Distant digraph domination

Tung Nguyen[1]
Princeton University,
Princeton, NJ 08544, USA

Alex Scott[2]
University of Oxford,
Oxford, UK

Paul Seymour[3]
Princeton University,
Princeton, NJ 08544, USA

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Abstract

A *k-kernel* in a digraph G is a stable set X of vertices such that every vertex of G can be joined from X by a directed path of length at most k. We prove three results about k-kernels.

First, it was conjectured by Erdős and Székely in 1976 that every digraph G with no source has a 2-kernel $|K|$ with $|K| \leq |G|/2$. We prove this conjecture when G is a "split digraph" (that is, its vertex set can be partitioned into a tournament and a stable set), improving a result of Langlois et al., who proved that every split digraph G with no source has a 2-kernel of size at most $2|G|/3$.

Second, the Erdős-Székely conjecture implies that in every digraph G there is a 2-kernel K such that the union of K and its out-neighbours has size at least $|G|/2$. We prove that this is true if $V(G)$ can be partitioned into a tournament and an acyclic set.

Third, in a recent paper, Spiro asked whether, for all $k \geq 3$, every strongly-connected digraph G has a k-kernel of size at most about $|G|/(k+1)$. This remains open, but we prove that there is one of size at most about $|G|/(k-1)$.

1 Introduction

A *digraph* is a finite directed graph with no loops or parallel edges (it may have directed cycles of length two). If G is a digraph, $X \subseteq V(G)$ is *stable* if there is no edge with both ends in X. In a digraph G, if $X, Y \subseteq V(G)$, we say X *k-covers* Y if for each $y \in Y$, there exists $x \in X$ and a directed path of length at most k from x to y. (If X is a singleton $\{x\}$ we write x for $\{x\}$ here, and the same for Y.) A *k-kernel* in a digraph G is a stable set X of vertices that k-covers $V(G)$. [1]

There are many interesting open questions about k-kernels; for instance, not every digraph has a 1-kernel, but every digraph has a 2-kernel [2], and the following was conjectured by P. L. Erdős and L. A. Székely [4] in 1976 (and remains open):

1.1 The small quasi-kernel conjecture: *Every digraph G with no source has a 2-kernel of size at most $|G|/2$.*

(A *source* is a vertex with in-degree zero.) There is a survey on this conjecture in [3], and the best bound on this seems to be a result of Spiro [7], that every digraph G with no source has a 2-kernel of size at most $|G| - \frac{1}{4}(|G| \log |G|)^{1/2}$, which is of course very far from the conjecture.

It is enough to prove 1.1 for *oriented graphs*, that is, digraphs with no directed cycle of length two; because deleting an edge from such a cycle makes the problem harder. (Unless this deletion makes a source; but if neither edge will work, delete both vertices and all their out-neighbours.) If G is a counterexample to 1.1, then, since it has a 2-kernel S say, it follows that $|S| > |G|/2$; and a natural special case is when $G \setminus S$ is a tournament. Let us say G is a *split digraph* if G is an oriented graph and its vertex set admits a partition into a stable set and a tournament. Ai, Gerke, Gutin, Yeo and Zhou [1] proved that 1.1 holds for split graphs in which all edges between the tournament and the stable set are directed towards the stable set. Langlois, Meunier, Rizzi, Vialette and Zhou [5] proved that every split digraph with no sources admits a 2-kernel of size at most $2|G|/3$. In section 2, we strengthen this:

1.2 *Every split digraph G with no sources admits a 2-kernel K with $|K| \leq |G|/2$.*

Our second result concerns a problem of Spiro [7], who observed that 1.1 implies:

1.3 Conjecture: *In every digraph G, there is a 2-kernel K such that at least half the vertices of G belong to K or have an in-neighbour in K.*

We discuss this in section 3, and prove that it holds for split digraphs, and indeed for digraphs with a vertex set that can be partitioned into a tournament and an acyclic subgraph.

Our third result concerns a different problem of Spiro [7], who asked whether:

1.4 Conjecture: *For all integers $k \geq 3$, every strongly-connected digraph G has a k-kernel of size at most $|G|/(k+1) + O_k(1)$.*

It seems that the best known bound in this case is due to Spiro, in the same paper, who proved that under the hypotheses of 1.4, there is a k-kernel of size at most about $|G|/\log k$. Our third result is that there is one of size at most $|G|/(k-1) + O_k(1)$. This as a consequence of 1.5 below.

