Methods in Applied PDEs (B5b - supplementary lectures), HT 2009. Example Sheet 1 (Solutions)

1. For convenience, use the notation $\delta(x-\xi)$ for what what we have been denoting δ_ξ, the δ function centered at ξ.

   a. Find the value of $x\delta(x)$ using the definition of $\delta(x)$.

   Solution: Consider $\int_{-\infty}^{\infty} x\delta(x)f(x)dx$, where $f(x)$ is continuous at $x = 0$. Write $xf(x) = g(x)$. This gives $g(0) = 0$, so that

   $$\int_{-\infty}^{\infty} x\delta(x)f(x)dx = 0 \tag{1}$$

   for all functions $f(x)$ that are continuous at $x = 0$. This justifies the equivalence of $x\delta(x)$ and the 0 function.

   b. Show that

   $$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{2}$$

   for $a \neq 0$ and use this to verify that $\delta(x)$ is an even function.

   Solution: Assume that $a > 0$ and write (using $ax = \xi$, $dx = (1/a)d\xi$):

   $$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \int_{-\infty}^{\infty} \delta(\xi)f(\xi/a)(1/a)d\xi = (1/a)f(0)\,. \tag{3}$$

   If $a < 0$ (again $ax = \xi$, $dx = (1/a)d\xi$), we obtain

   $$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \int_{\infty}^{-\infty} \delta(\xi)f(\xi/a)(1/a)d\xi = -(1/a)f(0)\,. \tag{4}$$

   Hence, we obtain in both cases that the result is $(1/|a|)f(0)$. Setting $a = -1$ shows that $\delta(x)$ is an even function.

   c. Show that

   $$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x+a) + \delta(x-a)] \tag{5}$$

   for $a > 0$.

Solution: Observe that $\delta(x^2-a^2)=[1/(2a)][\delta(x+a)+\delta(x-a)]$. Because $\delta(\xi)=0$ unless $\xi=0$, it follows that $\delta(x^2-a^2)=0$ except at $x=\pm a$. Therefore,

$$\int_{-\infty}^{\infty}\delta(x^2-a^2)f(x)dx=\int_{-a-\epsilon}^{-a+\epsilon}\delta[(x+a)(x-a)]f(x)dx \\ +\int_{a-\epsilon}^{a+\epsilon}\delta[(x+a)(x-a)]f(x)dx \quad (a>0)\,, \tag{6}$$

where $\epsilon\in(0,2a)$ can be arbitrarily small. In the neighborhood of $x=-a$, the factor $(x-a)$ can be replaced by $-2a$. This gives

$$\int_{-a-\epsilon}^{-a+\epsilon}\delta[(x+a)(x-a)]f(x)dx=\int_{-a-\epsilon}^{-a+\epsilon}\delta[(-2a)(x+a)]f(x)dx \\ =\int_{-a-\epsilon}^{-a+\epsilon}\frac{1}{|-2a|}\delta(x+a)f(x)dx=\int_{-\infty}^{\infty}\frac{1}{2a}\delta(x+a)f(x)dx\,. \tag{7}$$

The limits of $\pm\infty$ can be used because $\delta(x+a)=0$ except at $x=-a$. Similarly,

$$\int_{a-\epsilon}^{a+\epsilon}\delta[(x+a)(x-a)]f(x)dx=\int_{-\infty}^{\infty}\frac{1}{2a}\delta(x-a)f(x)dx\,. \tag{8}$$

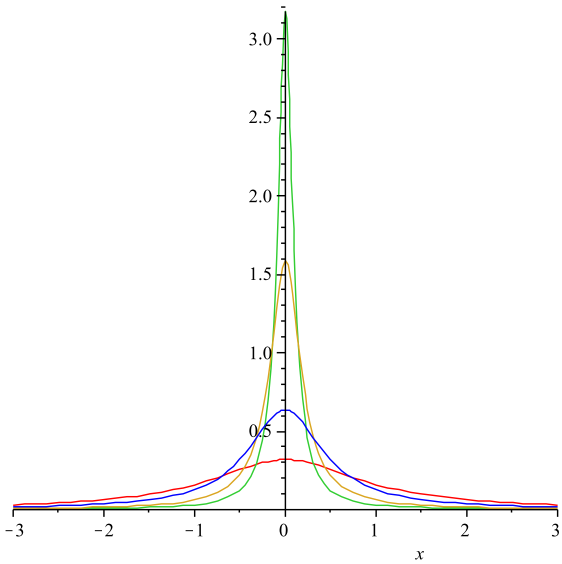
Note that the rule breaks down for $a=0$. There is apparently no way to interpret $\delta(x^2)$.

