delta \psi^*}, \frac{\partial \psi}{\partial z_0} \right\rangle & 0 \\ 0 & 0 \end{pmatrix} \tag{4.5}$$

$$D = \begin{pmatrix} \left\langle \frac{\partial \psi}{\partial z}, -z\psi \right\rangle & 0 \\ 0 & -\left\langle \psi, \frac{\partial \psi}{\partial \eta} \right\rangle \end{pmatrix} \tag{4.6}$$

Where $\langle *, * \rangle$ is the inner product given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^* g dz \tag{4.7}$$

and $\delta H_1/\delta\phi^*$ is a functional derivative given by[9]

$$\frac{\delta H_1}{\delta \psi^*} = \frac{\partial F}{\partial \psi^*} - \frac{d}{dz}\left(\frac{\partial F}{\partial(\partial_z \psi)}\right) + ... \tag{4.8}$$

where we take F to be the integrand of the perturbation Hamiltonian given by (2.4). The two terms of D can be easily calculated (I reproduced the calculations here for d_{11} and d_{22} which are given in the appendix. My calculation of d_{22} led to the discovery of an error in an intermediate equation in [2] although the final result is the same once the correct intermediate equation is used.) and the results are $d_{11} = \eta$ and $d_{22} = -1/\eta$. We give the details for m_{11} as this is not obvious to integrate. Firstly the functional derivative of the integrand of (2.4) is (we treat the fields ψ and ψ^* as though they were ordinary variables in order to calculate the functional derivative)

$$F = \sum_{n=-\infty}^{n=\infty} (\Theta(z - (nL + L_1)) - \Theta(z - (n+1)L))\psi\psi^*\psi \tag{4.9}$$

and the partial derivative of equation (3.8) with respect to the center z_0 is

$$\frac{\partial \psi}{\partial z_0} = \eta^2 \text{sech}[\eta(z - z_0)] \tanh[\eta(z - z_0)] e^{i\mu t} \tag{4.10}$$

Substituting (4.9) and (4.10) into the inner product (4.7) and interchanging the order of summation and integration (we assume this is valid) we find

$$\left\langle \frac{\delta H_1}{\delta \psi^*}, \frac{\partial \psi}{\partial z_0} \right\rangle = \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} (\Theta(z - (nL + L_1)) - ... \tag{4.11}$$
$$\Theta(z - (n+1)L))\eta^5 \text{sech}^4[\eta(z - z_0)] \tanh[\eta(z - z_0)] dz =$$
$$\eta^5 \sum_{n=-\infty}^{n=\infty} \int_{nL+L_1}^{\infty} \text{sech}^4[\eta(z - z_0)] \tanh[\eta(z - z_0)] dz -$$
$$\int_{(n+1)L}^{\infty} \text{sech}^4[\eta(z - z_0)] \tanh[\eta(z - z_0)] dz$$

where we have used the property of the Heaviside step function $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$. We then integrate this and take the partial derivative with respect to z_0 inside the summation sign. The result can be rewritten as in [2] and the final result is

$$m_{11} = -\eta^5 \sum_{n=-\infty}^{n=\infty} \text{sech}^5[\eta((n+1)L - z_0)] \sinh[\eta((n+1)L - z_0)] -$$

$$\text{sech}^5[\eta(nL + L_1 - z_0)]\sinh[\eta(nL + L_1 - z_0)] \tag{4.12}$$

Using the values of the matrices M and D just calculated, we find that to leading order the eigenfrequencies of the perturbed soliton are given by

$$\epsilon m_{11} - \omega^2 \eta = 0 \tag{4.13}$$

According to [2] the series in (4.13) converges rapidly due to the exponential factors in the terms of the series so we only need to consider the terms for which $n = -1, 0, 1$ and these give results accurate to 10 decimal places. Carrying out this calculation in Matlab (for parameter values $\mu = -1.0$ and $\Delta g_0 = -0.5$ as given in [1], and where the bright soliton is centered at a maximum of piecewise nonlinearity) shows that there are in fact a pair of eigenfrequencies which are complex conjugates to each other. Their imaginary part is given by $Im(\omega) = \pm 0.3973$ which shows that this configuration is unstable. When the bright soliton is centered in a region of minimum piecewise nonlinearity with $\Delta g = +0.5$ and $\mu = -1.0$, $Im(\omega) = 0$ and hence this configuration is stable.

4.1.1 Dark soliton solutions

A necessary condition for the dark soliton solution to equation (1.12) to survive this perturbation, the following integral must vanish [2](this is the analogous condition that the condition that the bright soliton solution remains an extremum/critical point of (2.4)[4])

$$M'(s) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\Delta g(z)}{dz}[\eta^4 - \psi^4(z-s)]dz \tag{4.14}$$

where $\psi(z-s)$ is the dark soliton solution to (1.12) which has been translated to $z = s$ in equation (4.14).This requirement is equivalent to the condition of the perturbed solution being centered in a region with nonlinearity coefficient g_0 or g_1[2].Solving equation (4.14) for s (which gives two roots as there are just two different types of regions) we then proceed to determine the sign of $\epsilon M''(s)$, where $\epsilon = \Delta g_0$(in homogeneity strength) and $M''(s)$ is the derivative of (4.14) with respect to s. The following cases are possible with the respective type of instability. Firstly if $\epsilon M''(s) > 0$

then there is instability and exactly one eigenfrequency quartet (four eigenvalues with nonzero imaginary part) and secondly if $\epsilon M^{''}(s) < 0$ then there is again instability but this time there is one eigenfrequency pair[2] (two eigenvalues with nonzero imaginary part).These conditions are related to the eigenvalues again through the relation $\omega^2 = -\lambda^2$. We can calculate the leading order eigenfrequencies using equation [2] (4.15)

$$\omega^2 - i\frac{\epsilon M^{''}(s_0)}{8}\omega + \frac{\epsilon M^{''}(s_0)}{16} = 0 \tag{4.15}$$

