EXAMPLES SHEET, FINITE DIMENSIONAL NORMED SPACES

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Exercises with daggers (†) are harder, which is not to say that the others are not. Answers and comments on some of the questions appear at the end.

1. Show that if X is a Banach space and X^* is separable and its elements separate points (*i.e.* if $x \neq y \in X$ then there is some $\phi \in X^*$ such that $\phi(x) \neq \phi(y)$) then X is separable.

2. Show that c_c is dense in ℓ_p for $1 \leqslant p < \infty$.

3. Write $\mathbf{Ban_1}$ for the category of Banach spaces in which the morphisms are short maps. As per Example 2.2 of the notes, write

$$\iota_{q\to p} : \ell_q \to \ell_p; x \mapsto x,$$

and show that the direct limit of the system $\langle \ell_q, \iota_{q\to p}\rangle$ over the directed set $[1, \infty)$ under the usual order is (isometrically isomorphic to) c_0.

4. Show that ℓ_1^2 is isometrically isomorphic to ℓ_∞^2 over $\mathbb{R}$. (Here ℓ_p^n is the vector space $\mathbb{R}^n$ endowed with the norm

$$\|x\|_{\ell_p^n} = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

with the usual convention for $p = \infty$.)

5. Show that if $\phi : X \to Y$ is a surjective linear isometry from X to Y then it is an isometric isomorphism.

6. Show that ℓ_1^* is isometrically isomorphic to ℓ_∞.

7. † Show that for $1 \leqslant p < q < \infty$ the spaces ℓ_p and ℓ_q are not isometrically isomorphic.

8. Show that c and c_0 are not isometrically isomorphic.

9. Prove the sequential Banach-Alaoglu theorem (Theorem 2.9) directly by the method of Proposition 2.8.

10. Prove that a linear operator $T : Y^* \to X^*$ is weak-* to weak-* continuous if and only if there is some $S \in L(X, Y)$ such that $T = S^*$.

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11. Show that $L(X, Y^*)$ is metrisable in the topology of pointwise convergence if and only if X and Y are finite dimensional.

12. Give an example of functions $(f_n)_n$ and f in ℓ_p $(1 < p < \infty)$ such that $\phi(f_n) \to \phi(f)$ for all $\phi \in \ell_p^*$ (we say that $f_n \to f$ in the weak topology) but $f_n \nrightarrow f$ in ℓ_p-norm.

13. Prove Schur's property that if $\phi(f_n) \to \phi(f)$ for every $\phi \in \ell_1^*$, then $f_n \to f$ in ℓ_1-norm.

14. Show that Δ and D_2^∞ from §3.3 of the notes are homeomorphic.

15. Prove that if T is a compact metrisable space and $t \in T$ then there are bounded continuous functions $(f_{n,t})_n$ such that $f_{n,t} \to 1_{\{t\}}$ pointwise *i.e.* $f_{n,t} \to 1_{\{t\}}$ in the bounded pointwise topology.

16. Prove the closed graph theorem for topological spaces: suppose X is a topological space, Y is a compact Hausdorff space, and $f : X \to Y$. Then f is continuous if and only if the graph of f, that is the set $\{(x, f(x)) : x \in X\}$ is closed as a subset of $X \times Y$.

17. Give an example of continuous linear maps $\Psi : C([0,1]) \to C([0,1])$ and $\Phi : C([0,1]) \to C([0,1])$ satisfying the hypotheses of the Robust Banach-Stone theorem (Theorem 3.10)) that are not multiples of isometries *i.e.* where $\|\Psi\|\|\Phi\| \neq 1$.

18. Give an example of an isometric isomorphism from c_0 into $C[0,1]$.

19. † Suppose that $\Psi : X \to Y$ is a surjective isometry with $\Psi(0_X) = 0_Y$. Show that Ψ is linear.

20. Prove Lemma 4.5.

21. Prove Lemma 4.6.

22. Show that $X \oplus Y$ with the maps $(x, y) \mapsto x$ and $(x, y) \mapsto y$ is a product in **TopVect**; and with the maps $x \mapsto (x, 0_Y)$ and $y \mapsto (0_X, y)$ is a coproduct in **TopVect**.

