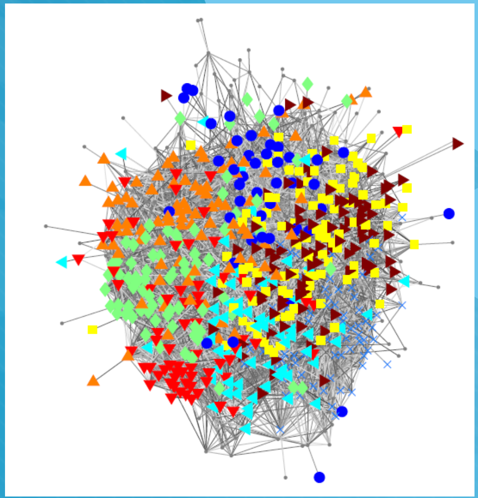

An Introduction to the Mathematics of Networks

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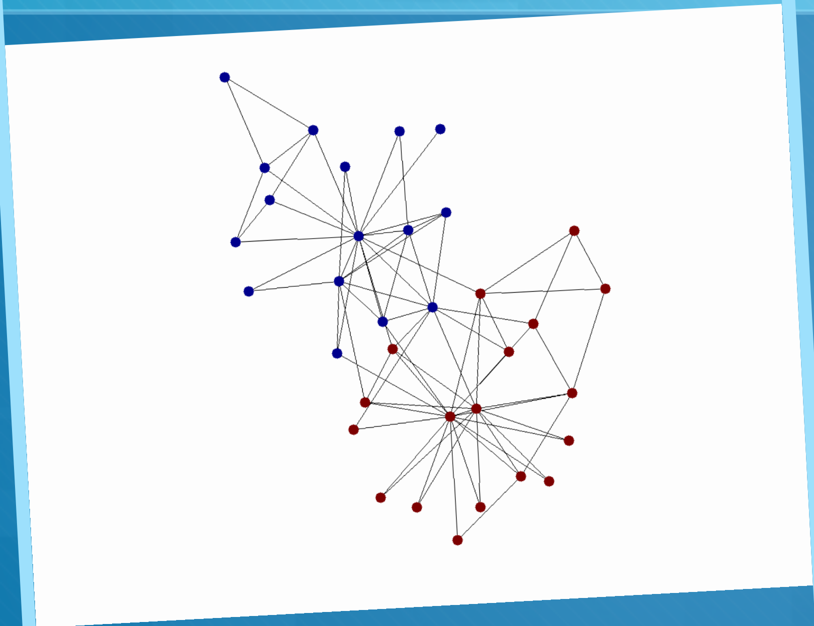
Outline

- Introduction
- What is *network science*?
- Discussion and Examples
- Conclusions



What is a network?

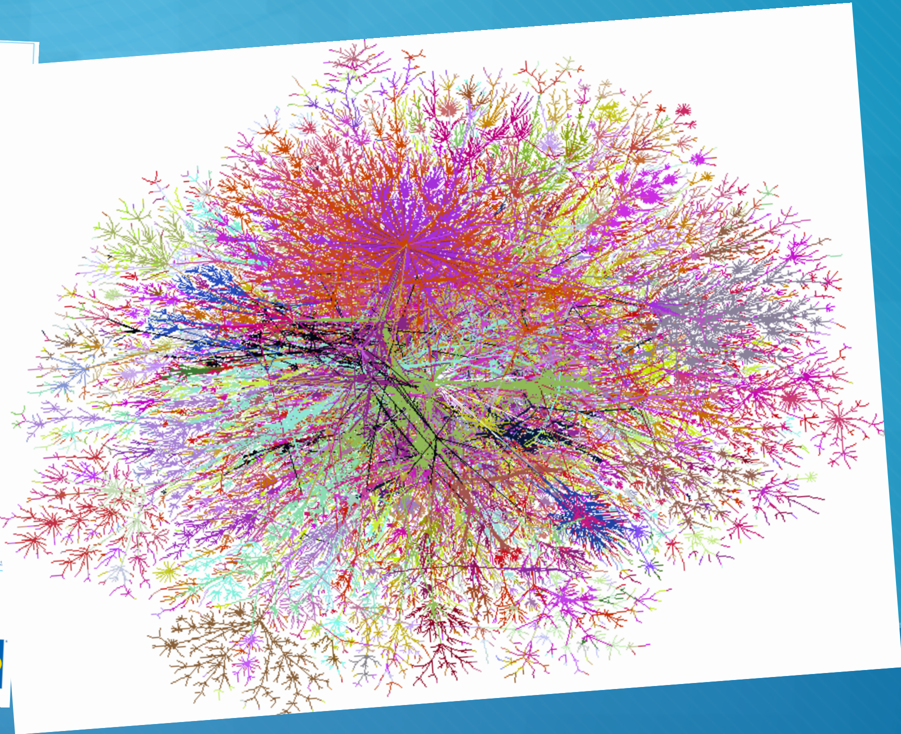
A network consists of *nodes* representing entities. Nodes are connected by *edges* representing ties between the entities.

Examples:

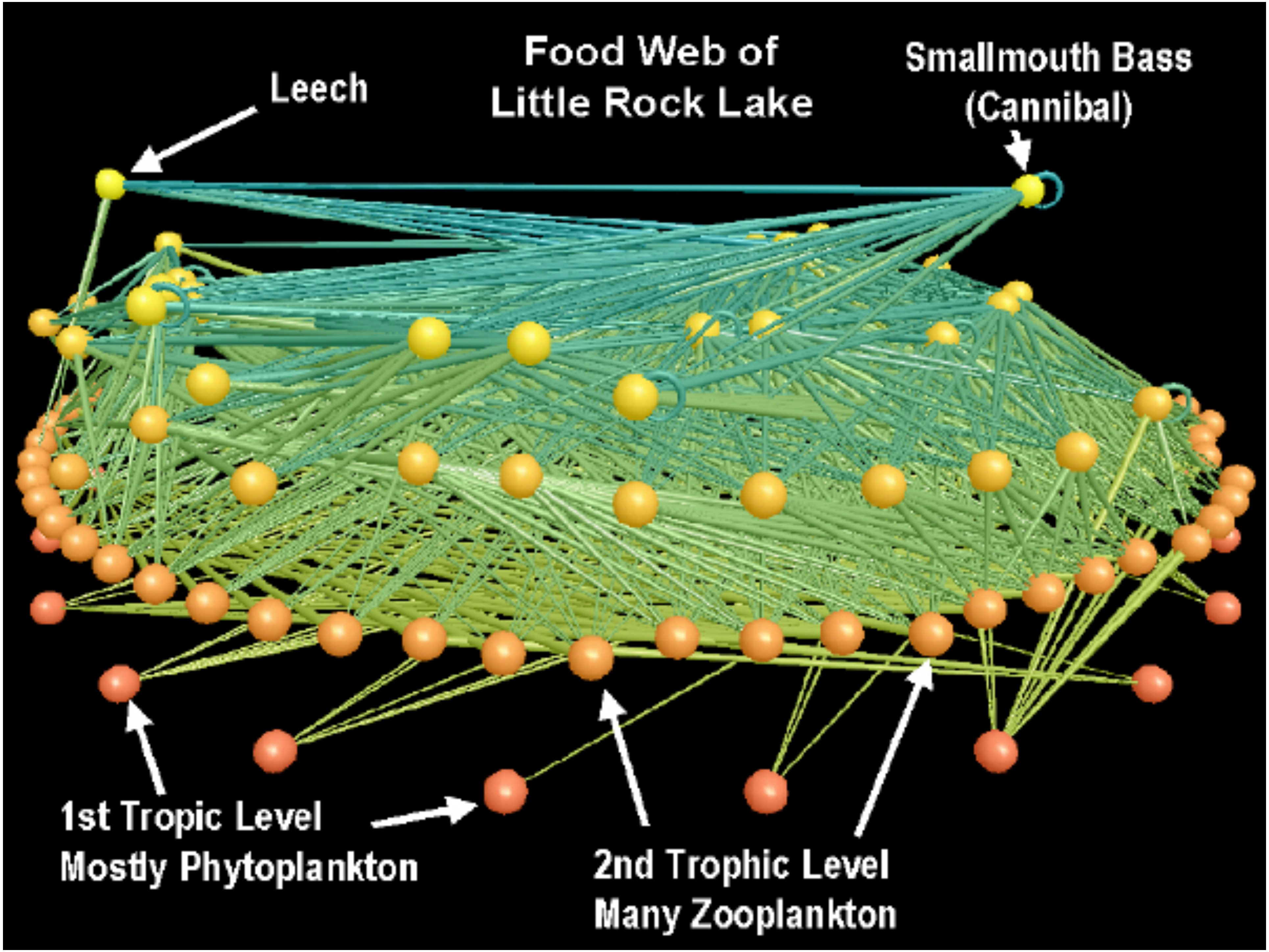
Individuals connected by Facebook "friendships"