where $s_0 = s_1, s_2$ are the roots of $M^{'}(s) = 0$ and we have used the relation $\omega^2 = -\lambda^2$ to transform the equivalent equation in [2] to the one given in (4.15). Now we will calculate the roots of $M^{'}(s)$.The calculation given here duplicates the results that were presented in [2]. Equation (4.14) becomes, noting that the derivative of the Heaviside step function is the delta function ($\Theta^{'}(z) = \delta(z)$)

$$\begin{aligned} M^{'}(s) &= \frac{\Delta g_0}{2}\sum_{n=-\infty}^{n=\infty}\int_{-\infty}^{\infty}\delta(z-(nL_1+L)) - \delta(z-(n+1)L)[\eta^4 - \psi^4(z-s)]dz \\ &= \frac{\Delta g_0}{2}\sum_{n=-\infty}^{n=\infty}[\eta^4 - \psi^4(nL_1+L-s)] - [\eta^4 - \psi^4((n+1)L-s)] \\ &= \frac{\eta^4\Delta g_0}{2}\sum_{n=-\infty}^{n=\infty}\tanh[\eta(nL_1+L-s-z_0)] - \tanh[\eta((n+1)L-s-z_0)] \end{aligned} \tag{4.16}$$

Now $M^{'}(s_0) = 0$ if and only if $s + z_0 = nL + L/2 + L_1/2$ (just set the general term in (4.14) equal to zero and solve for $s + z_0$ for all n to find the roots). Also we know that the solution needs to centered at that center of a region with nonlinearity coefficient g_0 or g_1 so we need to take $z_0 = -L_1/2$ or $z_0 = -L/2$. Hence the roots are $s_1 = nL + L/2 + L_1$ and $s_2 = nL + L_1/2$.Next we find the first derivative of (4.14) with respect to z and then evaluate it at these roots. So the first derivative of (4.14) is

$$\begin{aligned} M^{''}(s) = \frac{\Delta g_0}{2}\sum_{n=-\infty}^{n=\infty}\int_{-\infty}^{\infty}\delta(z-(nL_1+L)) - .. \\ \delta(z-(n+1)L)\left(-4\psi^3(z-s)\frac{\partial\psi}{\partial s}\right)dz \end{aligned}$$

$$= -2_0 \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} \delta(z-(nL_1+L)) - ..$$

$$\delta(z-(n+1)L)\tanh^3\eta(z-s-z_0)\text{sech}^2\eta(z-s-z_0)dz$$

$$= -2\Delta g_0 \sum_{n=-\infty}^{n=\infty} \tanh^3\eta(nL+L_1-s-z_0)\text{sech}^2\eta(nL+L_1-s-z_0) -$$

$$\tanh^3\eta((n+1)L-s-z_0)\text{sech}^2\eta((n+1)L-s-z_0)$$

Now we can use the two roots of $M'(s)$ found before to determine $M''(s_{1,2})$ which are (in order to simply the result we use the hyperbolic identity $1-\text{sech}^2 z = \tanh^2 z$)

$$M''(s_1) = 4\eta^5 \sum_{p=-\infty}^{p=\infty} \tanh^3\eta(pL+L_1/2) - \tanh^5\eta(pL+L_1/2) \tag{4.17}$$

$$M''(s_2) = 4\eta^5 \sum_{p=-\infty}^{p=\infty} \tanh^3\eta(pL+2L_1) - \tanh^5\eta(pL+2L_1) \tag{4.18}$$

4.2 *Bogoliubov linear stability analysis of Soliton solutions*

In this section we introduce Bogoliubov de Gennes linear stability analysis in order to confirm the results of our calculations previously via Hamiltonian perturbation theory and also so that we can analyze the stability of elliptic function solutions given later (We cannot use the Hamiltonian perturbation theory that we used for perturbed soliton solutions as that used the assumption that the solution to (1.12) with constant g was localised). This results in the Bogoliubov de Gennes (BdG) equations which provide us with two ordinary coupled differential equations for the eigenmodes u and v and the eigenfrequencies ω. To obtain the BdG equations we add an order ϵ perturbation to the ground state solution (solitonic, extend periodic or a general ground state solution) of equation (1.12) with constant nonlinearity and this results in the perturbed wave function given in equation (for solitons we know that this perturbed wave function is again either a bright or dark soliton with the perturbation that we use) (4.19)

$$\psi(z,t) = \psi_0(z)e^{i\mu t} + \epsilon(u(z)e^{i(-\omega+\mu)t} + v^*(z)e^{i(\omega^*+\mu)t}) \tag{4.19}$$

Then substitute equation (4.19) into equation (1.12) and take the leading order terms in ϵ and this gives us the BdG equations for a general ground state ψ_0 [3]

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} - \mu - 2g(z)\psi_0^2\right) u - g(z)\psi_0^2 v = \omega u \tag{4.20}$$

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} - \mu - 2g(z)\psi_0^2\right) v - g(z)\psi_0^2 u = -\omega v \tag{4.21}$$

Discretising equations (4.20) and (4.21) get equations (4.22) and (4.23)

$$-\frac{1}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{\delta z^2} - \mu u_i - 2g(z_i)\psi_0(z_i)^2 u_i - g(z_i)\psi_0(z_i)^2 v_i = \omega u_i \tag{4.22}$$

$$-\frac{1}{2}\frac{v_{i+1} - 2v_i + v_{i-1}}{\delta z^2} - \mu v_i - 2g(z_i)\psi_0(z_i)^2 v_i - g(z_i)\psi_0(z_i)^2 u_i = -\omega v_i \tag{4.23}$$

where δz is the spatial step size. We can rewrite the discretized system (4.22) and (4.23) as a matrix equation $A\mathbf{x} = \omega b$ where $\mathbf{x}$ has components u_i and v_i and then solve for the eigenfrequencies ω which are eigenvalues of the system. To do this we would have used the Matlab function eig.

