Methods in Applied PDEs (B5b - supplementary lectures), HT 2009. Example Sheet 2

1. **Weak solutions of Laplace's equation are necessarily classical.** Show that a solution of $\Delta u = 0$ in $\Omega = R_2$ cannot have a simple jump across a surface σ given by the line $x_2 = 0$.

   Solution: To start, we will consider general $\Omega \subset R_n$ and σ. (The more specific nature of the question was to simplify specific technical details below.)

   Suppose that σ divides Ω into two domains: Ω_+ and Ω_-. If u is a weak solution of $\Delta u = 0$ on R_n, then it follows for test function ϕ with support in Ω that

$$0 = \int_\Omega u\Delta\phi dx = \int_{\Omega_+} u\Delta\phi dx + \int_{\Omega_-} u\Delta\phi dx \,. \tag{1}$$

   If u and its first derivatives have limiting values on both sides of σ, then one finds by Green's formula that

$$\int_{\Omega_+} u\Delta\phi dx = \int_{\Omega_+} \phi\Delta u + \int_{\sigma^+} \left(u\frac{\partial\phi}{\partial n} - \phi\frac{\partial u}{\partial n}\right) dS \,, \tag{2}$$

   where σ_+ is the side of σ that bounds Ω_+, dS is a surface area element, and n is an outward unit normal on σ_+. There is no contribution from the boundary Γ of Ω because ϕ vanishes in a neighborhood of Γ. Combining (2) with a similar equation for Ω_- and using the assumption that u is a classical solution in Ω_+ and Ω_- yields

$$0 = \int_\sigma \left[(u_+ - u_-)\frac{\partial\phi}{\partial n} - \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n}\right)\phi \right] dS \,, \tag{3}$$

   where $\phi \in C_0^\infty(\Omega)$ and n is the outward unit normal to σ from Ω_+.

   The factors multiplying $\frac{\partial\phi}{\partial n}$ and ϕ in (3) must each vanish on σ. We will show this for the case $\Omega = R_2$ and σ given by the line $x_2 = 0$, but it is true in general. (One can adapt the argument for the more general case.) Consider the test function $\phi(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$, where $\phi_2(0) = 0$, $\phi_2'(0) \neq 0$, and $\phi_1(x) \geq 0$ has its support in a small neighborhood of the point $x_1 = \xi_1$. It then follows from (3) that $u(\xi_1, 0^+) = u(\xi_1, 0^-)$. Because ξ_1 is arbitrary, we have that $u(x_1, 0^+) = u(x_1, 0^-)$ for any point on σ. One similarly obtains that $\frac{\partial u}{\partial x_2}(x_1,0^+) = \frac{\partial u}{\partial x_2}(x_1,0^-)$. In the general case, the analogous conclusion is written $u_+ = u_-$ and $\frac{\partial u_+}{\partial n} = \frac{\partial u_-}{\partial n}$. The first of these relations also implies that tangential derivatives are continuous across σ. Thus, u and all its first derivatives are continous there. In fact, one can show that u is infinitely differentiable in Ω.

2. Find the causal fundamental solution for the diffusion (heat) equation ine one space dimension. [That is, find this solution for the diffusion equation with a delta function source term at $(x,t) = (0,0)$.]

   Solution: We want the solution $E(x,t;0,0)$ of

$$LE = \frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial x^2} = \delta(x,t)\,, \quad x,t \in \mathbb{R}\,, \tag{4}$$

with $E \equiv 0$ for $t < 0$ and $E \to 0$ as $|x| \to \infty$.