Web pages connected by hyperlinks

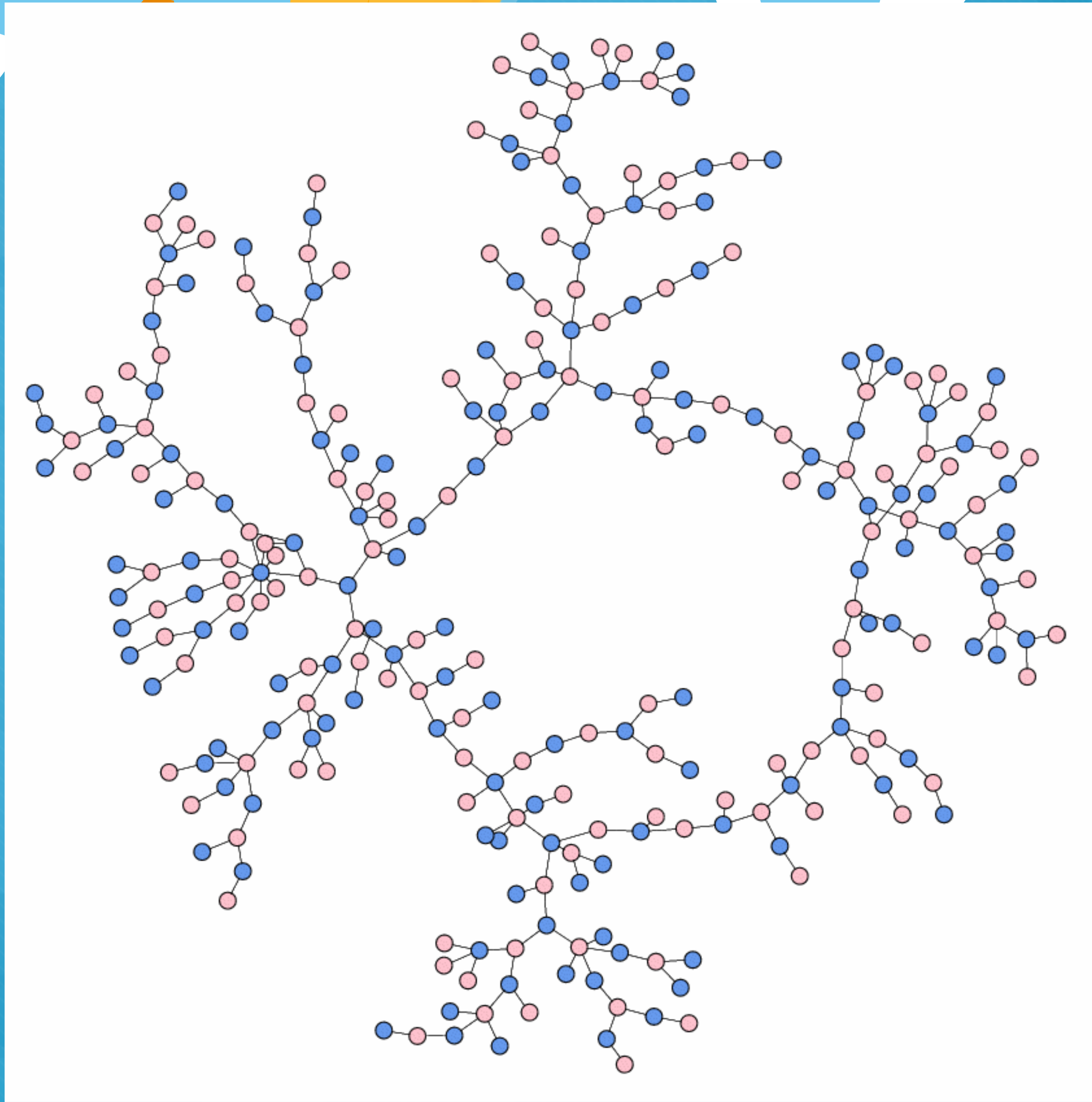
Contiguous cities on a train route



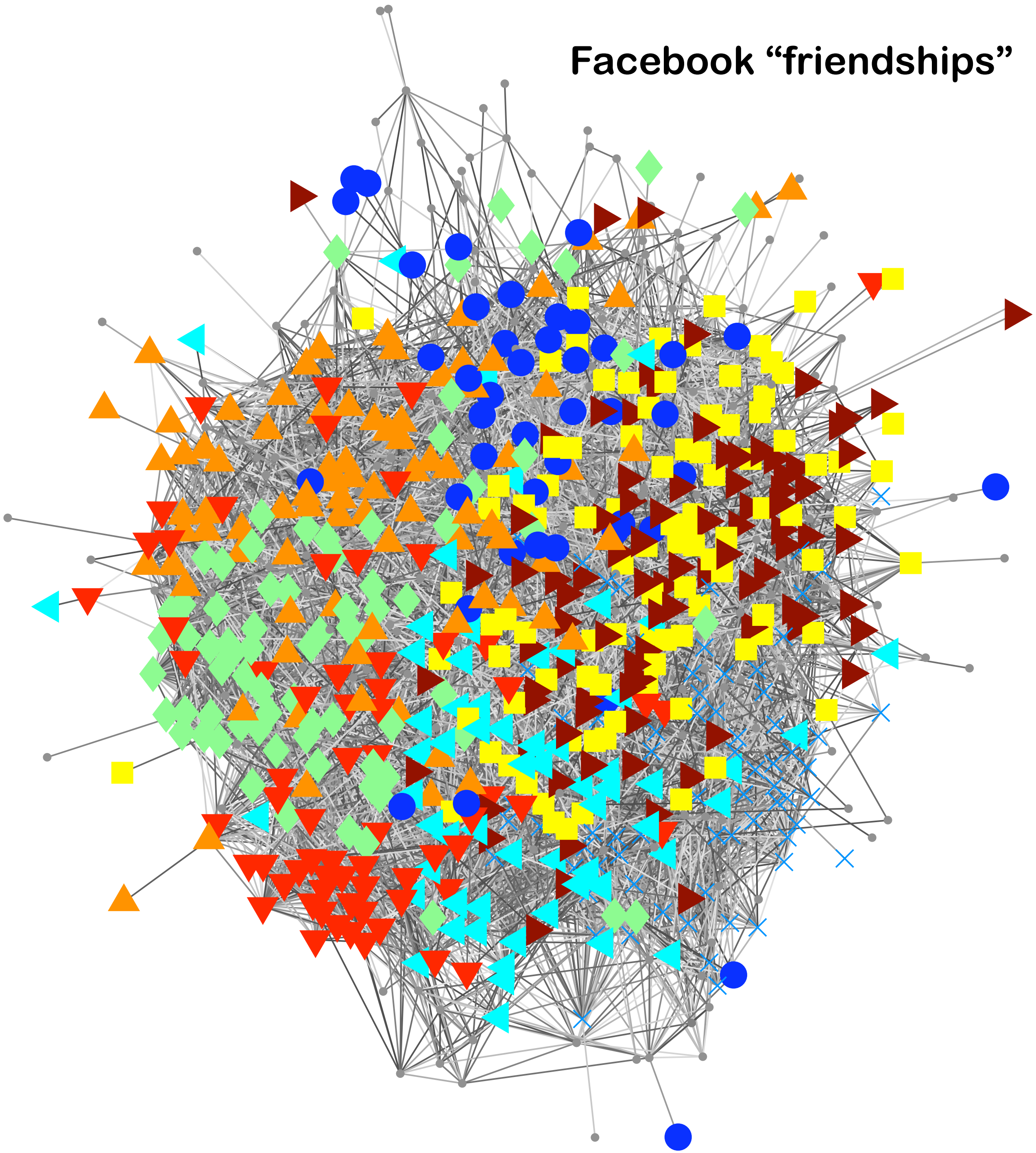
Networks are Everywhere

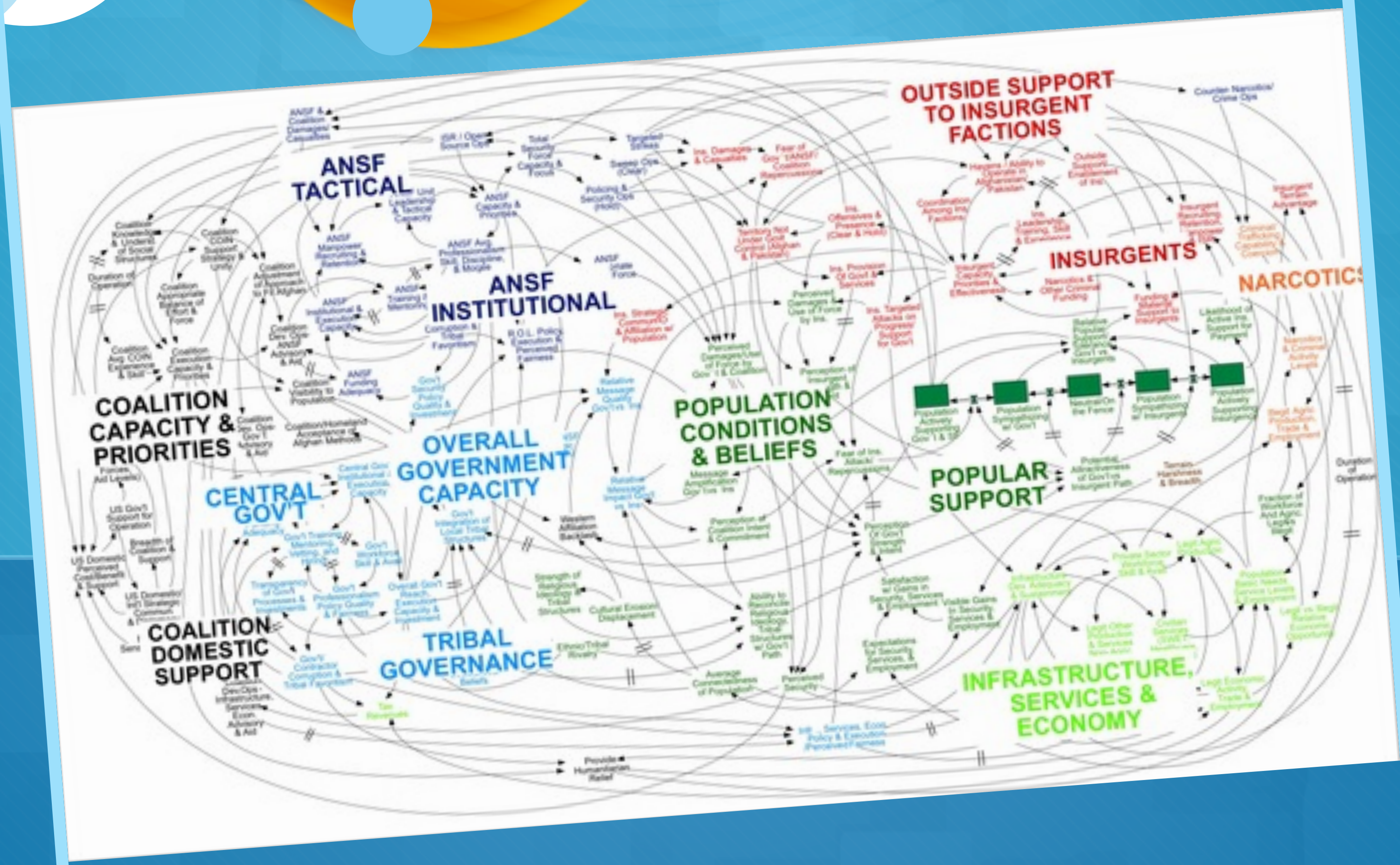


Food Web of
Little Rock Lake
Leech
Smallmouth Bass
(Cannibal)
1st Tropic Level
Mostly Phytoplankton
2nd Trophic Level
Many Zooplankton



Facebook “friendships”





“When we understand this slide, we’ll have won the war.”

Multirelational organization of large-scale social networks in an online world

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The capacity to collect fingerprints of individuals in online media has revolutionized the way researchers explore human society. Social systems can be seen as a nonlinear superposition of a multitude of complex social networks, where nodes represent individuals and links capture a variety of different social relations. Much emphasis has been put on the network topology of social interactions, however, the multidimensional nature of these interactions has largely been ignored, mostly because of lack of data. Here, for the first time, we analyze a complete, multirelational, large social network of a society