4.3 Numerical simulations Soliton solutions

In this section we show the numerical simulations performed with matlab for bright and dark solitons. The numerical simulations show that the analytical predictions for the eigenvalues agree with the observed behaviour and also the type of instability predicted. In fig (4.1) we see the onset of an oscillating instability at approximately the same time in the simulation as that given in [2]($t \geq 80$). In fig (4.2) we see again, agreement with the results given in [2]. Hamilitonian perturbation theory predicted that this solution would be stable and the numerical simulations confirm this at least up to the time the simulation was run. In fig (4.3) we again see instability which is again in agreement with [2] although here we have not explicitly calculated the eigenfrequencies.

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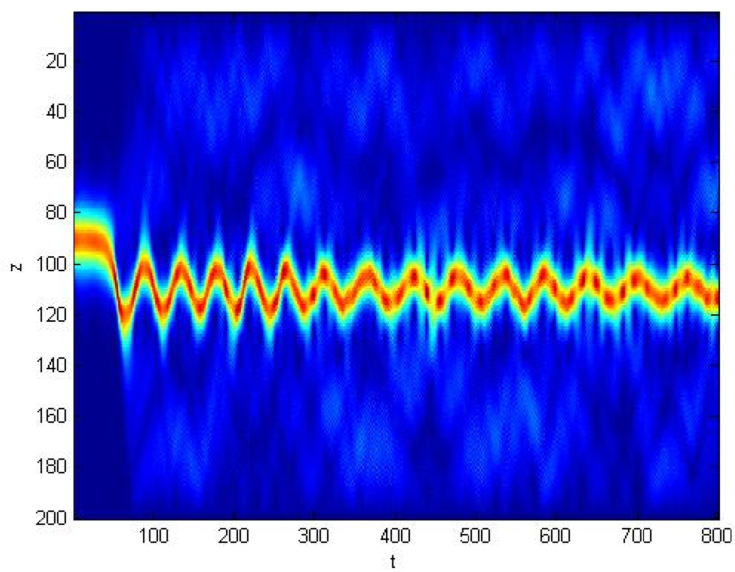

Fig. 4.1: Contour plot showing the oscillational instability of the stitched bright soliton solution of equation (2.3) centered in a region with maximum nonlinearity as it evolves. The parameters are the same as given in [2] but the scales are different. It can be seen the instability sets in a approximately the same time as that given in [2] which was $t \geq 80$

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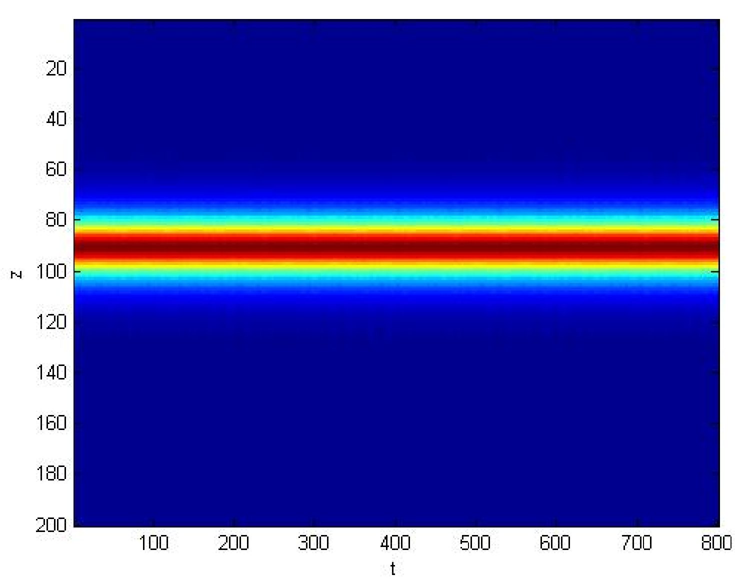

Fig. 4.2: plot of the evolution of stitched bright solition centered in a region with a minimum nonlinearity

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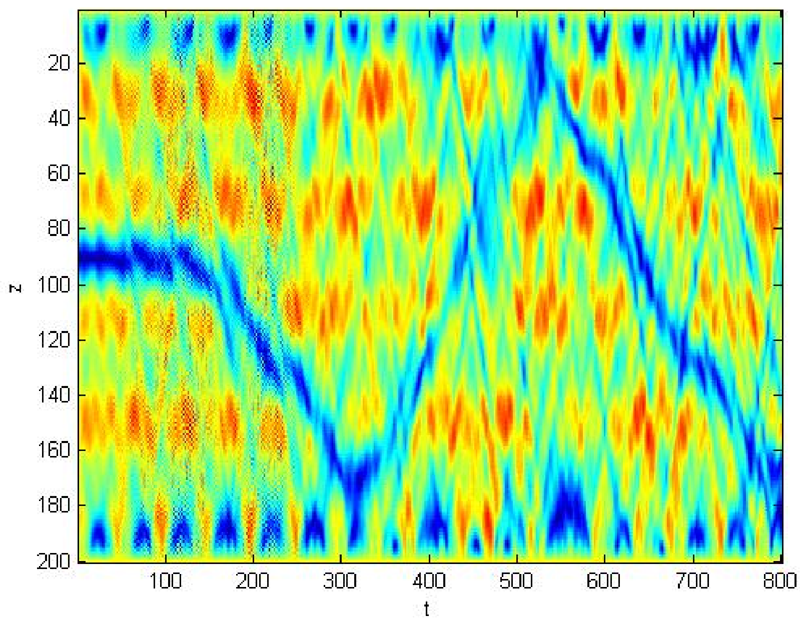

Fig. 4.3: Contour plot of a stitched dark soliton. We can see that again in this case the predicted instability sets in and is of the type suggested by the eigenfrequencies i.e. oscillating an instability. The parameter values here are $\mu = 1.0$ and $\Delta g_0 = -0.5$ Although the plot looks different(in terms of colour) to the one presented in [2] this is due to the fact that I have used a different mesh and possible a different algorithm.