Not worrying about rigor for now, we take a Fourier transform in the space coordinate:

$$\hat{E} = \int_{-\infty}^{\infty} e^{i\omega x} E dx\,. \tag{5}$$

Multiplying (4) by $e^{i\omega x}$ and integrating gives

$$\frac{d\hat{E}}{dt} + \omega^2 \hat{E} = \delta(t)\,, \tag{6}$$

with $\hat{E} = 0$ for $t < 0$. Thus, $\hat{E}$ jumps by 1 at $t = 0$, and

$$\hat{E} = e^{-\omega^2 t}\,, \quad t > 0\,, \tag{7}$$

Inverting the Fourier transform gives

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\hat{E} d\omega = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}\,, \quad t > 0\,. \tag{8}$$

This suggests that

$$E = H(t)\frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} \tag{9}$$

is a fundamental solution of (4). To verify this, we need to show that (8) satisfies

$$\langle E, L^*\phi\rangle = \phi(0,0) \tag{10}$$

for every test function $\phi(x,t)$

The left side of (10) is the convergent integral

$$\int_0^\infty dt \int_{-\infty}^\infty dx \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} L^*\phi\,, \tag{11}$$

which can be written as

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty dt \int_{-\infty}^\infty dx \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} L^*\phi\,. \tag{12}$$

Applying Green's formula to this integral, and noting that $\frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}$ satisfies the homogeneous diffusion equation for $t > 0$ by construction, we obtain

$$\int_\epsilon^\infty dt \int_{-\infty}^\infty dx \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}} L^*\phi = \int_{-\infty}^\infty \frac{e^{-x^2/(4t)}}{\sqrt{4\pi\epsilon}}\phi(x,\epsilon)dx\,. \tag{13}$$

One finishes the argument by showing that the limit as $\epsilon \longrightarrow 0$ is $\phi(0,0)$.

3. a. Show that for a continuous function f (which is not necessarily differentiable), the expression $u = f(x-ct)$ is a weak solution of the PDE

$$u_t + cu_x = 0\,. \tag{14}$$

Solution: A weak solution $u(x,t)$ of (14) must satisfy

$$\int_{\mathbb{R}^2}(u_t + cu_x)\phi(x,t)dtdx = 0 \tag{15}$$

for all differentiable ϕ with compact support. This gives

$$\int_{\mathbb{R}^2} u_t\phi dtdx = -c\int_{\mathbb{R}^2} u_x\phi dtdx = -c\int_{\mathbb{R}^2} u_x\phi dxdt\,, \tag{16}$$

where the order of the integration variables can be switched by Fubini's theorem. Now we integrate both sides of the equation by parts and (noting that the boundary terms vanish because ϕ has compact support) we obtain

$$\int_{\mathbb{R}} \left([\phi u]_{t=-\infty}^{t=\infty} - \int_{\mathbb{R}} u\phi_t dt \right) dx = -c \int_{\mathbb{R}} \left([\phi u]_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} u\phi_x dx \right) dt \,. \tag{17}$$

That is,

$$-\int_{\mathbb{R}^2} u\phi_t dt dx = c \int_{\mathbb{R}^2} u\phi_x dx dt \,, \tag{18}$$

where we remark that u need not be differentiable for this integral equation to be satisfied. Also, we can use Fubini's theorem to switch the order of integration on the left side of (18), so that a weak solution of (14) must satisfy

$$\int_{\mathbb{R}^2} u(\phi_t + c\phi_x) dx dt = 0 \tag{19}$$

for all differentiable ϕ with compact support. We also see with (19) that the differential operator is self-adjoint.

In particular, consider $u(x,t) = f(x-ct)$. This gives

$$\begin{aligned} \int_{\mathbb{R}^2} u(\phi_t + c\phi_x) dx dt &= \int_{\mathbb{R}^2} f(x-ct)(\phi_t + c\phi_x) dx dt \\ &= \int_{\mathbb{R}^2} f(x_1)(\phi_{t_1} - c\phi_{x_1} + c\phi_{x_1}) 1 dx_1 dt_1 \,, \end{aligned} \tag{20}$$

where $x_1 = x - ct$ and $t_1 = t$. We now want to compute the Jacobian J. Its inverse is

$$J^{-1} = \det \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} = 1 \,, \tag{21}$$

which gives $J = 1$. The chain rule gives $\phi_x = \phi_{x_1}$ and $\phi_t = \phi_{t_1} - c\phi_{x_1}$, so that

