Methods in Applied PDEs (B5b - supplementary lectures), HT 2009. Example Sheet 2

1. **Weak solutions of Laplace's equation are necessarily classical.** Show that a solution of $\Delta u = 0$ in $\Omega = R_2$ cannot have a simple jump across a surface $\sigma$ given by the line $x_2 = 0$.

2. Find the causal fundamental solution for the diffusion (heat) equation ine one space dimension. [That is, find this solution for the diffusion equation with a delta function source term at $(x, t) = (0, 0)$.]

3. a. Show that for a continuous function $f$ (which is not necessarily differentiable), the expression $u = f(x - ct)$ is a weak solution of the PDE

$$u_t + cu_x = 0\,. \tag{1}$$

   b. Show that the function $u(x_1, x_2)$ defined by $u = 1$ for $x_1 > \xi_1\,, x_2 > \xi_2$ and $u = 0$ for all other $(x_1, x_2)$ is a fundamental solution for the operator

$$L = \frac{\partial^2}{\partial x_1 \partial x_2} \tag{2}$$

   with pole $(\xi_1, \xi_2)$.

4. **Green's functions and continuous spectra.** Consider an infinite stretched string subject to an external harmonic force per unit length. The equation of motion is

$$u_{xx} - \frac{1}{c^2} u_{tt} = -\frac{1}{T} F(x, t)\,, \tag{3}$$

   where

$$F(x, t) = f(x) e^{-\omega_0 t}\,. \tag{4}$$

   Given a bounded solution

$$u(x, t) = y(x) e^{-i\omega_0 t}\,, \tag{5}$$

   derive the ODE that $y(x)$ must satisfy and find its solution using Green's functions.

5. **Multi-dimensional Green's functions.** Consider the (3+1)-dimensional D'Alembert equation

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = f(\mathbf{r}, t)\,, \tag{6}$$

where $c$ is the speed of light, $\mathbf{r} = (x, y, z)$, and the function $f$ is given.

a. Let

$$f(\mathbf{r}, t) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0) \tag{7}$$

be a four-dimensional Dirac delta function. What conditions must be satisfied by the Green's function (fundamental solution) $G$ corresponding to (6,7)? Derive this Green's function.

b. Use your answer to (a) to write down the general solution of (6).