5. SEMI-ANALYTICALLY STITCHED SOLITON SOLUTIONS

5.1 Stitched bright soliton solutions

The major purpose of the introduction of a piece-wise constant nonlinearity by [2] is to obtain semi-analytical approximations to the perturbed soliton profile instead of just numerical simulations. One reason for this is so that further theoretical results can be obtained and this can also aid experimentalists by given them more control over their experiments. The procedure involves stitching the wave function and its first partial derivative with respect to z at the interface Z between two neighboring regions but in which g is takes two different values. Firstly we can find an explicit bright soliton solution for each value of the nonlinearity coefficient g. We also know that the bright soliton profile extending over all the regions persists following the introduction of the perturbation given in equation (2.2) and so we may write down an analytical expression for the spatial part of the wave function in equation (5.1)

$$\psi_s(z) = \frac{\eta_s}{\sqrt{g_s}} \text{sech}\eta_s(z - z_s) \tag{5.1}$$

where η_s is the sth amplitude, z_s is the sth centre and g_s is the sth value of the nonlinearity(of which there is only two values here but this can be extended to include more than too) in the sth region. If we also introduce the variable Z_s which is the point of intersection between region s and $s+1$ then matching the solutions (5.1) in the respective regions at these points as well as their first partial derivatives results in the follow relations between η_s, z_s ,η_{s+1} and z_{s+1} where η_{s+1} and z_{s+1} are to be determined.

$$\frac{\eta_s}{\sqrt{g_s}} \text{sech}[\eta_s(Z_s - z_s)] = \frac{\eta_{s+1}}{\sqrt{g_{s+1}}} \text{sech}[\eta_{s+1}(Z_{s+1} - z_s)] \tag{5.2}$$

$$-\frac{\eta_s^2}{\sqrt{g_s}} \text{sech}[\eta_s(Z_s - z_s)] \tanh[\eta_s(Z_s - z_s)] = \tag{5.3}$$

$$-\frac{\eta_{s+1}^2}{\sqrt{g_{s+1}}} \text{sech}[\eta_{s+1}(Z_{s+1} - z_{s+1})] \tanh[\eta_{s+1}(Z_{s+1} - z_{s+1})]$$

If we then divide (5.2) into (5.3) then we get equation (5.4)

$$\tanh \eta_{s+1}(Z_{s+1} - z_{s+1}) = \frac{\eta_s}{\eta_{s+1}} \tanh \eta_s(Z_s - z_s) \tag{5.4}$$

and then, if we take the inverse of equation (5.4) arrive at equation (5.5)

$$\eta_{s+1}(Z_{s+1} - z_{s+1}) = \operatorname{arctanh}\left(\frac{\eta_s}{\eta_{s+1}} \tanh \eta_s(Z_s - z_s)\right) \tag{5.5}$$

You can use a nonlinear solver on equations (5.2) and (5.3) such as Newton's method or if (η_s, z_s) and η_{s+1} are known then you can use (5.5) to find z_s[2]. This system of nonlinear equations is over determined because η_s is determined also by μ_s and g_s. However it gives reasonably accurate results [2].

5.2 Semi Analytical Stitched Dark Soliton Solutions

Repulsive BECs (positive nonlinearity) which possess dark soliton also persist if they are centered in a region with nonlinearity coefficient g_0 or g_1 and so we can also semi analytically stitch these profiles too as in section (0.11). We can again write down the analytical form for the dark soliton solution in the sth region as

$$\psi_s(z) = \frac{\eta_s}{\sqrt{g_s}} \tanh \eta_s(z - z_s) \tag{5.6}$$

Using the same matching conditions in section (0.11) we get the corresponding relations between the sth amplitudes η_s and the sth centres z_s in equations (0.51) and (0.52)

$$\frac{\eta_s}{\sqrt{g_s}} \tanh[\eta_s(Z_s - z_s)] = \frac{\eta_{s+1}}{\sqrt{g_{s+1}}} \tanh[\eta_{s+1}(Z_{s+1} - z_s)] \tag{5.7}$$

$$\frac{\eta_s^2}{\sqrt{g_s}} \operatorname{sech}^2[\eta_s(Z_s - z_s)] = \frac{\eta_{s+1}^2}{\sqrt{g_{s+1}}} \operatorname{sech}^2[\eta_{s+1}(Z_{s+1} - z_{s+1})] \tag{5.8}$$

Equations (0.51) and (0.52) are equivalent as can be seen via the identity $1 - \operatorname{sech}^2 z = \tanh^2 z$ and so we have two unknowns and one equation relating them. One possibility to solve this problem could be the use of the method of least squares. This has not been done in this dissertation.

5.3 Numerically stitched bright and dark solitons

Here we give the numerically stitched bright and dark soliton solutions to the Quasi 1D GP equation. We can see in fig (5.1) that the numerically stitched bright soliton is indistinguishable from the corresponding bright soliton for a collisonally homogeneous BEC given in [2]. The dark soliton in fig (5.2) is in agreement with the corresponding figure in [2].