Let T be a subdigraph with underlying graph a tree, such that for some vertex r of T, every edge of T is directed away from r in the natural sense. We call T an *arborescence*, and r is its *root*. Every

[1] In some papers a k-kernel is defined with edges reversed: every vertex of G is joined *to* X by a short directed path.

strongly-connected digraph has a subdigraph that is a spanning arborescence (*spanning* means that the arborescence contains all vertices of the digraph). In section 4 we will prove:

1.5 *For all integers $k \geq 2$, every digraph G with $|G| > 1$ and with a spanning arborescence has a k-kernel of size at most $1 + (|G| - 2)/(k - 1)$.*

This follows easily from a result about acyclic digraphs (*acyclic* means there is no directed cycle):

1.6 *For every integer $k \geq 1$, if G is an acyclic digraph with $|G| \geq 2$ and with only one source, then G has a k-kernel of size at most $1 + (|G| - 2)/k$.*

This result is tight, as can be seen from the digraph shown in figure 1.

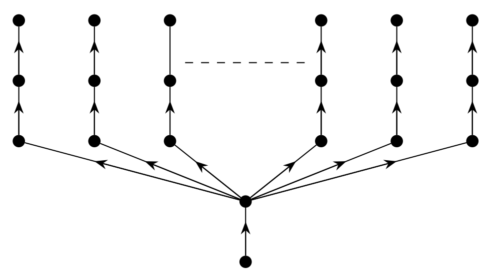

Figure 1: All 3-kernels have size $\geq 1 + (|G| - 2)/3$. For $k > 3$ make the vertical paths longer.

2 Split digraphs

If G is a digraph, we use $G[X]$ to denote the subdigraph induced on $X \subseteq V(G)$. We say "u is adjacent to v" to mean that u is an in-neighbour of v, and "adjacent from" to mean it is an out-neighbour. A *neighbour* of v means a vertex that is either an in-neighbour or an out-neighbour of v. We sometimes use "G-in-neighbour" to mean "in-neighbour in the digraph G", and so on (this is helpful because we sometimes work with different digraphs that have the same vertex set.) For a vertex v of a digraph G, $N_G^+(v)$ denotes the set of all out-neighbours of v, and $N_G^-(v)$ is its set of in-neighbours. A *split* in an oriented graph G is a pair (S, T), where $S \cup T = V(G)$, $S \cap T = \emptyset$, S is a stable set, and $G[T]$ is a tournament. (We will often write T for $G[T]$.)

In this section we prove 1.2, but it is convenient to prove a slightly stronger statement, that the same conclusion holds just assuming that no vertex in S is a source. Now there is a difficulty, because this is false for the 1-vertex digraph with $S = \emptyset$, but this is the only exception. We will prove:

2.1 *Let (S, T) be a split of an oriented graph G, such that $S \neq \emptyset$ and no vertex in S is a source. Then there is a 2-kernel K with $|K| \leq |G|/2$.*

For the proof, we begin with some lemmas. A 2-kernel K is *strong* if for every vertex $v \in T$, either there is a vertex in K that 1-covers v, or a vertex in $K \cap T$ that 2-covers v. (We do not know whether 1.2 remains true if we ask for a strong 2-kernel of size at most $|G|/2$.) If $v \in T$, we say $s \in S$ is a *problem* for v if v is adjacent from s, and v does not 2-cover s, and no non-neighbour of v in S 2-covers s. If v has a problem, then v is contained in no 2-kernel.

2.2 *Let G, T, S be as above, and let $v \in V(T)$. If v is contained in no strong 2-kernel, then there exists $w \in V(T) \setminus \{v\}$, adjacent to v, such that $N^-_G(w) \subseteq N^-_G(v)$; and either $w \in S$ and w is a problem for v, or $w \in T$.*

Proof. Since the set consisting of v and all non-neighbours of v in S is not a strong 2-kernel, there exists $w \in V(G) \setminus \{v\}$ such that v does not 2-cover w, and either $w \in T$ and no non-neighbour of v in S 1-covers w, or $w \in S$ and no non-neighbour of v in S 2-covers w. In the first case, since v does not 2-cover w, $N^-_G(w) \cap T \subseteq N^-_G(v)$. If $s \in N^-_G(w) \cap S$, then since no non-neighbour of v in S 1-covers w, it follows that $s \in N^+_G(v) \cup N^-_G(v)$; and since v does not 2-cover w, $s \notin N^+_G(v)$, and so $s \in N^-_G(v)$. This proves that $N^-_G(w) \subseteq N^-_G(v)$ as required. In the second case, w is a problem for v. Moreover, every in-neighbour of w is an in-neighbour of v: because if $u \in T$ is adjacent to w, then u is not adjacent from v since v does not 2-cover w, and so u is adjacent to v. Hence, again, $N^-_G(w) \subseteq N^-_G(v)$. This proves 2.2. ∎