consisting of the 300,000 odd players of a massive multiplayer online game. We extract networks of six different types of one-to-one interactions between the players. Three of them carry a positive connotation (friendship, communication, trade), three a negative (enmity, armed aggression, punishment). We first analyze these types of networks as separate entities and find that negative interactions differ from positive interactions by their lower reciprocity, weaker clustering, and fatter-tail degree distribution. We then explore how the interdependence of different network types determines the organization of the social system. In particular, we study correlations and overlap between different types of links and demonstrate the tendency of individuals to play different roles in different networks. As a demonstration of the power of the approach, we present the first empirical large-scale verification of the long-standing structural balance theory, by focusing on the specific multiplex network of friendship and enmity relations.

complex networks | multiplex relations | quantitative sociology

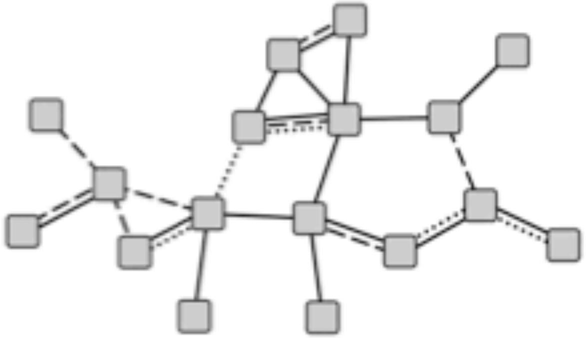
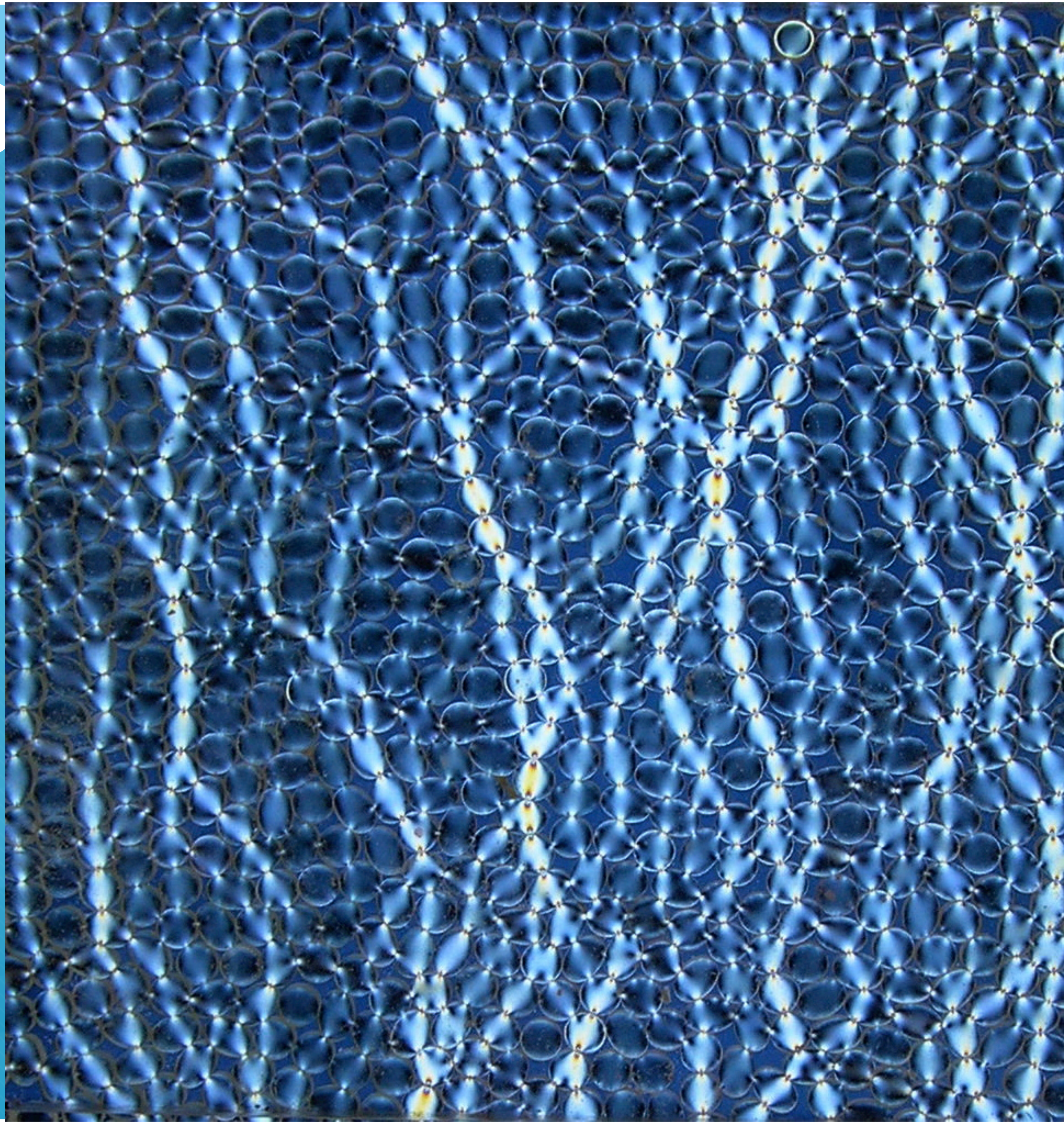