$$\int_{\mathbb{R}^2} u(\phi_t + c\phi_x) dx dt = \int_{\mathbb{R}^2} f(x_1)\phi_{t_1} dx_1 dt_1 = \int_{-\infty}^{\infty} \left[f(x_1) \int_{-\infty}^{\infty} \phi_{t_1} dt_1 \right] dx_1 = 0 \tag{22}$$

because ϕ has compact support. Hence, for a continuous function f, $u = f(x-ct)$ is a weak solution of the PDE $u_t + cu_x = 0$. QED. (Note that one can also do this problem using the method of characteristics, which you have seen in other courses.)

b. Show that the function $u(x_1, x_2)$ defined by $u = 1$ for $x_1 > \xi_1\,, x_2 > \xi_2$ and $u = 0$ for all other (x_1, x_2) is a fundamental solution for the operator

$$L = \frac{\partial^2}{\partial x_1 \partial x_2} \tag{23}$$

with pole (ξ_1, ξ_2).

Solution: Note first that $u(x_1, x_2)$ is discontinuous at $(x_1, x_2) = (\xi_1, \xi_2)$, so that there is a pole there (using the terminology discussed in lecture). By definition, u is a fundamental solution for L if

$$\int_{\mathbb{R}^2} Lu\phi(x_1, x_2)dx = \int_{\mathbb{R}^2} \phi(x_1, x_2)\delta_\xi dx = \phi(\xi_1, \xi_2)\,. \tag{24}$$

[Notation: $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$.] This implies that $\int_{\mathbb{R}^2} uL^*\phi(x_1, x_2)dx = \phi(\xi_1, \xi_2)$ for all test functions ϕ, which we recall are smooth functions with compact support and compactly supported derivatives. [Notation: Recall that L^* represents the adjoint operator of L.] In fact, $L = \frac{\partial^2}{\partial x_1 \partial x_2}$ is self-adjoint, so that $L^* = L$. Hence, we just need to show that u satisfies

$$\int_{\mathbb{R}^2} u\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \int_{\mathbb{R}^2} \phi\frac{\partial^2 u}{\partial x_1 \partial x_2} = \phi(\xi_1, \xi_2)\,. \tag{25}$$

Note that in (25) the equality on the left holds because of equality of mixed partial derivatives.

We have

$$\begin{aligned}\int_{\mathbb{R}^2} u\frac{\partial^2 \phi}{\partial x_1 \partial x_2}dx &= \int_{\xi_2}^{\infty}\int_{\xi_1}^{\infty} 1\frac{\partial^2 \phi}{\partial x_1 \partial x_2}dx_1dx_2 = \int_{\xi_2}^{\infty} \frac{\partial \phi}{\partial x_2}(x_1, x_2)\Big|_{x_1=\xi_1}^{x_1=\infty} dx_2 \\ &= \int_{\xi_2}^{\infty}\left[0 - \frac{\partial \phi}{\partial x_2}(\xi_1, x_2)\right]dx_2 = -\int_{\xi_2}^{\infty} \frac{\partial \phi}{\partial x_2}(\xi_1, x_2)dx_2 \\ &= -\phi(\xi_1, x_2)\Big|_{x_2=\xi_2}^{x_2=\infty} = \phi(\xi_1, \xi_2)\,. \end{aligned} \tag{26}$$

(Note that we use in (26) that ϕ and its first derivative have compact support.)

Therefore, $u(x_1, x_2)$ is a fundamental solution of L with a pole at (ξ_1, ξ_2). QED.

4. **Green's functions and continuous spectra.** Consider an infinite stretched string subject to an external harmonic force per unit length. The equation of motion is

$$u_{xx} - \frac{1}{c^2} u_{tt} = -\frac{1}{T} F(x,t)\,, \tag{27}$$

where

$$F(x,t) = f(x) e^{-\omega_0 t}\,. \tag{28}$$

Given a bounded solution

$$u(x,t) = y(x) e^{-i\omega_0 t}\,, \tag{29}$$

derive the ODE that $y(x)$ must satisfy and find its solution using Green's functions.