pictures/brightsoliton1.jpg

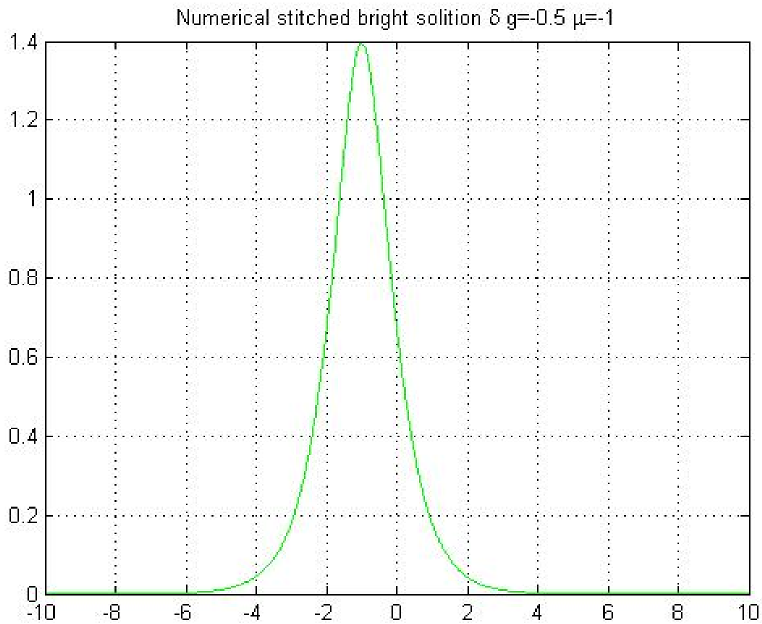

Fig. 5.1: Numerically stitched bright soliton

pictures/darksoliton1.jpg

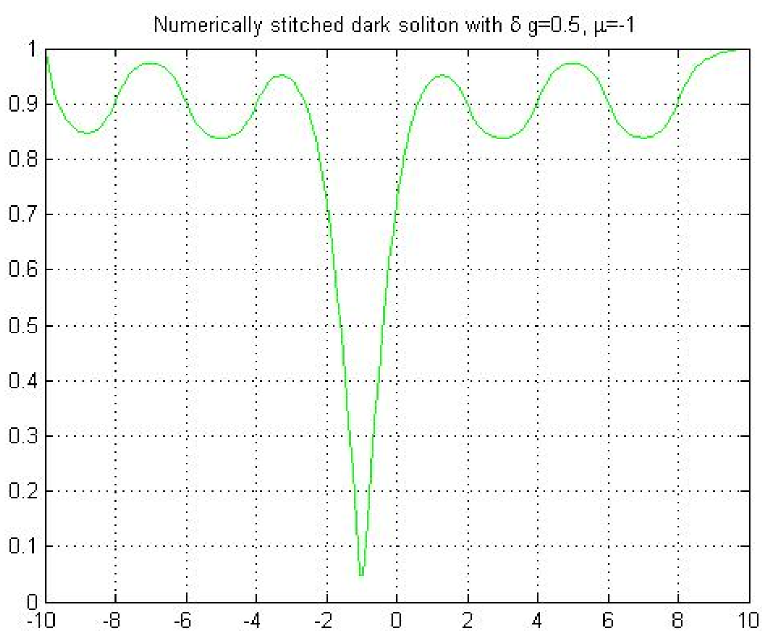

Fig. 5.2: Numerically stitched dark soliton

6. EXTENDED PERIODIC SOLUTIONS IN TERMS OF JACOBI ELLIPTIC FUNCTIONS

In the following subsections we shall give a brief introduction to the theory of Jacobi elliptic functions which can be used to represent the extended periodic solutions to equation (1.12) with constant g. Then after this we shall derive solutions to equation (1.12) with constant g, for attractive and repulsive BECs with box and periodic boundary conditions. In section (6.2) we derive the attractive solution to equation (1.12) with box boundary conditions from the first principles. This method is described in [1] but is given for repulsive solutions where as we give it for attractive solutions (the corresponding paper for attractive solutions uses an ansatz directly to find the solutions there in) and the calculation here is not exactly the same. We also reproduce the constraints on the parameters s (the Jacobi elliptic modulus) and j (this is an integer that appears when we make the ansatz in the following sections satisfy the particular boundary conditions for that case) given in [1] but the details of the calculations were not given.

6.1 Brief Introduction to Jacobi Elliptic functions

The J.E.Fs used in this dissertation are $\text{sn}(x,l)$,$\text{cn}(x,l)$ and $\text{dn}(x,l)$ where l satisfies $0 \leq l \leq 1$ [9]. JEFs are a general class of functions which include trignometric($t = 0$ and $\text{cn}(x,0) = \cos\, x$) and hyperbolic($l \to 1^-$ and $\text{cn}(x,1^-)=$ sech x) functions as limiting situations. If we fix $l = l_0$ where l_0 is constant, then this picks out a particular JEF i.e. $\text{cn}(x,l_0)$ for example. To define them analytically we first define equation (6.1)

$$x = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - l\sin^2\theta}} \tag{6.1}$$

where ϕ is called the Jacobi amplitude[9]. From (6.1) we can define the JEFs used in this dissertation[9] by

$$\text{cn}(x,l) = \cos\phi \tag{6.2}$$

$$\mathrm{sn}(x,l) = \sin\phi \tag{6.3}$$

$$\mathrm{dn}(x,l) = \sqrt{1 - l\mathrm{sn}^2(x,s)} \tag{6.4}$$

The period of these functions is $4K(l)$ (which corresponds to the period 2π of sin and cos or if $l \to 1^-$ the infinite period of a hyperbolic function) where

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - l\sin^2\theta}} \tag{6.5}$$

6.2 Attractive elliptic function Solutions

Here we shall derive solutions for the attractive case (i.e. $g < 0$) where again we take $g = -1$. We shall derive two types of solutions here where we have box boundary conditions ($f(0) = f(1) = 0$) and periodic boundary conditions ($f(0) = f(1)$ and $f^{'}(0) = f^{'}(1)$).