2. a. Sketch the function $f_n(x)=\frac{n}{\pi}\frac{1}{1+n^2x^2}$ for several values of n. Show that it converges weakly with

$$\lim_{n\longrightarrow\infty}\int_{-\infty}^{\infty}f_n(x)g(x)dx=g(0)\,. \tag{9}$$

(In other words, show that this a delta sequence.)

Solution: By definition, a sequence of functions $f_n(x)$ converges weakly (i.e., converges distributionally) as $n\to\infty$ if it converges in inner product. That is, for any test function $g(x)$, we have



$\langle f_n(x), g(x)\rangle \to \langle f(x), g(x)\rangle$ for some distribution f. In integral notation (with $x \in \mathbb{R}$), one can write

$$\int_{\mathbb{R}} f_n(x)g(x)dx \to \int_{\mathbb{R}} f(x)g(x)dx \tag{10}$$

for all test functions $g(x)$.

The figure below shows $f_n(x)$ for $n = 1, 2, 5, 10$.

As $n \to \infty$, it does not converge to any function. However, it converges weakly and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx = g(0) \tag{11}$$

for any test function $g(x)$, as we shall now prove.

$$\int_{-\infty}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx = \int_{-\infty}^{-1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx + \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx + \int_{1/\sqrt{n}}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx\,. \tag{12}$$

Let B the bound for $g(x)$ so that $|g(x)| \leq B$ for all x. Consequently,

$$\left|\int_{1/\sqrt{n}}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx\right| \leq B\int_{1/\sqrt{n}}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}dx = B\left(\frac{1}{2}-\frac{1}{\pi}\arctan(\sqrt{n})\right) \to 0 \tag{13}$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_{1/\sqrt{n}}^{\infty} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx = 0\,. \tag{14}$$

Similarly,

$$\lim_{n\to\infty}\int_{-\infty}^{-1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx = 0\,. \tag{15}$$

Using the mean value theorem (and noting that $f_n(x) > 0$), we obtain

$$\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx = g(\xi)\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}dx = g(\xi)\frac{2}{\pi}\arctan(\sqrt{n})\,, \tag{16}$$

where $\xi \in [-1/\sqrt{n}, 1/\sqrt{n}]$. This gives

$$\lim_{n\to\infty}\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \frac{n}{\pi}\frac{1}{1+n^2x^2}g(x)dx = g(0)\,, \tag{17}$$

completing the proof.

b. Consider the function $F_n(x)$ given by

$$F_n(x) = \exp\left\{-\frac{1}{[1-4(nx-1)^2]}\right\} \tag{18}$$

for $|x-1/n| < 1/(2n)$ and $F_n(x) = 0$ for $|x-1/n| \geq 1/(2n)$. Now define

$$f_n(x) = \frac{F_n(x)}{\int_{1/(2n)}^{3/(2n)} F_n(x)dx} \tag{19}$$

to normalize the function. Show that the sequence $f_n(x)$ converges pointwise to 0. Then show that $f_n(x)$ converges weakly as a distribution. Comment on the significance of your observations to models of physical systems.

Solution: First, we'll show that $f_n(x)$ converges pointwise to zero. If $x \leq 0$, then $f_n(x) = 0$ for all n. If $x > 0$, choose N sufficiently large so that $N > 3/(2x)$. Then $f_n(x) = 0$ for all $n \geq N$, and we have pointwise convergence. However, the convergence is *not* uniform.

Now, let's show that $f_n(x)$ converges weakly as a distribution. For any test function $g(x)$, it follows from the mean value theorem (noting that $f_n \geq 0$) that

$$\int_{-\infty}^{\infty} f_n(x)g(x)dx = \int_{1/(2n)}^{3/(2n)} f_n(x)g(x)dx = g(\xi)\int_{1/(2n)}^{3/(2n)} f_n(x)dx = g(\xi) \tag{20}$$

for $\xi \in [1/(2n), 3/(2n)]$. If $n \to \infty$, then $\xi \to 0^+$. By the continuity of test functions, $\lim_{\xi\to 0} g(\xi) = g(0)$. (This means that $\{f_n(x)\}$ is a delta sequence but it is actually a convergent one (it converges to zero). This was able to happen because the peak of $f_n(x)$ shifts while it becomes higher and narrower.) We have thus shown that $\lim_{n\to\infty} f_n(x) = 0$, so that

$$\int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x)g(x)dx = 0 \tag{21}$$

for all $g(x)$ but

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)g(x)dx = g(0) \tag{22}$$

for all $g(x)$.