Solution: We assume that $u(x,t)$ is bounded everywhere, so that $|u(x,t)| < \infty$. With (29), we get the ODE

$$y_{xx} + k_0^2 y = -\frac{f(x)}{T}\,, \tag{30}$$

where $k_0 = \omega_0 / c$. The Green's function must satisfy

$$G_{xx} + k_0^2 G = \delta(x - \xi)\,. \tag{31}$$

If we examine the eigenvalue problem

$$y_{xx} = \lambda y \tag{32}$$

for bounded functions y, we see that the only restriction is $\lambda \le 0$ so that the spectrum is continuous. Taking $\lambda = -k^2$, the eigenfunctions can be taken as

$$\phi_\lambda = e^{\pm ikx}\,. \tag{33}$$

Instead of a series expansion for the Green's function $G(x|\xi)$, we need to represent it as a Fourier integral,

$$G(x|\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k,\xi) e^{-ikx} dk\,. \tag{34}$$

With this in mind, we Fourier transform both sides of (31). Multiplying by $(1/\sqrt{2\pi})e^{ikx}$ and integrating over x from $-\infty$ to ∞ gives

$$-k^2 g(k,\xi) + k_0^2 g(k,\xi) = \frac{1}{\sqrt{2\pi}} e^{ik\xi}\,, \tag{35}$$

where $g(k,\xi)$ is the Fourier transform of $G(x|\xi)$.

Solving (35) gives

$$g(k,\xi) = \frac{1}{\sqrt{2\pi}} \frac{e^{ik\xi}}{k_0^2 - k_2}\,, \tag{36}$$

which gives

$$G(x|\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik\xi} - e^{-ikx}}{k_0^2 - k^2} dk\,. \tag{37}$$

We have thereby obtained an integral representation of the Green's function for this case of continuous spectrum. If we desire a closed-form expression as a summation, we are faced with a dilemma: What should we do about the poles of the integrand at $k = \pm k_0$. Without specifying an integration contour, one can'tt properly define the Green's function.

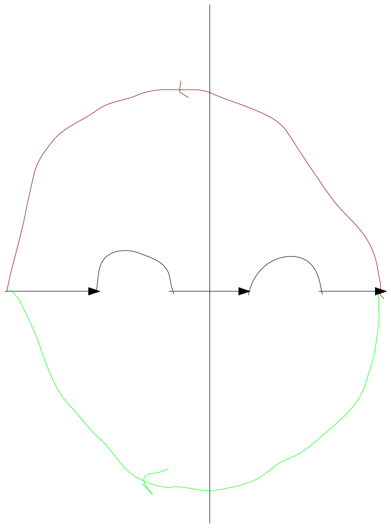
Let's suppose that $x < \xi$, which means physically that the point of observation is to the left of the point of disturbance (where the delta function source is). The contour used to evaluate $G(x,\xi)$ using residue calculus must then be closed upwards. (Otherwise, the large semicircular part of the contour won't vanish.) If the poles at $k = -k_0$ and $k = +k_0$ are allowed to contribute to $G(x|\xi)$, then their residues will contain the respective factors

$$e^{-ik_0(\xi - x)}\,, \qquad e^{ik_0(\xi - x)}\,. \tag{38}$$

The first expression corresponds to a wave travelling to the right, and the second to a wave travelling to the left. The original function is recovered using

$$u(x,t) = y(x)e^{-\omega_0 t} = e^{-i\omega_0 t} \int_{-\infty}^{\infty} G(x|\xi) f(\xi) d\xi\,. \tag{39}$$

Physical intuition tells us that the wave travelling to the right from point ξ should not influence the response at point x lying to the left of ξ. (It's a radiative process, so we expect outgoing waves.) Thus, for $x < \xi$, the contour should bypass the pole $k = -k_0$ from above and the pole $k = +k_0$ from below. As a result, only the latter pole contributes. We show the contour in the figure. (One closes on the top for $x < \xi$ and on the bottom for $x > \xi$.)