2.3 *Let G, T, S be as above, and suppose that G, S, T form a smallest counterexample to 2.1. Suppose also that $v \in V(T)$ is contained in no strong 2-kernel, and let w be as in 2.2. If $w \in T$, then there is no problem for w.*

Proof. Suppose that $w \in T$, and $s \in S$ is a problem for w. Let $A = N^+_G(v)$. Since $N^-_G(w) \subseteq N^-_G(v)$, no vertex in A is adjacent to w, and in particular $s \notin A$. Make a digraph G' from G by deleting v and making w complete to A. So G' has no sources.

(1) *$N^-_{G'}(w) \subseteq N^-_G(v)$.*

Let $u \in N^-_{G'}(w)$. So $u \notin A$, and so $u \in N^-_G(w) \subseteq N^-_G(v)$. This proves (1).

Let K be a 2-kernel of G'. We will show that K is also a 2-kernel of G. Certainly it is stable in G.

(2) *$w \notin K$.*

Suppose that $w \in K$. Then $s \notin K$, so there is a directed path P of G', of length one or two, from some $x \in K$ to s. Since s is a problem for w in G, some edge of P is not an edge of G, which is impossible since $s \notin A$. This proves (2).

So $w \notin K$. Since K 2-covers w in G', (1) implies that K 2-covers v in G, and 1-covers v in G if it 1-covers w in G'. Let $a \in A$. We must show that K 2-covers a in G. If $a \in K$ this is true, so we assume there is a directed path P of G' of length one or two, from some $x \in K$ to a. If P is a path of G then K 2-covers a in G, so we may assume that the last edge of P is an edge of G' not in G. But $w \notin K$ and $x \in K$, so $w \neq x$, and therefore P has length two with middle vertex w. By (1), x-v-a is a path of G, so K 2-covers a in G.

This proves that every 2-kernel of G' is a 2-kernel of G. Since G, S, T form a smallest counterexample to 2.1, and G' has fewer vertices than G, and $(S, T \setminus \{v\})$ is a split for G', with $S \neq \emptyset$, and no vertex in S is a source in G', it follows that G' has a 2-kernel of size at most $|G'|/2$; but this is also a 2-kernel for G, which is impossible. This proves that there is no problem for v, and so proves 2.3. ∎

Now we prove the main theorem, which we restate:

2.4 *Let (S,T) be a split of an oriented graph G, such that $S \neq \emptyset$ and no vertex in S is a source. Then there is a 2-kernel K with $|K| \leq |G|/2$.*

Proof. We may assume that G, S, T form a smallest counterexample. Let B be the set of all vertices in T with problems. For each $b \in B$, select a problem z_b for b, and let Z be the set $\{z_b : b \in B\}$. Let Q be the set of all $q \in S \setminus Z$ with $N^-_G(q) \subseteq B$. For each $q \in Q$, it has an in-neighbour in B, since it is not a source; select one such in-neighbour b_q. Similarly, for each $s \in S \setminus (Q \cup Z)$, choose some $t_s \in T \setminus B$ adjacent to s.

For each $z \in Z$, let $\Phi(z)$ be the set of $q \in Q$ such that $z = z_{b_q}$. For each $t \in T \setminus B$, let $\Phi(t)$ be the union of $\{t\}$ and the set of $s \in S \setminus (Q \cup Z)$ such that $t = t_s$. Thus, the sets $\Phi(v)$ $(v \in V(H))$ are pairwise disjoint and have union $V(G) \setminus (B \cup Z)$. Some of the sets $\Phi(z)$ $(z \in Z)$ may be empty.