Fig. 1. Multiplex networks consist of a fixed set of nodes connected by different types of links. This multirelational aspect is usually neglected in the analysis of large social networks. In our MMOG dataset, six types of social links can exist between any two players, representing their friendship or enmity relations, their exchanged private messages, their trading activity, their one-to-one aggressive acts against each other (attacks), and their placing of head money (bounties) on other players as, e.g., means of punishment.

in the organization of large-scale networks. For example, the existence of different link types between agents explains the overlap of community structures observed in social networks, where nodes may belong to several communities, each associated to one different type of interaction (5, 6). Methodological work on multiplex networks includes the development of multiplex community detection (7), clustering (8), and other network analysis algorithms (9). The role of multiple relation types in

Networks in MMORPGs



Types of Networks

- Binary networks: 1 if there is a connection and 0 if there isn't
- Weighted networks: Some value if there is a connection (representing strength of connection) and otherwise 0
- Directed networks
- Bipartite networks: only nodes of different types are connected to each other (e.g., an actor connected to a movie in which he/she appeared)
- More …

Representing a Network

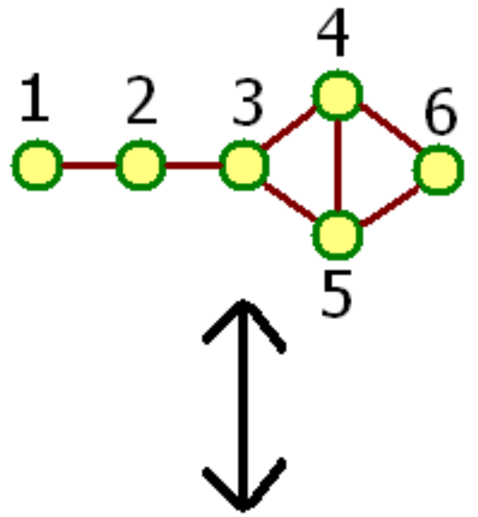
Adjacency matrix A

This example: binary ("unweighted")

A_{ij} = 1 if there is a connection between nodes i and j

A_{ij} = 0 if no connection

How do we generalize the mathematical representation to weighted, directed, and bipartite examples? What happens for networks that depend on time? What changes in other situations?



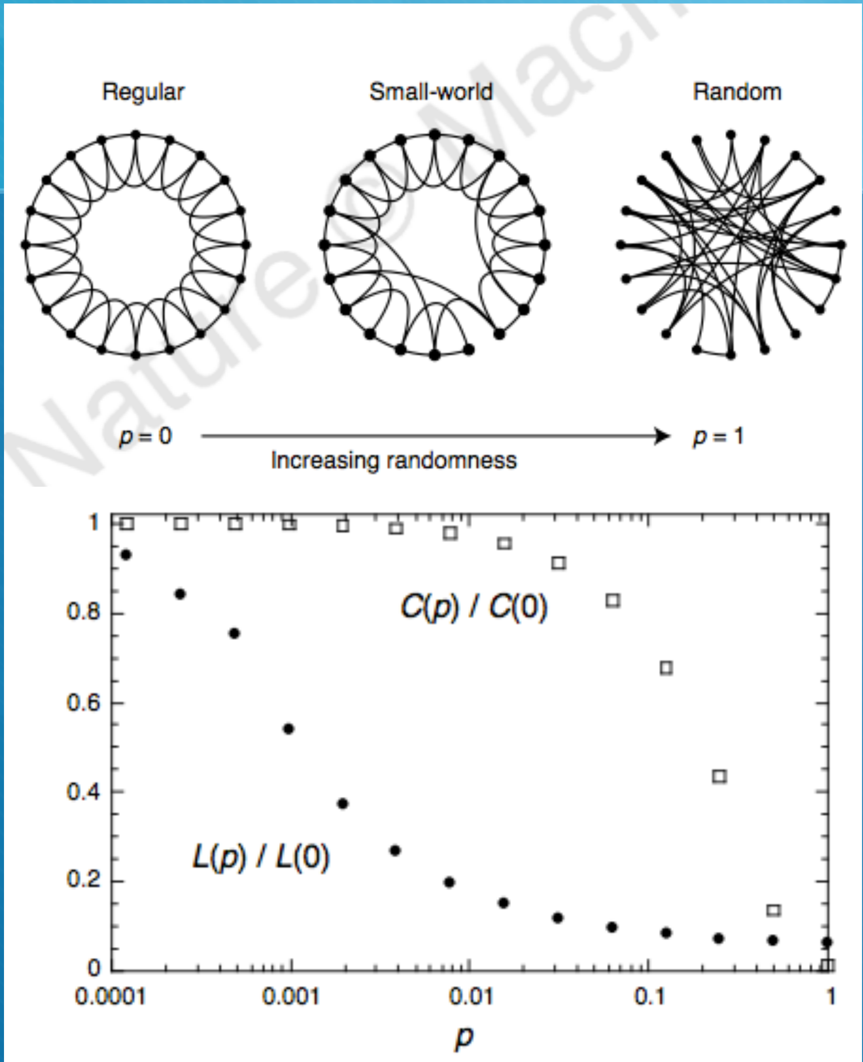
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Goals of Network Science

- 1. Basic principles
  - Microscopic, mesoscopic, and macroscopic structures
- 2. Function = structure + dynamics
  - Dynamics *on* networks, dynamics *of* networks, interactions between the two
    - Time-evolution, robustness, etc.
- 3. Application
  - Inference/prediction of structure, demographics, etc.
  - Control and design

Basic Principles

- Microscopic structure
  - Properties and roles of individual nodes and edges
  - E.g., local clustering properties, node roles
- Macroscopic and mesoscopic structure
  - Macro: Distributions of various microscopic properties of all nodes
    - Caveats: sensitive to noise, missing data, etc.
  - Meso: e.g., modules and hierarchies, core-periphery structure