6.3 Attractive elliptic function solutions satisfying Box boundary conditions

Firstly we multiply (3.3) by $f^{'}$ as we did before and integrate to get (3.4) again. However now we factorize the quartic and assume for simplicity that it has the factorization given in (6.6)(If it has four distinct factors then we may apply a linear fractional transformation to the integrand and then another transformation to reduce it to the case given here using methods described in [10])

$$(f^{'})^2 = (f_1 - f^2)(f_2 - f^2) \tag{6.6}$$

Rearranging (6.6) to give (6.7)

$$dz = \frac{df}{\sqrt{(f_1 - f^2)(f_2 - f^2)}} \tag{6.7}$$

We can then use the substitution $f^2 = f_1 \sin^2\theta$ to put (6.7) in the form (6.8).We can then integrate to get

$$\alpha z + \kappa = \int_0^{f/\sqrt{f_1}} \frac{d\theta}{\sqrt{1 - l\sin^2\theta}} \tag{6.8}$$

Where $l = f_1/f_2$. So by using the defintion of the cn function given in the previous section we can see that the most general extended periodic solution to (3.3) is

$$f(z) = A\mathrm{cn}(\alpha \mathrm{z} + \kappa) \tag{6.9}$$

Where A, α and κ are constants of integration to be determined from the box boundary conditions. Substituting (6.9) into equation (3.3) we get

$$\mu A\mathrm{cn} = \frac{1}{2}\alpha^2 A\mathrm{cndn}^2 - \frac{1}{2}\alpha^2 lA\mathrm{cnsn}^2 - A^3\mathrm{cn}^3 \tag{6.10}$$

Using identities (1) an (2) for elliptic functions given in the appendix we reduce (6.10) to powers of cn only and obtain the following identity

$$\mu A\mathrm{cn} = \frac{1}{2}\alpha^2(1-2\mathrm{l})\mathrm{Acn} + (\alpha^2\mathrm{lA} - \mathrm{A}^3)\mathrm{cn}^3 \tag{6.11}$$

Equating like powers of cn we obtain the following relations for A and μ

$$\mu = \frac{1}{2}\alpha^2(1-2l) \tag{6.12}$$

$$A = \alpha\sqrt{l} \tag{6.13}$$

Applying box boundary conditions to (6.9) we find that $\mathrm{cn}(\alpha+\kappa) = \mathrm{cn}(\kappa) = 0$ from which we determine that $\alpha = 2jK(l)$ and $\kappa = -K(l)$ where $j \in \{1, 2, 3, ...\}$ because $\mathrm{cn}(jK(l)) = 0$ where j is odd. Hence solution and chemical potential are

$$f(z) = 2jK\sqrt{l}\mathrm{cn}(K(2j(z-z_0)-1)) \tag{6.14}$$

$$\mu = \frac{1}{2}(2jK)^2(1-2l) \tag{6.15}$$

We need to constrain (equation (1.3) provides the constraint) equation (6.14) so that the normalized particle number is conserved and so that the equation (6.14) is physically meaningful [1]. This constraint will restrict our freedom in choosing certain pairs for the parameters (l, j)(If we just wanted to satify equation (3.3) with box boundary conditions without modeling an attractive BEC we would be free to choose any values for l and j). Applying (1.3) to equation (6.14) we find that (note that as cn is translationally invariant under the transformation $z \to z - z_0$ we may leave out z_0 without loss of generality)

$$4j^2K(l)^2 l\int_0^1 \mathrm{cn}^2(K(2jz-1))dz = 1 \tag{6.16}$$

Next if we use the substitution $x = K(2jz-1)$ (6.16) becomes

$$2jKl\int_{-K}^{K(2j-1)} \mathrm{cn}^2 x dx = 1 \tag{6.17}$$

to simplify (6.17) so that it involves only functions of l(the quarter period $K(l)$ and the complete elliptic integral of the second kind $E(l) = \int_0^{\pi/2} \sqrt{1 - l\sin^2\theta}\mathrm{d}\theta$)

and j we use the identity $1 - sn^2 = cn^2$ to give (6.18)

$$2jKl - 2jKl\int_{-K}^{K(2j-1)} \mathrm{sn}^2 x dx = 1 \tag{6.18}$$

Then if we use another substitution given by $\mathrm{sn}x = \sin\theta$ we find that the limits for θ $\theta = 0$ and $\theta = \pi/2$ (remember that $\mathrm{sn}x = \sin\phi$ so that to find the limits for θ we solve the equations $\sin\theta = \sin 0$ and $\sin\theta = \sin\pi/2$). The differential dx becomes

$$dx = \frac{cos\theta d\theta}{\sqrt{1 - \sin^2\theta}\sqrt{1 - l\sin^2\theta}} \tag{6.19}$$

Substituting all this into the integral in (6.18) we find that

$$\begin{aligned} \int_{-K}^{K(2j-1)} \mathrm{sn}^2(x)dx &= \\ \int_0^{\pi/2} \frac{cos\theta\sin^2\theta d\theta}{\sqrt{1-\sin^2\theta}\sqrt{1-l\sin^2\theta}} & \\ &= \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\sqrt{1-l\sin^2\theta}} \\ &= \frac{1}{l}\int_0^{\pi/2} \frac{1-(1-l\sin^2\theta)d\theta}{\sqrt{1-l\sin^2\theta}} \\ &= \frac{1}{l}\int_0^{\pi/2} \frac{1}{\sqrt{1-l\sin^2\theta}} - \frac{1-l\sin^2\theta}{\sqrt{1-l\sin^2\theta}}d\theta \\ &= \frac{1}{l}(K(l) - E(l)) \end{aligned} \tag{6.20}$$

Substituting equation (6.20) into equation (6.18) we get the constraint on equation (6.14)

$$2jK(E - (1 - 2jl^2)K) = 1 \tag{6.21}$$

6.3.1 Attractive elliptic function solutions satisfying Periodic Boundary conditions

The derivation here follows that given in [1]. The ansatz used is again

$$f(z) = A\mathrm{cn}(\alpha z + \kappa) \tag{6.22}$$

Where the parameters A, α and κ are parameters to be determined from the constraints imposed by having to satisfy equation (3.3), the boundary conditions $f(0) =$