This suggests that the concept of pointwise convergence may have very little meaning from the perspective of a physicist. For example, if the functions $f_n(x)$ represent successive approximations to a certain physical quantity and the formula

$$\int_{-\infty}^{\infty} f_n(x)dx = 1 \tag{23}$$

remains true at all times, then how is it possible to reconcile physical intuition with the statement that $f_n(x)$ "converges to zero?" Essentially, for certain situations, distributions may be more suitable than conventional functions for the description of physical quantities. For example, constraints like (23) are very common in thermodynamics, as one always takes limits under the condition that certain quantities remain constant.

3. **A calculus for functions with jumps:** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable except at points $a_1, \cdots, a_k$ where it has jumps. (All the left-hand and right-hand derivatives exist at those points.) Denote by $[f^{(m)}]$ the function obtained by differentiating f m times without regard to the jumps.

   a. Derive the expression for f' (i.e., including the jumps).

   Solution: The idea behind this problem is to generalize $H' = \delta$ to more general functions with jumps. Note that $[H'] = 0$.

   The piecewise continuous function $[f']$ will not usuall coincide with the distributional derivative of f. If f has jumps of amount $\Delta f_1, \cdots, \Delta f_k$ at $a_1, \cdots a_k$ (respectively), then the function

$$g = f - \sum_{i=1}^{k} \Delta f_i H(x - a_i) \tag{24}$$

   is continuous and has piecewise continuous derivative g' that coincides with its distributional derivative. Applying differentiation in the sense of distributions gives

$$g' = f' - \sum_{i=1}^{k} \Delta f_i \delta(x - a_i) = [f'], . \tag{25}$$

   This gives

$$f' = [f'] + \sum_{i=1}^{k} \Delta f_i \delta(x - a_i)\,. \tag{26}$$

   b. Suppose the only point of discontinuity is $a_1 = 0$. Write down the expression for $f^{(m)}$.

Solution: Denote by $\Delta f^{(m)}$ the jump in $[f^{(m)}]$ at $x = 0$. Successive differentiating of (26) gives

$$f^{(m)} = [f^{(m)}] + \Delta f^{(m-1)}\delta + \Delta f^{(m-2)}\delta' + \cdots + \Delta f^{(0)}\delta^{(m-1)} \,. \quad (27)$$

This can be proven by induction.

c. Calculate f', f'', and f''' for $f(x) = e^{-|x|}$.

Solution:

$$\begin{aligned} f' &= [f'] = \begin{cases} -e^{-x}\,, & x > 0\,, \\ e^{x}\,, & x < 0\,. \end{cases} \quad , \\ f'' &= [f''] - 2\delta(x) = f - 2\delta(x)\,, \\ f''' &= f' - 2\delta'(x)\,. \end{aligned} \quad (28)$$

4. a. Consider $f(x) = H(x)\log(x)$, where $x \in \mathbb{R}$ and the Heaviside "function" $H(x - \xi)$ equals 1 when $x > \xi$ and equals 0 when $x < \xi$. Calculate the derivative of $f(x)$ in the distributional sense. **Hint:** To do this rigorously, you will need to recall the definition of the Cauchy principal value integral.

Solution: The function $f(x) = H(x)\log x$ defines the distribution

$$\langle f, \phi\rangle = \int_0^\infty \phi(x)\log x dx \,. \quad (29)$$

Using integration by parts, we have

$$\langle f', \phi\rangle = -\langle f, \phi'\rangle = -\int_0^\infty \phi'(x)\log x dx \,, \quad (30)$$

which we would like to transform to examine the function $H(x)(1/x)$, which must be somehow related to $f'(x)$. However, $H(x)/x$ is not locally integrable, so it doesn't define a distribution and some qualification is needed.

The right side of (30) is a convergent integral, so we can write

$$\langle f', \phi\rangle = -\lim_{\epsilon\to 0}\int_\epsilon^\infty \phi'(x)\log x dx \,. \quad (31)$$

Integration by parts gives

$$\langle f', \phi \rangle = \lim_{\epsilon \to 0} \left[\int_\epsilon^\infty \frac{\phi(x)}{x} dx + \phi(\epsilon) \log \epsilon \right] = \left[\int_\epsilon^\infty \frac{\phi(x)}{x} dx + \phi(0) \log \epsilon \right] . \tag{32}$$