Small Worlds

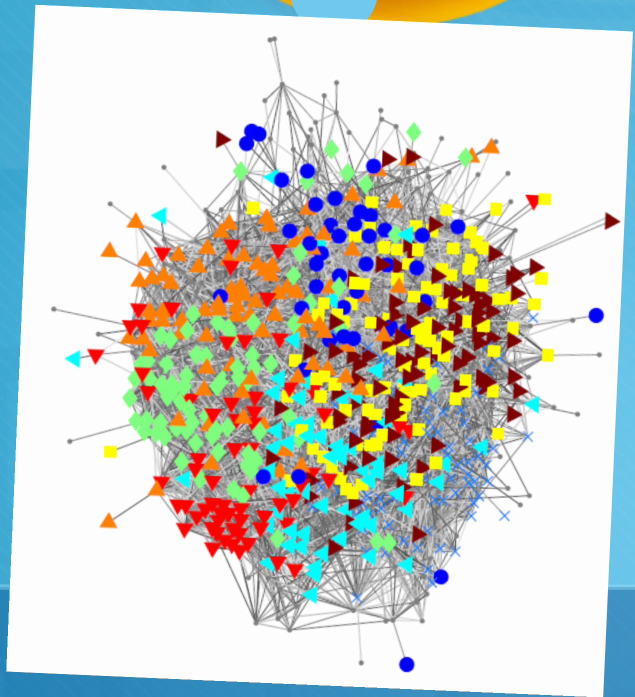
6 degrees of separation (psychologist Stanley Milgram)

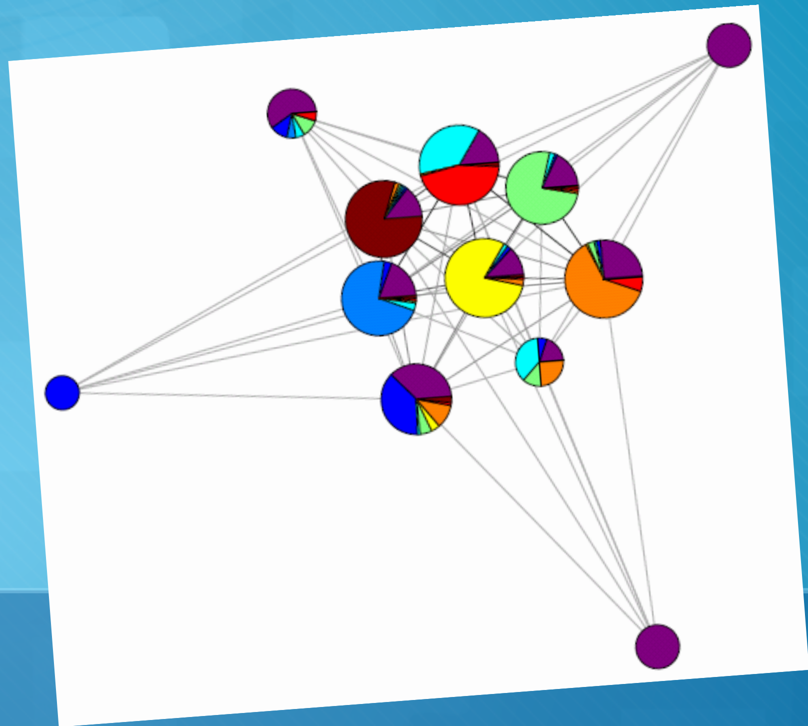
6 degrees of Kevin Bacon

Erdös numbers

Mathematical models developed starting in late 1990s to study this (starting with Watts & Strogatz, 1998)

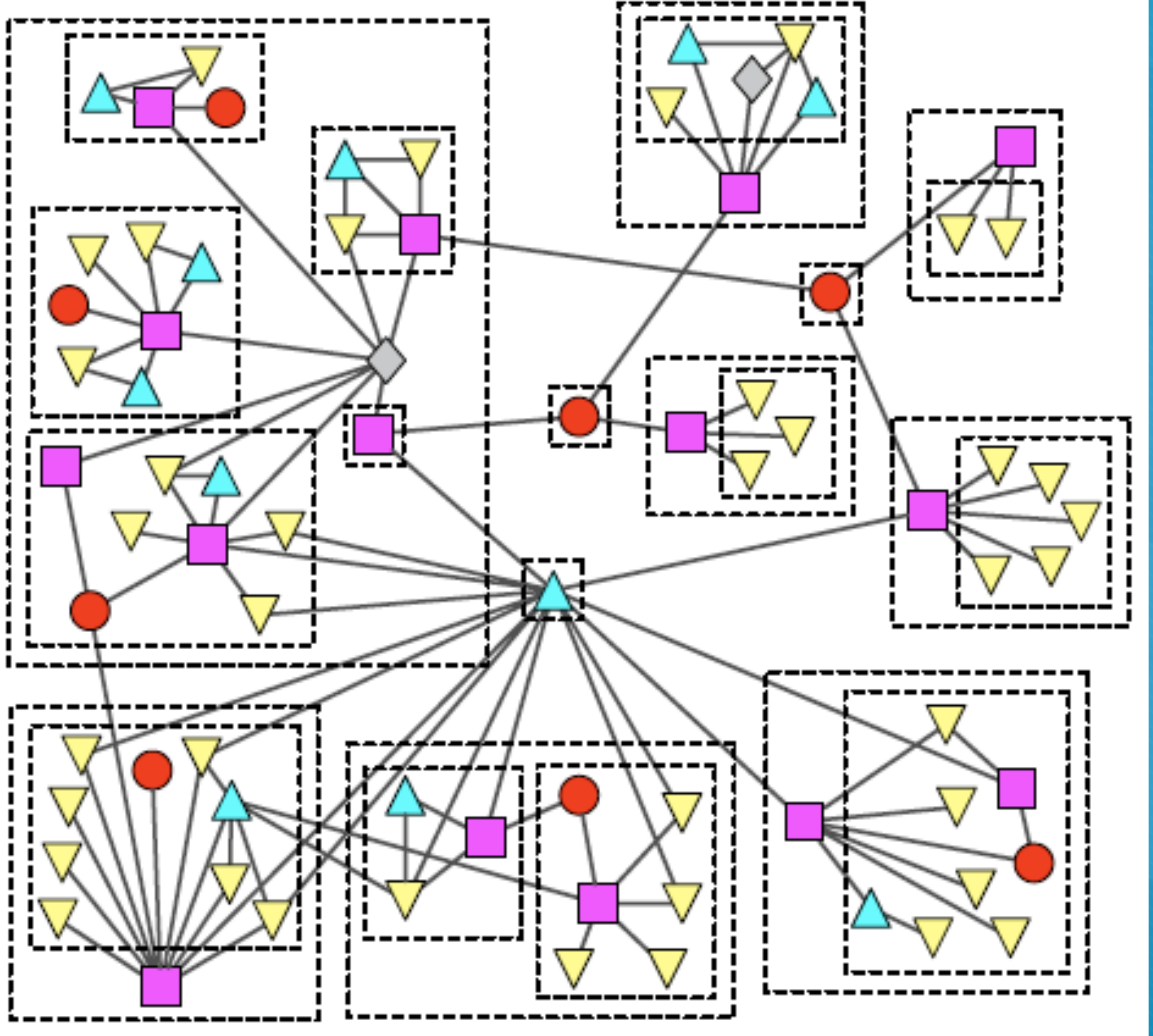
How to navigate small worlds?

