1. Prove that \( \tau(n) = n^{o(1)} \).

2. Prove that \( \sum_{x>n} \frac{\chi(x)}{x} = O(1/n) \).

3. Prove that \( \phi(n) = \Omega(n/\log \log n) \).

4. Prove that
   \[
   \sum_{p \leq n} \frac{1}{p} \sim \log \log n.
   \]

5. More generally, prove that
   \[
   \sum_{p \leq n : p \equiv a \pmod{q}} \frac{1}{p} \sim \frac{1}{\phi(q)} \log \log n,
   \]
   where \((a, q) = 1\).

6. Suppose that \((a_n)_n\) is a sequence of complex numbers with \( \sum_n |a_n| < \infty \). Show that the product
   \[
   \prod_{n=1}^{\infty} (1 - a_n) := \lim_{N \to \infty} \prod_{n \leq N} (1 - a_n)
   \]
   converges and is zero if and only if \(a_n = 1\) for some \(n\).

7. Show that if \(A \subset \mathbb{Z}/p\mathbb{Z}\) has \(|A| < \log p\) then
   \[
   \sup_{\gamma \neq 0} |\widehat{1}_A(\gamma)| = \Omega(|A|).
   \]

8. Suppose that \(G\) is a finite abelian group, \(A \subset G\) has density \(\alpha\), and
   \[
   S \subset \{ \gamma \in \hat{G} : |\widehat{1}_A(\gamma)| \geq \epsilon \alpha \}
   \]
   for some \(\epsilon \in (0, 1]\). Show that
   \[
   |S| \leq \epsilon^{-2} \alpha^{-1}.
   \]

9. Show that if \( p \) is prime and \( w \in \mathbb{Z}/p\mathbb{Z} \) then there are elements \( x, y, z \) such that \( w \equiv x^2 + y^2 + z^2 \pmod{p} \).

10. Show that there is some function \( p_0(\alpha) \) such that if \( p > p_0(\alpha) \) is prime and \( A \subset \mathbb{Z}/p\mathbb{Z} \) has density \( \alpha \), then every \( x \in \mathbb{Z}/p\mathbb{Z} \) has \( x \equiv u^2 + a_1 + a_2 \pmod{p} \) for some \( a_1, a_2 \in A \) and \( u \in \mathbb{Z}/p\mathbb{Z} \).

11. Show that there is an absolute constant \( C > 0 \) such that if \( x > C \) is odd and \( N > C \) is a natural then \( x \equiv u_1 + u_2 + u_3 \pmod{N} \) where \( u_1, u_2, u_3 \) are all coprime to \( N \).

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