The individual terms in the brackets have no limit as $\epsilon \to 0$ (unless ϕ happens to vanish at 0), each bracket taken as a whole *does* have a limit and it is this limit that defines the distribution. The right side of (32) can be interpreted as a regularization of the divergent integral $\int_0^\infty (\phi(x)/x)dx$. One can write

$$\frac{d}{dx} H(x) \log x = \mathrm{pf} \frac{H(x)}{x} , \tag{33}$$

where "pf" stands for "pseudofunction." The right side of (32) can thus be written

$$\langle \mathrm{pf}[H(x)/x], \phi \rangle . \tag{34}$$

One finds similarly that $H(-x)\log(-x)$ coincides with $\log|x|$ for $x < 0$ and vanishes for $x > 0$. One obtains

$$\frac{d}{dx} H(-x) \log(-x) = \mathrm{pf}[H(-x)/x] . \tag{35}$$

We thus obtain

$$\frac{d}{dx} \log|x| = \mathrm{pf}[H(x)/x] + \mathrm{pf}[H(-x)/x] = \mathrm{pf}[1/x] , \tag{36}$$

where the pseudofunction $1/x$ is defined as the distribution

$$\langle \mathrm{pf}[1/x], \phi \rangle = \lim_{\epsilon \to 0} \left[\int_\epsilon^\infty \frac{\phi(x)}{x} dx + \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} dx \right] . \tag{37}$$

The right side of (37) converges as $\epsilon \to 0$ and defines a distribution but the individual terms diverge unless $\phi(0) = 0$. With this construction, we have assigned a value, known as the *Cauchy principal value*, to the otherwise-divergent integral $\int_{-\infty}^\infty [\phi(x)/x]dx$.

b. Consider $f(x) = 1/|x|$, where $x \in \mathbb{R}^3$. Calculate $\Delta f(x)$ in the distributional sense.

Solution: The operator Δ is formally self-adjoint, so

$$\left\langle \Delta \frac{1}{|x|}, \phi \right\rangle = \left\langle \frac{1}{|x|}, \Delta\phi \right\rangle = \int_{\mathbb{R}^3} \frac{\Delta\phi}{|x|} dx \,. \tag{38}$$

The integral on the right side of (38) converges because the singularity at the origin is very weak and we may write

$$\int_{\mathbb{R}^3} \frac{\Delta\phi}{|x|} dx = \lim_{\epsilon\to 0} \int_{|x|>\epsilon} \frac{\Delta\phi}{|x|} dx \,. \tag{39}$$

We calculate the integral on the right of (39) using a Green identity and the fact that $\phi \equiv 0$ outside of a bounded region. This gives

$$\int_{|x|>\epsilon} \frac{\Delta\phi}{|x|} = \int_{|x|>\epsilon} \phi\Delta\left(\frac{1}{|x|}\right) dx + \int_{|x|=\epsilon} \left[\frac{1}{|x|}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)\right] dS \,, \tag{40}$$

where n is the outward unit normal to the domain $\{|x| > \epsilon\}$. Letting $r := |x|$, noting that $\partial/\partial n = -(\partial/\partial r)$, and using the fact that $\Delta(1/r) = 0$ for $r \neq 0$, we obtain

$$\int_{|x|>\epsilon} \frac{\Delta\phi}{|x|} dx = -\int_{|x|=\epsilon} \left(\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\phi}{r^2}\right) dS \,. \tag{41}$$

All of the derivatives of a test function are bounded. Therefore,

$$\left|\frac{\partial\phi}{\partial r}\right| < M \tag{42}$$

for all x, and we obtain

$$\left|\int_{|x|=\epsilon} \frac{1}{r}\frac{\partial\phi}{\partial r} dS\right| \leq \frac{M}{\epsilon}(4\pi\epsilon^2) = 4\pi\epsilon M \to 0 \tag{43}$$

as $\epsilon \to 0$. Additionally,

$$\int_{|x|=\epsilon} \frac{\phi}{r^2} dS = \int_{|x|=\epsilon} \frac{\phi(0) + [\phi(x) - \phi(0)]}{r^2} dS = 4\pi\phi(0) + \int_{|x|=\epsilon} \frac{\phi(x) - \phi(0)}{r^2} dS \,. \tag{44}$$

Because $\phi(x)$ is continuous at $x = 0$, the last integral tends to 0 as $\epsilon \to 0$. Consequently,

$$\left\langle \Delta \left(\frac{1}{|x|} \right), \phi \right\rangle = -4\pi\phi(0), . \tag{45}$$

This means that we have

$$\Delta \left(\frac{1}{|x|} \right) = -4\pi\delta(x) \tag{46}$$

in the sense of distributions.

5. **Fourier transforms in the complex plane:**

a. Using $\omega = u + iv$, show that one can write

$$f(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} \hat{f}(\omega) e^{-i\omega x} d\omega \tag{47}$$

for appropriate $v \in \mathbb{R}$ (that you should specify to the extent possible).