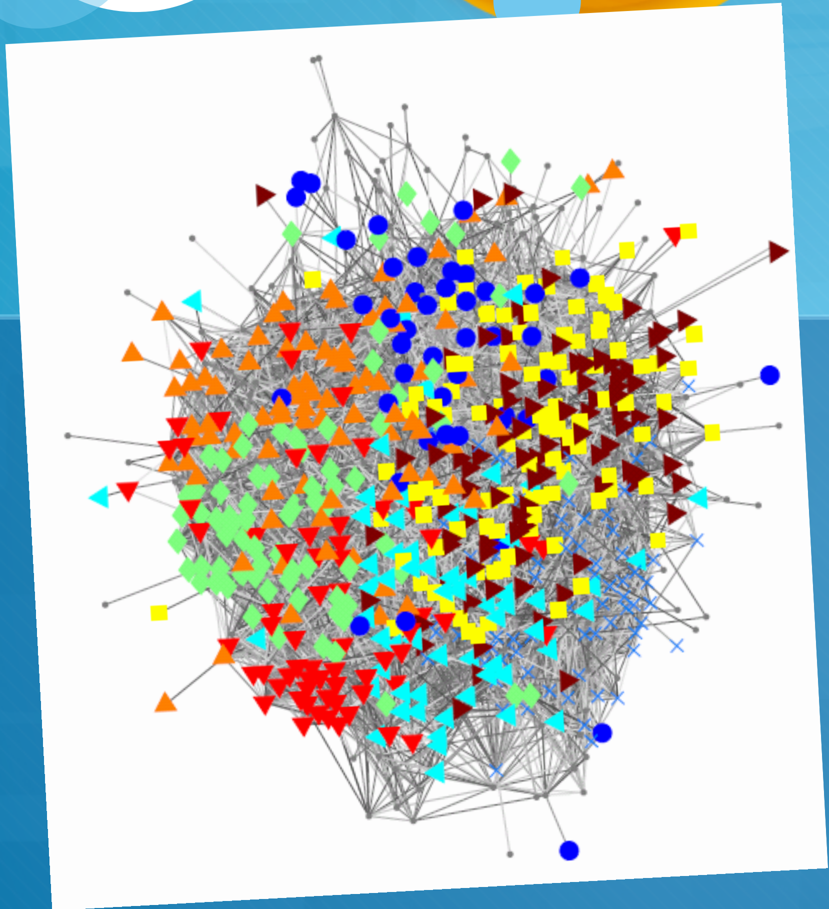




Community Detection (clustering)

Develop and use computer algorithms to group nodes (e.g. circles of friends) in an automated fashion.





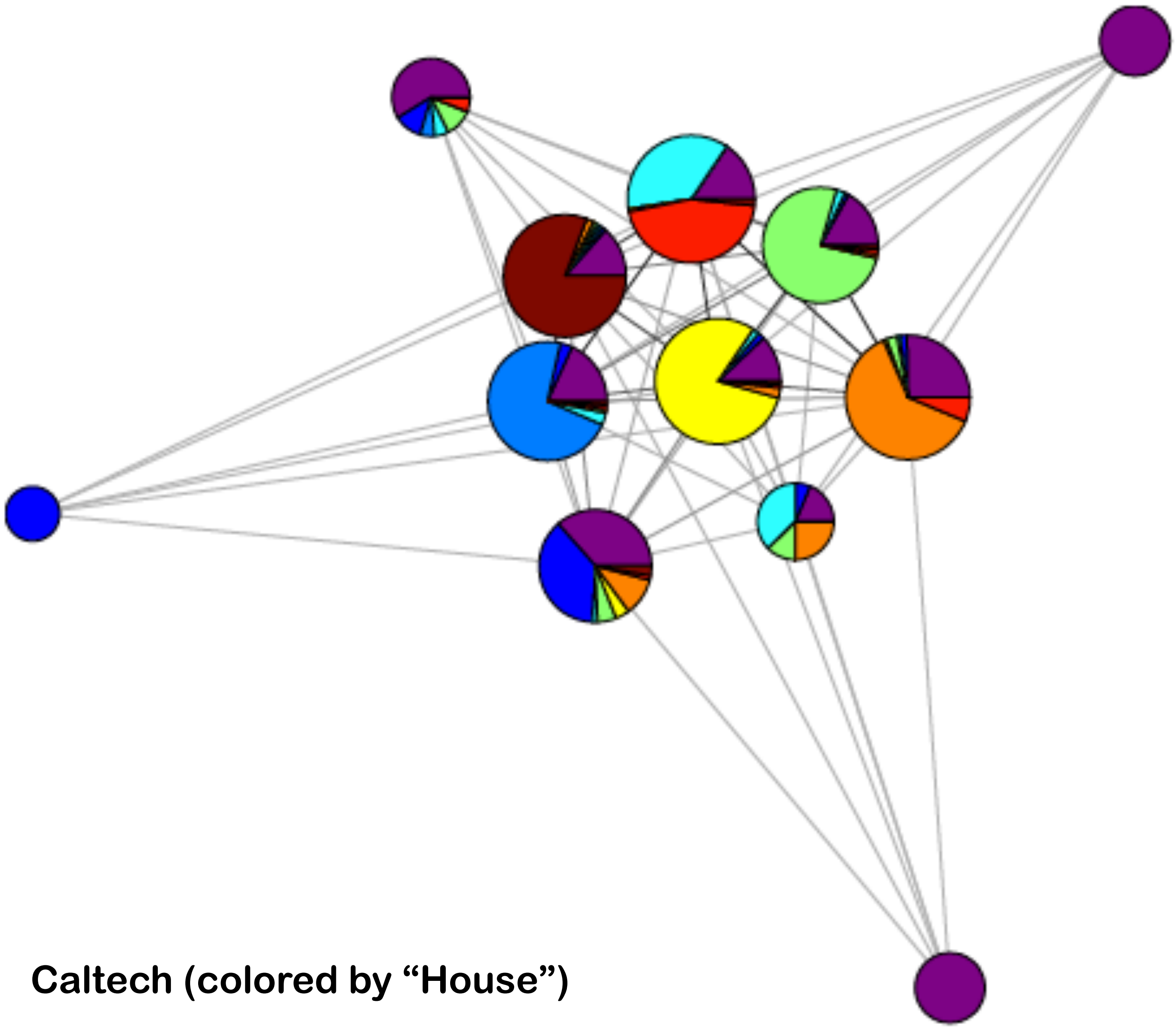
Facebook Networks

Nodes = individuals

Links = self-identified friendships (1 or 0)

Data

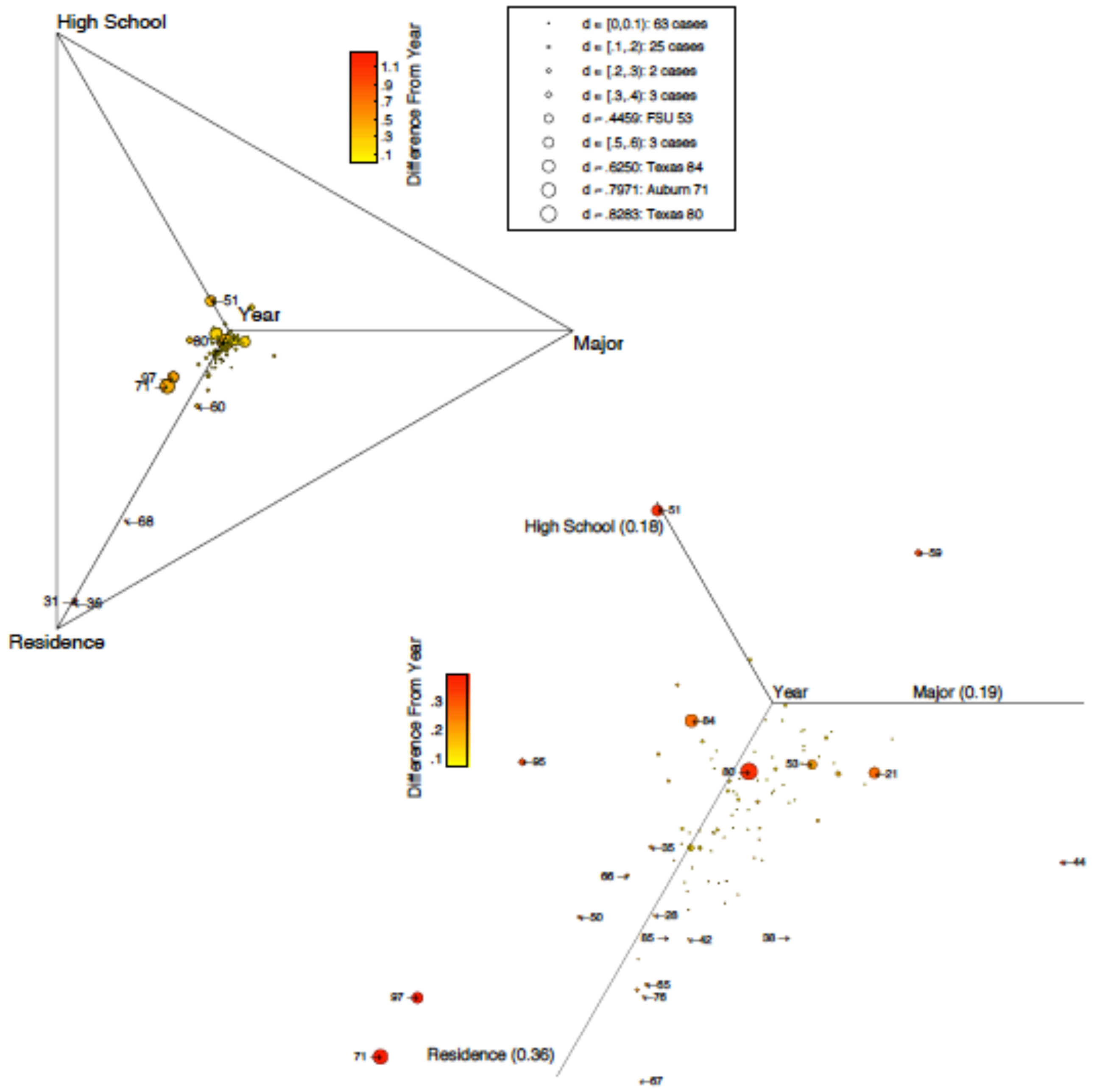
- 100 different universities (full networks)
- Single-time snapshot: September 2005
  - Facebook was university-only
- Self-reported demographics
  - Gender, class year, high school, major, dormitory/"House"
- Provided by Adam D'Angelo & Facebook