We can now straightforwardly calculate for $x < \xi$ that

$$\begin{aligned} G(x|\xi) &= -2\pi i \mathrm{Res}\left[-\frac{1}{2\pi}\frac{e^{ik(\xi-x)}}{(k+k_0)(k-k_0)}\right]_{k=k_0} \\ &= -\frac{i}{2k_0}e^{ik_0(\xi-x)} \quad (x<\xi)\,. \end{aligned} \tag{40}$$

For $x > \xi$, we close the contour in the lower half plane and find

$$\begin{aligned} G(x|\xi) &= -2\pi i \mathrm{Res}\left[-\frac{1}{2\pi}\frac{e^{ik(\xi-x)}}{(k+k_0)(k-k_0)}\right]_{k=-k_0} \\ &= -\frac{i}{2k_0}e^{-ik_0(\xi-x)} \quad (x > \xi)\,. \end{aligned} \tag{41}$$

Instead of bypassing the poles as just discussed, we could alternatively define $G(x,\xi)$ by keeping the path of integration straight but displacing the pole $k = -k_0$ slightly below the real axis and $k = +k_0$ slightly above the axis. We can then unambuously write

$$G(x|\xi) = \frac{1}{4\pi k_0}\int_{-\infty}^{\infty}\left[\frac{e^{ik(\xi-x)}}{k-(-k_0-i\epsilon)} = \frac{e^{ik(\xi-x)}}{k-(k_0+i\epsilon)}\right]dk \tag{42}$$

for both $x < \xi$ and $x > \xi$.

Note 1: The obtained Green's function,

$$G(x,\xi) = \begin{cases} -\frac{i}{2k_0}e^{ik_0(\xi-x)}\,, & x < \xi\,, \\ -\frac{i}{2k_0}e^{-ik_0(\xi-x)}\,, & x > \xi \end{cases}\,, \tag{43}$$

is symmetric rather than Hermitian under the interchange of x and ξ (i.e., there isn't the usual complex conjugation).

Note 2: Despite the complex notation, recall that $G(x|\xi)$ is used to calculate a *real* function $u(x,t)$ using

$$u(x,t) = \int_{-\infty}^{\infty}\mathrm{Re}[G(x,\xi)e^{-i\omega_o t}f(\xi)]d\xi\,. \tag{44}$$

If $f(x)$ is real, then the actual Green's function is the real part of $G(x|\xi)e^{-i\omega_0 t}$. This gives

$$\mathrm{Re}\left[G(x|\xi)e^{-i\omega_0 t}\right] = \begin{cases} \frac{1}{2k_0}\sin[k_0(x-\xi)+\omega_0 t]\,, & x < \xi\,, \\ \frac{1}{2k_0}\sin[k_0(\xi-x)+\omega_0 t]\,, & x > \xi \end{cases}\,. \tag{45}$$

Note 3: Although real strings are finite, one expects the analysis of an infinite string to be relevant for the dynamics near the middle, especially for very long strings.

5. **Multi-dimensional Green's functions.** Consider the (3+1)-dimensional D'Alembert equation

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = f(r,t)\,, \tag{46}$$

where c is the speed of light, $r = (x, y, z)$, and the function f is given.

a. Let

$$f(r,t) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0) \tag{47}$$

be a four-dimensional Dirac delta function. What conditions must be satisfied by the Green's function (fundamental solution) G corresponding to (46,47)? Derive this Green's function. need to do some contour integration.

Solution: Before we start solving the problems, note that (46) is pervasive in electromagnetic theory. In a nonpolarizable and nonmagnetic medium, the electric field E and the magnetic field B must satisfy Maxwell's equations. One way to study Maxwell's equations is to introduce scalar and vector potentials in such a way that they satisfy a condition due to Lorentz. One can then show that the three components (in Cartesian coordinates) of the vector potential each satisfy an equation of the form (46). Hence, one can reduce a wide class of electromagnetic problems to investigations of D'Alembert equations.