Solution: With $\omega = u + iv$, we obtain

$$\hat{f}(\omega) = \hat{f}(u + iv) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx = \int_{-\infty}^{\infty} e^{iux} e^{-vx} f(x) dx\,. \tag{48}$$

Thus, $\hat{f}(\omega)$ amounts to taking a real-parameter Fourier transform (with transform variable u) of the function $g(x) = e^{-vx} f(x)$. If one chooses v such that $g(x) \in L_1(-\infty, \infty)$, then we get

$$e^{-vx} f(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{f}(u + iv) e^{-iux} du\,. \tag{49}$$

The last integral can be interpreted as an integral in the complex ω-plane along a line parallel to the real axis. In fact, we have

$$f(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{iv-R}^{iv+R} \hat{f}(\omega) e^{-iux} du\,. \tag{50}$$

With a slight abuse of notation, this gives

$$f(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} \hat{f}(\omega) e^{-iux} du \,, \tag{51}$$

where v is any real number such that

$$\int_{-\infty}^{\infty} |e^{-vx} f(x)| dx < \infty \,. \tag{52}$$

Note that the factor e^{-vx} isn't necessarily helpful. If $v > 0$, the factor improves convergence in the upper limit but impairs it in the lower limit (and converesely for $v < 0$). Even for $f(x) = 1$, there isn't any value of v for which (52) holds, so one ultimately needs to be more sophisticated. (See the lecture notes.) However, there are situations in which (52) is satisfied on a strip—i.e., for $v \in (v_1, v_2)$—in which case $\hat{f}(\omega)$ is an analytic function on that strip.

b. Find the Fourier transform of $f(x) = e^{-|x|}$ and then reobtain $f(x)$ with an inverse transform.

Solution: With $f(x) = e^{-|x|}$, it follows that $f(x)e^{-vx} \in L_1(-\infty, \infty)$ for all $v \in (-1, 1)$. This gives

$$\hat{f}(\omega) = \int_{-\infty}^{0} e^{i\omega x} e^{x} dz + \int_{0}^{\infty} e^{i\omega x} e^{-x} dx = \frac{1}{1 + i\omega} + \frac{1}{1 - i\omega} = \frac{2}{1 + \omega^2} \,. \tag{53}$$

We'll use (47), with $v = 0$ to recover $f(x)$ by contour integration. (Most of you will have seen this for Laplace and Fourier transforms before.) For $x > 0$, the function $2e^{-i\omega x}/(1 + \omega^2)$ is exponentially small in the lower half of the ω-plane (i.e., $v < 0$). Consider the contour C_R that consists of the boundary of a large semicircle of radius R (with diameter between $(-R, 0)$ and $(R, 0)$ on the real axis). By Cauchy's theorem, we get

$$\lim_{R \to \infty} \int_{C_R} \frac{1}{2\pi} \frac{2}{1 + \omega^2} e^{-i\omega x} d\omega = 2\pi i r \,, \tag{54}$$

where r is the sum of the residues of $e^{-i\omega x}/(1 + \omega^2)$ in the region bounded by C_R. The only pole is a simple pole at $\omega = -i$, and the corresponding residue is $ie^{-x}/(2\pi)$. As $R \to \infty$, the contribution from the curved portion of C_R vanishes (why?). We thus find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+\omega^2}e^{-i\omega x}d\omega = e^{-x}, \quad x > 0\,. \tag{55}$$

Similarly, by considering a semicircle in the upper half-plane, we recover e^x for $x < 0$.

c. Find the Fourier transform of $f(x) = -2H(x)\sinh(x)$. How do you reconcile the results of (b) and (c)?

Solution: The function $f(x)$ is given by

$$f(x) = \begin{cases} -2\sinh x\,, & x > 0, \\ 0\,, & x < 0\,. \end{cases} \tag{56}$$

Because $2\sinh x = e^x - e^{-x}$, we see that $e^{-vx}\sinh x \in l_1(0,\infty)$ for $v > 1$. Hence, $e^{-vx}f(x) \in L_1(-\infty,\infty)$ for $v > 1$. One can calculate

$$\hat{f}(\omega) = \int_0^{\infty}(e^{-x} - e^{x})e^{i\omega x}dx = \frac{2}{1+\omega^2}\,, \quad v > 1\,. \tag{57}$$

As you can see, this gives the same functional form as the Fourier transform in part (b). However, this is fine because these transforms were taken in *different, non-overlapping regions* of the ω-plane. If you invert this in the appropriate region, you can recover the original function.