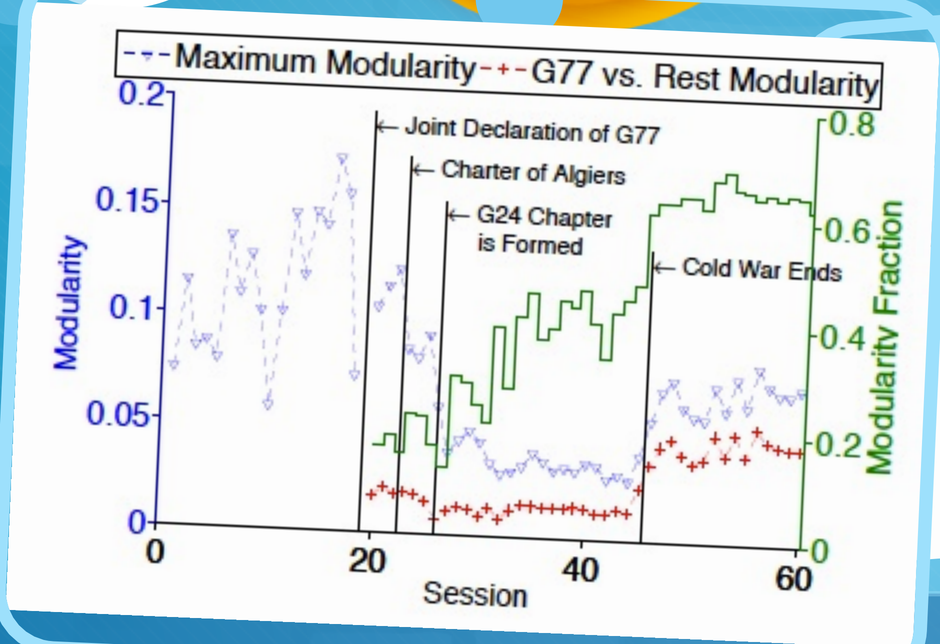


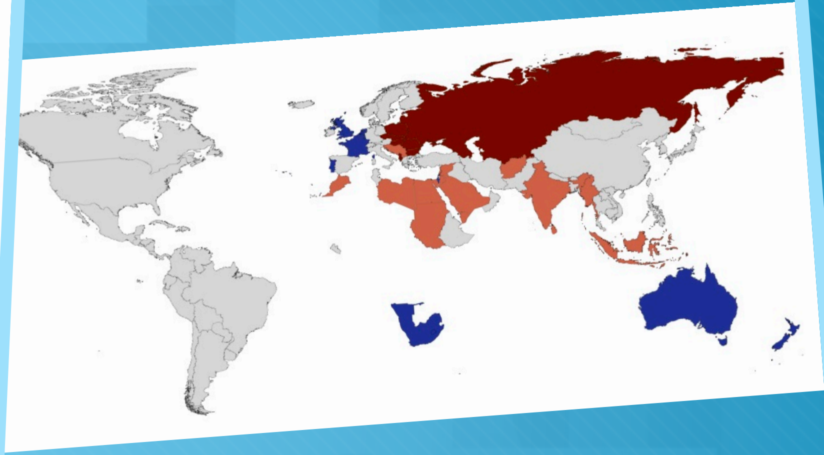
Caltech (colored by “House”)

How do universities organize?

- Houses are important at Caltech (reality check for methodology)
- High school is more important at large universities
- Class year is the most important factor at most universities and dorm is often a very strong secondary factor
- Major has varying importance at different universities
- Our work suggests interesting future research projects for social scientists


High School
Year
Major
Residence
Difference From Year
d ∈ [0,0.1): 63 cases
d ∈ [.1,.2): 25 cases
d ∈ [.2,.3): 2 cases
d ∈ [.3,.4): 3 cases
d = .4459: FSU 53
d ∈ [.5,.6): 3 cases
d = .6250: Texas 84
d = .7971: Auburn 71
d = .8283: Texas 80
High School (0.18)
Major (0.19)
Residence (0.36)