To find the solution of (46), we start by constructing a Green's function $G = G(x, y, z, t|x_0, y_0, z_0, t_0)$ that satisfies

$$\Delta G - \frac{1}{c^2}G_{tt} = \delta(r - r_0)\delta(t - t_0)\,, \tag{48}$$

where $\delta(r - r_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$. Because G is a distribution, it will need to be bounded as $x, y, z \to \infty$. It will also need to be bounded for a given r as $t \to \infty$ and must vanish identically for $t < t_0$. (This last condition arises because of the *principle of causality.*)

To solve this problem, we will take Fourier transforms with both space and time. (An alternative would be to take Fourier transforms in space but a Laplace transform in time.) We shift the time and space coordinates by defining the variables $X = x - x_0$, $Y = y - y_0$, $Z = z - z_0$, and $T = t - t_0$. Equation (48) then becomes

$$G_{XX} + G_{YY} + G_{ZZ} - \frac{1}{c^2} G_{TT} = \delta(X)\delta(Y)\delta(Z)\delta(T) \,. \qquad (49)$$

It follows that $G(r, t|r_0, t_0)$ is in fact a function of $R = (X, Y, Z)$ and T—i.e., it only depends on the differences $(r - r_0)$ and $(t - t_0)$, which means that it is invariant with respect to translations in both time and space (which follows directly from the PDE it satisfies).

The Fourier transform of G is

$$g(K, \omega) = \frac{1}{(\sqrt{2\pi})^4} \int_{\mathbb{R}^4} G(R, T) e^{i(K \cdot R + \omega T)} dT dR \,, \qquad (50)$$

where $K = (K_x, K_y, K_z)$ is the Fourier wavevector. Once we obtain g, Fourier inversion will give

$$G(R, T) = \frac{1}{(\sqrt{2\pi})^4} \int_{\mathbb{R}^4} g(K, \omega) e^{-i(K \cdot R + \omega T)} d\omega dK \,. \qquad (51)$$

We thus have

$$G_{XX} + G_{YY} + G_{ZZ} - \frac{1}{c^2} G_{TT} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left(\frac{\omega^2}{c^2} - |K|^2 \right) g(K, \omega) e^{-i(K \cdot R + \omega T)} d\omega dK \,. \qquad (52)$$

Additionally,

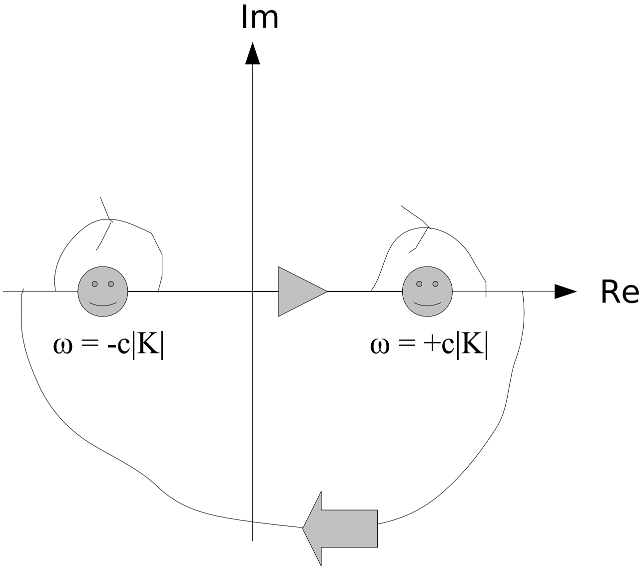
$$\delta(R)\delta(T) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{-i(K \cdot R + \omega T)} d\omega dK \,. \qquad (53)$$

Substituting into the PDE gives

$$g(K, \omega) = \frac{c^2}{(2\pi)^2} \frac{1}{\omega^2 - c^2 |K|^2} \,, \qquad (54)$$

so that the desired Green's function is given by the integral

$$G(R, T) = \frac{c^2}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{e^{-iK \cdot R} e^{-i\omega T}}{(\omega + c|K|)(\omega - c|K|)} d\omega dK \,. \qquad (55)$$



We will do the integral over $d\omega$ first, where the treatment of the poles in the contour integration is dictated by causality (i.e., we need to use the boundary condition that G must vanish for $T < 0$). We show the contour in a figure.

