

1. Recall from the lecture that  $\delta(x - \xi) = \delta_\xi$ .
  - a. Find the value of  $x\delta(x)$  using the definition of  $\delta(x)$ .
  - b. Show that

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad (1)$$

for  $a \neq 0$  and use this to verify that  $\delta(x)$  is an even function.

- c. Show that

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x + a) + \delta(x - a)] \quad (2)$$

for  $a > 0$ .

2. a. Sketch the function  $f_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$  for several values of  $n$ . Show that it converges weakly with

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx = g(0). \quad (3)$$

- b. Consider the function  $F_n(x)$  given by

$$F_n(x) = \exp \left\{ -\frac{1}{[1 - 4(nx - 1)^2]} \right\} \quad (4)$$

for  $|x - 1/n| < 1/(2n)$  and  $F_n(x) = 0$  for  $|x - 1/n| \geq 1/(2n)$ . Now define

$$f_n(x) = \frac{F_n(x)}{\int_{1/(2n)}^{3/(2n)} F_n(x)dx} \quad (5)$$

to normalize the function. Show that the sequence  $f_n(x)$  converges pointwise to 0. Then show that  $f_n(x)$  converges weakly as a distribution. Comment on the significance of your observations to models of physical systems.

3. **A calculus for functions with jumps:** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable except at points  $a_1, \dots, a_k$  where it has jumps. (All the left-hand and right-hand derivatives exist at those points.) Denote by  $[f^{(m)}]$  the function obtained by differentiating  $f$   $m$  times without regard to the jumps.

- a. Derive the expression for  $f'$  (i.e., including the jumps).
  - b. Suppose the only point of discontinuity is  $a_1 = 0$ . Write down the expression for  $f^{(m)}$ .
  - c. Calculate  $f'$ ,  $f''$ , and  $f'''$  for  $f(x) = e^{-|x|}$ .
4. a. Consider  $f(x) = H(x) \log(x)$ , where  $x \in \mathbb{R}$  and the Heaviside “function”  $H(x - \xi)$  equals 1 when  $x > \xi$  and equals 0 when  $x < \xi$ . Calculate the derivative of  $f(x)$  in the distributional sense. **Hint:** To do this rigorously, you will need to recall the definition of the Cauchy principal value integral.
- b. Consider  $f(x) = 1/|x|$ , where  $x \in \mathbb{R}^3$ . Calculate  $\Delta f(x)$  in the distributional sense.

5. **Fourier transforms in the complex plane:**

- a. Using  $\omega = u + iv$ , show that one can write

$$f(x) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} \hat{f}(\omega) e^{-i\omega x} d\omega \quad (6)$$

for appropriate  $v \in \mathbb{R}$  (that you should specify to the extent possible).

- b. Find the Fourier transform of  $f(x) = e^{-|x|}$  and then reobtain  $f(x)$  with an inverse transform.
- c. Find the Fourier transform of  $f(x) = -2H(x) \sinh(x)$ . How do you reconcile the results of (b) and (c)?