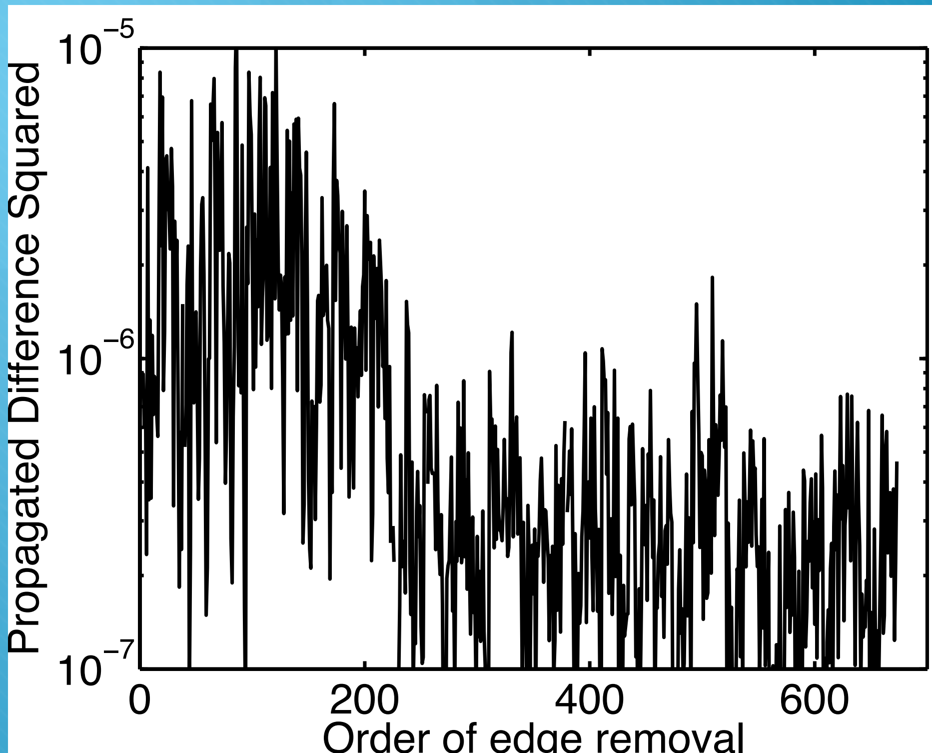
Voting Networks

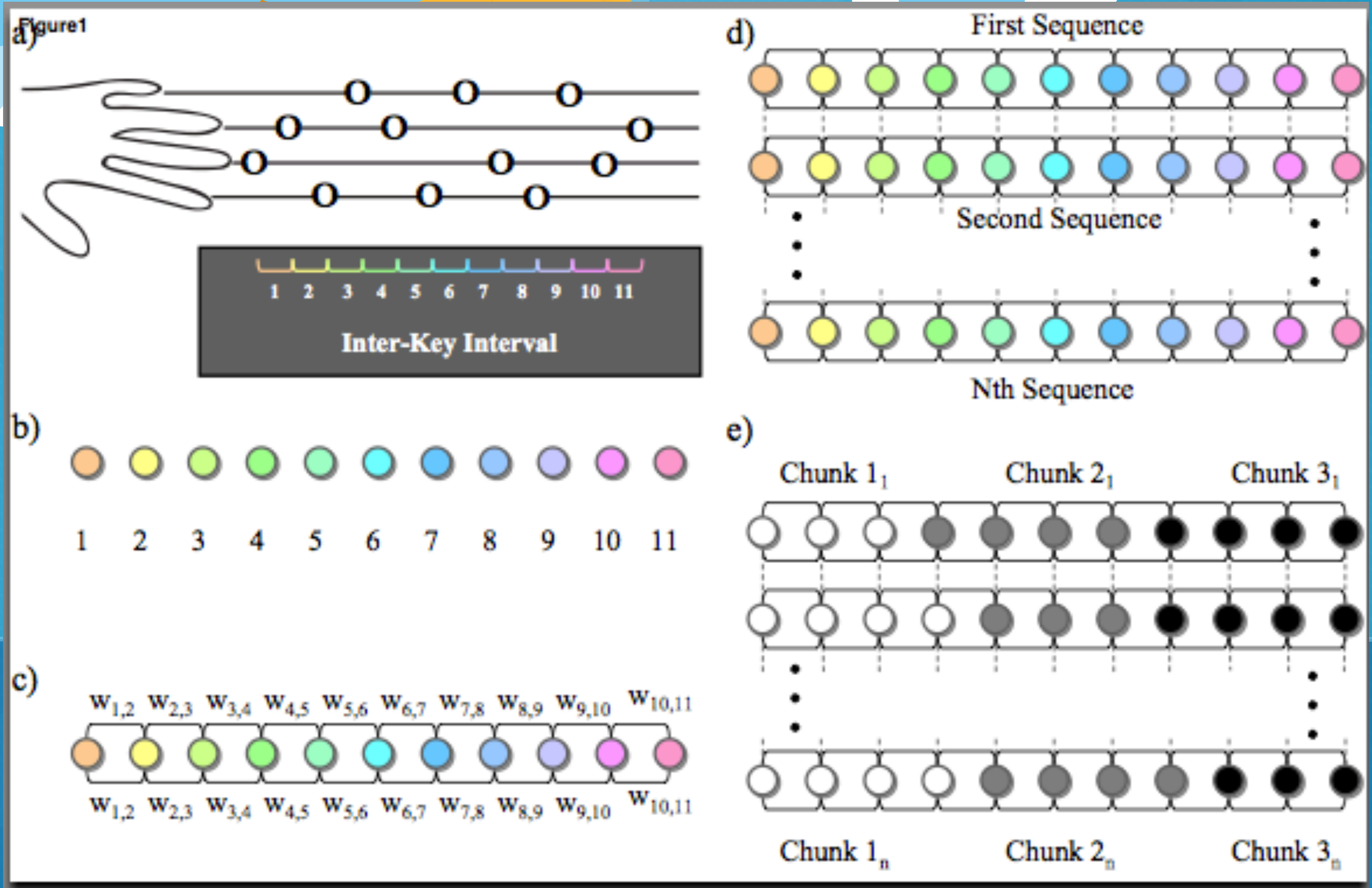
Example: Voting on resolutions in the United Nations General Assembly.

Ranking Sports Teams

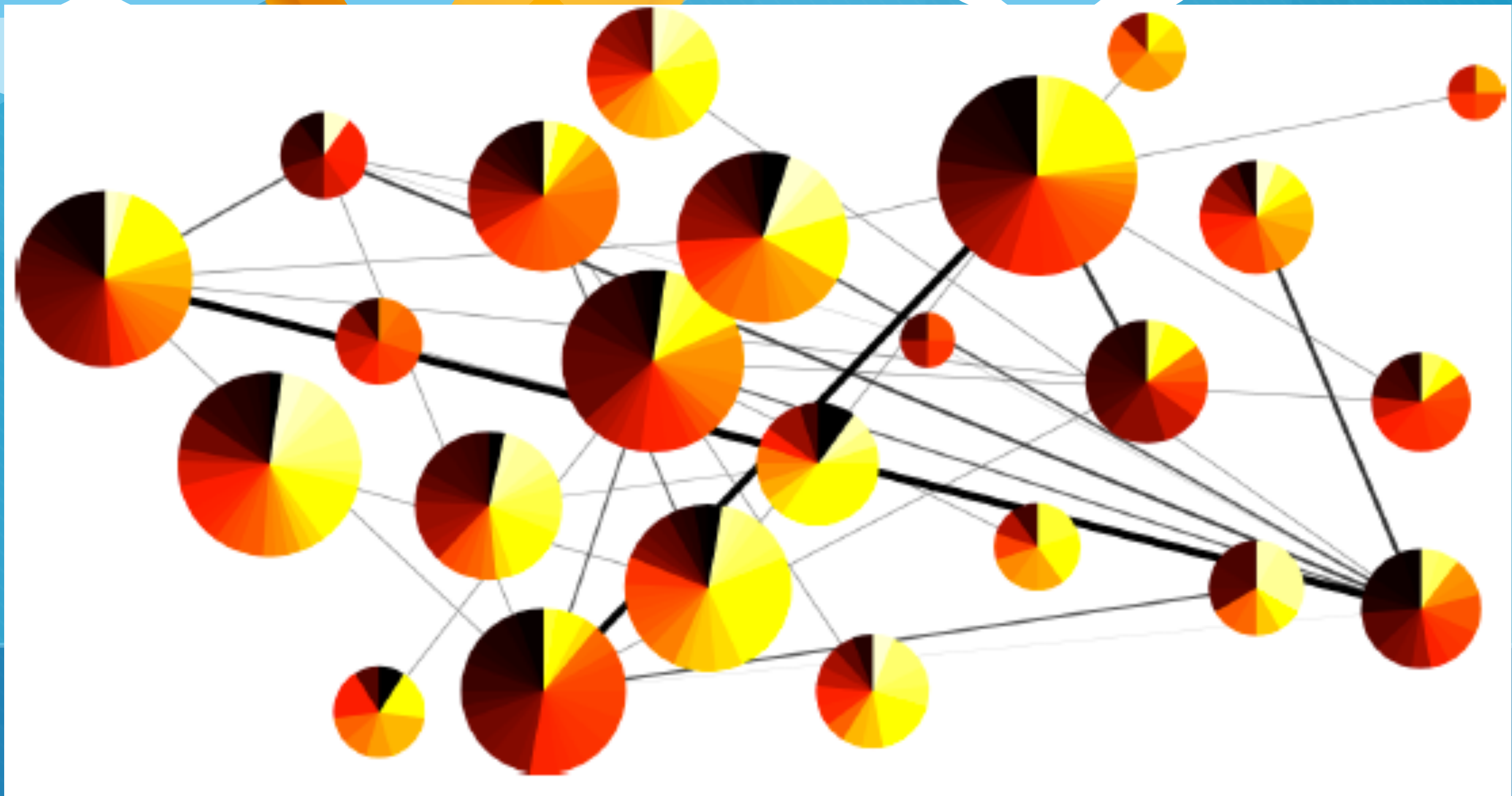
(this example: American college football)

- Network:
  - nodes = teams
  - edges = games
- Effect on ranking of reversing the outcome of games depends on how important the game is for the network structure





Chunking: Can networks tell you something about how you learn to get better at Guitar Hero?



Given a network, how should you measure which nodes are most important?

Example: How would you measure which railway stations are most important? What about Hollywood actors?

Some References

- M. E. J. Newman [2008] *Physics Today* 61(11): 33—38
- S. H. Strogatz [2001] *Nature* 410, 268—276.
- Lots of entries on Wikipedia and Scholarpedia

Conclusions

- Network science is a fascinating subject that draws from mathematics, physics, computer science, sociology, biology, and numerous other fields.
- A statistical mechanics perspective: Start with local information and appropriately coarse-grain/average over things to find global insights.
- Mathematics is a vibrant area that shows up *everywhere* in your life.