23. Suppose that X and Y are Banach spaces over a field $\mathbb{F}$. Explain why, despite the apparently greater number of morphisms in **TopVect**, their product in $\mathbf{Ban_1}$ is a product in **TopVect**, and similarly for their coproduct. Hence conclude that this product and coproduct are continuously isomorphic. (This provides another proof of the last part of Lemma 4.5, at least up to the specific values of the norms.)

24. Show that $\ell_p(X)$ is a Banach space.

25. Prove Lemma 4.9.

26. Show that the map defined in Example 4.10 really is an isometric isomorphism.

27. Check that you believe Lemma 4.12.

28. Show that the map in Exercise 4.14 really is a short projection.

29. Show that if X is a Banach space and $\phi : X \to X$ has $\|\phi - I\| < 1$ in the operator norm then ϕ is invertible. Hence, or otherwise, show that the set of invertible maps is open in $L(X, X)$ endowed with the topology induced by the operator norm.

30. Verify the triangle inequality for the Banach-Mazur distance *i.e.* show that $d_{\mathrm{BM}}(X, Z) \leqslant d_{\mathrm{BM}}(X, Y)d_{\mathrm{BM}}(Y, Z)$. Verify that it is also symmetric.

31. Suppose that $\phi : \ell_1 \to c_0$ is a continuous linear map. Show that $\ker \phi$ is *not* complemented in ℓ_1.

32. † Show that if $d(X, \ell_1^n) \leqslant K$ then there is a subspace $Y \leqslant X$ with $\dim Y = k := \lfloor\sqrt{n}\rfloor$ such that $d(Y, \ell_1^k) \leqslant \sqrt{K}$.

33. Show that if X is an n-dimensional Banach space and $x \in X$ then there is some $\phi \in X^*$ such that $\phi(x) = \|\phi\|\|x\|$. Hence show that Φ_X is an isometric isomorphism between X and $X^{*}*$.

34. Show that if X is an n-dimensional Banach space then X^* is an n-dimensional Banach space.

35. For $i \in \{0, 1\}$ defined two norms on c_0 as follows

$$\|x\|_i := \sup_n |x_n| + \left(\sum_{n=1}^{\infty} |2^{-n}x_{n+i}|^2\right)^{1/2}.$$

Show that $(c_0, \|\cdot\|_i)$ is a Banach space. By considering the maps

$$T_n : (c_0, \|\cdot\|_0) \to (c_0, \|\cdot\|_1); (x_1, \dots) \mapsto (x_n, x_1, \dots, x_{n-1}, x_{n+1}, \dots)$$

or otherwise show that $d_{\mathrm{BM}}((c_0, \|\cdot\|_0), (c_0, \|\cdot\|_1)) = 1$, but that these spaces are *not* isometrically isomorphic.

36. Show that if $d_{\mathrm{BM}}(X^*, Y^*) = d_{\mathrm{BM}}(X, Y)$ if X and Y are finite dimensional.

37. Show that ℓ_2 is isometrically isomorphic to $L_2([0, 1])$ as claimed in Example 6.1. More generally use the Gramm-Schmidt process to show that any infinite dimensional separable Hilbert space is isometrically isomorphic to ℓ_2.

38. Show that O_n acts transitively on S^{n-1}.

39. Show that the norm on $X \in \mathrm{Sub}(\Omega)$

$$\|X\|_{\mathrm{SG}'} := \sup\{p^{-1/2}\|X\|_{L_p(\mathbb{P})} : p \geqslant 1\},$$

is, indeed, a norm and that it is equivalent to the norm $\|\cdot\|_{\mathrm{SG}}$ defined in lectures.

40. Show that if X is a random variable such that $\mathbb{P}(|X| > t) \leqslant \exp(-ct^2)$ whenever $t > C$ then X is $(O_{C,c}(1), c/2)$-sub-exponential.

41. Show that X is sub-exponential if and only if $\mathbb{E}|X^p| = O(p)$ for all $p \geqslant 1$.

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