$f(1)$ and $f'(0) = f'(1)$ and the constraint (1.3). Substituting the ansatz into (3.3) and using various derivatives of elliptic functions and identities given [9] we arrive at the follow identity

$$\mu A\text{cn} = \frac{1}{2}\alpha^2(1-2l)A\text{cn} + (\alpha^2 lA - A^3)\text{cn}^3 \tag{6.23}$$

Comparing like powers of cn we obtain the relations for the parameters

$$\mu = \frac{1}{2}\alpha^2(1-2l) \tag{6.24}$$

$$A = \alpha\sqrt{l} \tag{6.25}$$

We still have to determine α and κ from the boundary conditions to satisfy $f(0) = f(1)$ we need $\text{cn}(\kappa) = \text{cn}(\alpha+\kappa)$. This can be easily satisfied if we take α to be integer multiples of the period $4jK$ and κ is then seen to be arbitrary, which for simplicity I will take to be 0. Finally in order to get unique, physically meaningful solutions, we need $f(z)$ to satisfy the constraint (1.3). This will constrain the elliptic modulus l and integer pairs (l, j) that are available to us and will also be useful slightly later when constructing stitched solutions as the stitched solution will also need to satisfy this condition (in this case there are $2j$ nodes). The constraint of $f(z)$ following from (1.3) in this case can be found in exactly the same way as the section (6.3.1) and is in this case

$$16j^2K(E-(1-l)K) = 1 \tag{6.26}$$

6.4 Repulsive elliptic function solutions

These types of solutions for repulsive BECs are characterized by a nonlinearity with $g > 0$ which we take here to be $g = 1$ (homogeneous case). The equation we solve in this case is

$$\frac{d^2f}{dz^2} = -2\mu f + f^3 \tag{6.27}$$

with the corresponding box boundary conditions (f(0)=f(1)=0) and periodic boundary conditions (f'(0)=f'(1) and f(0)=f(1)). The following subsections follow the same method given in [1] although the intermediate steps have been carried out here to obtain the results presented in [1].

6.4.1 Box boundary solutions

The ansatz used in this case is

$$f(z) = A\mathrm{sn}(\alpha z + \kappa) \tag{6.28}$$

If we substitute (6.28) into equation (6.27) and use the derivatives identities given by [9]we can reduce the result to include only powers of sn we get

$$\mu A\mathrm{sn} = \frac{1}{2}\mathrm{\alpha^2 A(1+l)sn - (l\alpha^2 A + A^3)sn^3} \tag{6.29}$$

Equating equal powers of sn we find that

$$A = \alpha\sqrt{l} \tag{6.30}$$

$$\mu = \frac{1}{2}\alpha^2(1+l) \tag{6.31}$$

applying $f(0) = f(1) = 0$ we find that $\mathrm{sn}(\kappa) = \mathrm{sn}(\alpha + \kappa)$. This shows that $\alpha = 2jK(m)$ and $\kappa = 0$. The constraint on $f(z)$ following from equation (1.3) results in

$$4j^2K^2\left(1 - \frac{E}{K}\right) = 1 \tag{6.32}$$

6.4.2 Periodic Boundary solutions

These types of solutions found using the same ansatz as the previous subsection but with periodic boundary conditions $f(0) = f(1)$ and $f'(0) = f'(1)$. I will just state the solution to equation (6.27) in this case along with the corresponding chemical potential

$$f(z) = 4jK(m)\sqrt{l}sn(4jKz) \tag{6.33}$$

$$\mu = \frac{1}{2}(4jK)^2(1+l) \tag{6.34}$$

The constraint on $f(z)$ gives the relation (0.84) between l and j

$$16j^2K^2\left(1 - \frac{E}{K}\right) = 1 \tag{6.35}$$

7. STITCHED ATTRACTIVE ELLIPTIC FUNCTION SOLUTIONS SATISFYING PERIODIC BOUNDARY CONDITIONS

Here we shall consider a method by which we can semi analytically stitch attractive elliptic function solutions satisfying periodic boundary conditions to the Quasi 1D GP equation. The same procedure can be considered to apply to box boundary conditions as well. Firstly we shall match the function values and gradients at the interfaces Z like the soliton case. Doing this we get the following relations (Note that again, here we denote with a subscript s, the various parameters specifying a solution in the region s)

$$4j_sK_s\sqrt{l_s}\text{cn}(4j_sK_s(z-z_s)) = 4j_{s+1}K_{s+1}\sqrt{l_{s+1}}\text{cn}(4j_{s+1}K_{s+1}(z-z_{s+1})) \quad (7.1)$$

$$-(4j_sK_s)^2\sqrt{l_s}\text{sn}(4j_sK_s(z-z_s))\text{dn}(4j_sK_s(z-z_s)) \quad (7.2)$$

$$= -(4j_{s+1}K_{s+1})^2\sqrt{l_{s+1}}\text{sn}(4j_{s+1}K_{s+1}(z-z_{s+1}))\text{dn}(4j_{s+1}K_{s+1}(z-z_{s+1}))$$

If we set $\alpha_1 = 4j_sK_s\sqrt{l_s}\text{cn}(4j_sK_s(z-z_s))$, and $\alpha_2 = -(4j_sK_s)^2\sqrt{l_s}\text{sn}(4j_sK_s(z-z_s))\text{dn}(4j_sK_s(z-z_s))$ then we can square equations (8.1) and (8.2) and using $\alpha_{1,2}$ we can eliminate the elliptic functions to get a relation between $\alpha_{1,2}, m_{s+1}, K_{s+1}, j_{s+1}$ which is

$$\alpha_2^2 = (m_{s+1}(4j_{s+1}K_{s+1})^2 - \alpha_1^2)((1-m_{s+1})(4j_{s+1}K_{s+1})^2 + \alpha_1^2) \quad (7.3)$$