Integration yields

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega T}}{(\omega + c|K|)(\omega - c|K|)} d\omega = (-2\pi i)\left[\frac{e^{iC|K|T}}{(-2c|K|)} + \frac{e^{-icKT}}{2c|K|}\right] = -2\pi\frac{\sin(c|K|T)}{c|K|}\,, \tag{56}$$

which implies that

$$G = -\frac{c^2}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\sin(c|K|T)}{c|K|} e^{-i(K\cdot R)} dK\,. \tag{57}$$

The easiest way to evaluate (57) is using spherical polar coordinates, in which $K\cdot R = |K||R|\cos(\theta)$ and $dK = |K|^2 \sin\theta d[|K|]d\theta d\phi$. The integral (57) then becomes

$$G = -\frac{c}{(2\pi)^3}\int_0^{\infty} \frac{\sin(c|K|T)}{|K|}|K|^2 d[|K|]\int_0^{\pi} e^{-|K||R|\cos\theta}\sin\theta d\theta\int_0^{2\pi} d\phi\,, \quad (T > 0)\,. \tag{58}$$

The rightmost integral can be evaluated trivially and we can use the change of variables $\alpha = \cos\theta$ to evaluate the center integral. We thus obtain

$$G = -\frac{c}{4\pi^2}|R| \int_0^\infty 2\sin(c|K|T)\sin(|K||R|)d[|K|]\,, \quad (T > 0)\,. \tag{59}$$

We now write

$$2\sin(c|K|T)\sin(|K||R|) = \cos(c|K|T - |K||R|) - \cos(c|K|T + |K||R|) \tag{60}$$

and also use the formula (defined as a distribution)

$$\int_0^\infty \cos(|K|\alpha)d[|K|] = \pi\delta(\alpha)\,, \tag{61}$$

which can be established from the integral representation

$$\delta(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i|K|\alpha}d[|K|]\,. \tag{62}$$

This gives

$$G(R,T) = -\frac{c}{4\pi}\frac{\delta(cT - |R|)}{|R|} + \frac{c}{4\pi}\frac{\delta(cT + |R|)}{|R|}\,. \tag{63}$$

Because $|R| \geq 0$, the second term in (63) is identically zero (one can also see this in advance from the contour over which we integrated). Using $\delta(a\alpha) = (1/a)\delta(\alpha)$ $((a > 0))$, we obtain that $G(r - r_0, t - t_0)$ for $t < t_0$ and

$$G(r - r_0, t - t_0) = -\frac{1}{4\pi}\frac{\delta\left(t - t_0 - \frac{|r - r_0|}{c}\right)}{|r - r_0|} \tag{64}$$

for $t > t_0$. Note from (64) that if there is a disturbance created at $r = r_0$ at time $t = t_0$, then it takes a time interval of $(1/c)|r - r_0|$ for its effects to reach point r. Once the wave passes, the solution returns to its original value.

b. Use your answer to (a) to write down the general solution of (46).

Solution: All of the real work for this problem has already been done in part (a). At this point, you basically just need to remember how Green's functions work. We obtain

$$\begin{aligned} \psi(r,t) &= \int_{\mathbb{R}^4} G(r-r_0,t-t_0)f(r_0,t_0)dr_0dt_0 \\ &= \int_{\mathbb{R}^3} \frac{f\left(r_0, t-\frac{|r-r_0|}{c}\right)}{|r-r_0|}dr_0\,, \end{aligned} \tag{65}$$

where for the last equality we have done the integration over t_0 explicitly. Formula (65) defines what is often called a it retarded potential in electromagnetism. For a lot more detail on the use of Green's functions and similar concepts in electromagnetism, see the textbook by Jackson (which is the standard advanced advanced undergrad/beginning graduate level textbook in electromagnetism).