Now looking at equation (8.3) we might obviously assume that we have three parameters to determine plus a fourth parameter z_s but this is not the case if we remember the two facts. Firstly, in order to obtain physically meaningful solutions to (3.3) subject to periodic boundary conditions, we had to constrain the possible pairs (l, j) to satisfy equation(6.26).The second fact is that the choice of elliptic modulus l dictates the value of the quarter period K. Therefore we can see that we have only two parameters that need to be determined by the relation (8.3) and equations (8.1) and (8.2) which are l_{s+1} and z_{s+1}. The strategy to find l_{s+1} from relation (8.3) is to search through the possible range of values for l_{s+1} which all lie in $[0, 1)$ and then to determine j_{s+1}, K_{s+1} from the relations (6.26) and (6.5). Then we test to see whether these satisfy (8.3) in which case they will give the correct parameters for the stitching. This is not as difficult or time consuming as you would imagine as there are few choices for the pairs (l, j) which lay in the middle ground between $l = 0$ and $l \to 1^-$. However as we approach $l = 0$ we can see that there are many choices for l and this is where the search might take time.

8. CONCLUSIONS AND EXTENSIONS

In this dissertation we have reproduce most of the results which were first obtained in [2]. We have rederived the Quasi 1D GP equation. We then carried out a simple phase plane analysis for attractive nonlinearity, to show why we would expect both solitonic and extend periodic solutions (expressible as elliptic function solutions). We found these solutions of the integrable 1D equation and then introduced a piecewise constant nonlinearity coefficent(just for the soliton solutions) as in [2] which breaks the integrability. We described conditions under which soliton solutions for constant coeffiecents remained solutions of the perturbed equation via Hamiltonian perturbation theory without showing the corresponding conditions on extended periodic solutions(we cant use that type Hamiltonian perturbation theory used for the solitons here as the extended periodic solutions are not localised solutions which was an assumption of the theory used for solitons) we calculated them numerically(for 3 cases with soliton solutions only) and showed that they survived the perturbation.

We then proceded to reproduce the calculation in [2] to find the stability eigenfrequencies for soliton solutions via the Hamiltonian perturbation theory. We described how to use the BdG equations to confirm the results we obtain from Hamiltonian perturbation theory, but we did not carry out the analysis(solving the system of equations, however correspondence with Dr A.S. Rodrigues who performed the calculations in [2] confirms that this was the method used to produce the spectral planes in [2]).We found that when the bright soliton solution is centered at $\max[g(z)]$ it is stable and when centered at $\min[g(z)]$ it is unstable and we found an eigenfrequency mode with positive imaginary part which indicated the instability would be present.We arrived at the final stage in the calculation of the eigenfrequencies for dark solitons but did not calculate them. We did however simulate the evolution of a numerically stitched dark soliton and found that it was unstable in agreement with [2] and with the predicted type of instability.

Then we described the method to semi analytically stitch together bright soliton [2] (and carried out the stitching numerically) and attractive extended periodic solutions(carried out numerically in case where the elliptic moduls l and the integer j where approximately equal in each region) by requiring that the function values and gradients matched at the interfaces between adjacent regions were the nonlinearity coeffcient is constant.In the bright soliton case it was straight forward to carry out this stitching but in the extended periodic case it was more difficult because now we have two more parameters and we were effectively matching different types of functions across interfaces in contrast to soliton solutions were all the functions were a transformed sech function. This difficulty in suggesting a method was overcome by the fact that we required the parameters to satisfy a constraint(conservation of particle number etc) in order to obtain physically meaningful solutions. We could then proceed to search for the correct parameters which would satisfy all the constraints and hence possibly give us our stiched solution although this was not performed.

Extensions which could be followed are, we could perform the stitching for elliptic function solutions which was suggested in this dissertation. We could run numerical simulations to examine the numerical stability of the elliptic function solutions in the presence of a piecewise constant nonlinearty. We could then carry out the linear stability analysis using the BdG equations on Matlab. We could consider an fourier representation for the periodic piecewise-constant nonlinearity and examine the behaviour of soliton solutions as the number of terms is taken to infinity. A further suggestion given in [2], is to examine the same problem studied in [2] but in 2Dimensions.

BIBLIOGRAPHY

[1] L.D.Carr, C.W.Clark and W.P.Reinhardt, Phys. Rev. **62** (2000)

[2] A.S.Rodrigues,P.G. Kevrekidis,Mason.A.Porter,D.J.Frantzeskakis,P.Schmelcher, and A.R.Bishop arXiv:0712.0986v2 [nlm.PS] (2008)

[3] R.Carretero-Gonzalez, D.J.Frantzeskakis and P.G. Kevrekidis, arXiv:0805.0761v1 [cond-mat.other] (2008)

[4] J.C.Bronski, L.D.Carr,R.Carretero-Gonzalez,B.Deconinck,J.N.Kutz and K.Promislow, Phys. Rev. **64** (2001)

[5] J.C.Bronski, L.D.Carr,B.Deconinck,J.N.Kutz and K.Promislow, Phys. Rev. **63** (2001)

[6] Todd Kapitula, Panayotis G.Kevrekidis, and Bjorn Sandstede, Physcia. D. **195** (2004)263-282

[7] J.C.Bronski, L.D.Carr,B.Deconinck and J.N.Kutz, Phys. Rev.Lett. **Vol 86 8** (2001)

[8] F.Dalfovo,S.Giorgini,L.P.Pitaevskii,and S.Stringari Reviews of modern physics **Vol 71 3** (1999)

[9] P.G.Drazin and R.S. Johnson Solitons: an introduction (1999) cambrige texts in applied mathematics

[10] M.R. Speiegel Theory and Problems of Advanced Calculus,Chpt 16,(1974) Schaums outline series

[11] L.I.Schiff Quantum Mechanics,(1968) McGraw-Hill International Editions

[12] R.Baierlein American Journal of Physics 69 (4): 423434 (2006)

[13] Griffiths, David J., "Introduction to Quantum Mechanics", 2nd ed. Pearson Education, Inc., 2005.