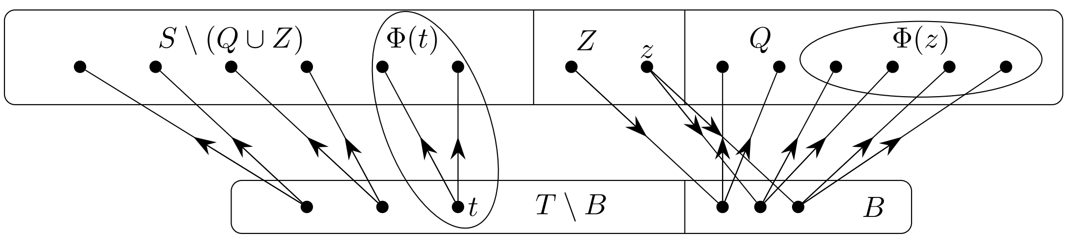

Figure 2: Definitions of $\Phi(z)$ and $\Phi(t)$.

Let H be the oriented graph obtained from $G[(T \setminus B) \cup Z]$ by adding all possible edges from $T \setminus B$ to Z; that is, if $t \in T \setminus B$ and $z \in Z$ are nonadjacent in G then we add an edge tz.

For each $v \in V(H)$, let $N^0_H(v)$ be the set of vertices that are neither out- nor in-neighbours of v (including v itself). Thus $N^0_H(v) = Z$ if $v \in Z$, and $N^0_H(v) = \{v\}$ if $v \in V(T) \setminus B$. Define $\phi^+(v) = \sum_{u \in N^+_H(v)} |\Phi(u)|$ and define $\phi^-(v), \phi^0(v)$ similarly. We call $\phi^-(v) + \phi^0(v)/2$ the *score* of v. If $V(H) = \emptyset$, then $T \setminus B = \emptyset$ and $B = \emptyset$ (since $Z = \emptyset$); so $T = \emptyset$, which implies that $S = \emptyset$ (since there are no sources), a contradiction. So $V(H) \neq \emptyset$. We have

$$\sum_{u \in V(H)} |\Phi(u)|\phi^+(u) = \sum_{uw \in E(H)} |\Phi(u)||\Phi(w)| = \sum_{w \in V(H)} |\Phi(w)|\phi^-(w),$$

and therefore

$$\sum_{u \in V(H)} |\Phi(u)|(\phi^-(u) - \phi^+(u)) = 0.$$

We claim that there exists $v \in V(H)$ such that $\phi^+(v) \geq \phi^-(v)$. If $|\Phi(u)|(\phi^-(u) - \phi^+(u)) \neq 0$ for some $u \in V(H)$, then $|\Phi(u)|(\phi^-(u) - \phi^+(u)) > 0$ for some $u \in V(H)$ and the claim is true. If not, then either $|\Phi(u)| = 0$ for each $u \in V(H)$, or $\phi^-(u) - \phi^+(u) = 0$ for some $u \in V(H)$, and in either case the claim is true. This proves that there exists $v \in V(H)$ such that $\phi^+(v) \geq \phi^-(v)$.

Since

$$\phi^+(v) + \phi^-(v) + \phi^0(v) = |G| - |Z| - |B| \leq |G| - 2|Z|,$$

it follows that $\phi^-(v) + \phi^0(v)/2 \leq |G|/2 - |Z|$. Choose $v \in V(H)$ with score as small as possible (and consequently with score at most $|G|/2 - |Z|$).

A vertex in T is *pure-up* if it has no in-neighbour in S. The case when v has score exactly $|G|/2-|Z|$ is troublesome, so let us first handle that.

(1) *We may assume that either v has score strictly less than $|G|/2-|Z|$, or $v \in Z$ and $\Phi(v) \neq \emptyset$, or $|\Phi(v)| \geq 2$.*

We assume that v has score exactly $|G|/2-|Z|$. It follows that $|B|=|Z|$, and every vertex $u \in V(H)$ has score at least $|G|/2-|Z|$, and so satisfies $\phi^+(u) \leq \phi^-(u)$. But

$$\sum_{u \in V(H)} |\Phi(u)|(\phi^-(u)-\phi^+(u)) = 0.$$

It follows that for every $u \in V(H)$, $|\Phi(u)|(\phi^-(u)-\phi^+(u))=0$, so either $\Phi(u)=\emptyset$ (and hence $u \in Z$) or $\phi^+(u)=\phi^-(u)$ (and hence u has the same score as v). In particular, if $\Phi(u) \neq \emptyset$ for some $u \in Z$, then we may replace v by u and the claim holds. Similarly, if some $u \in T \setminus B$ satisfies $|\Phi(u)| \geq 2$, we can replace v by u. So we may assume that $\Phi(u)=\emptyset$ for all $u \in Z$ (and hence $Q=\emptyset$), and $\Phi(u)=\{u\}$ for each $u \in T \setminus B$ (and hence $S \setminus (Q \cup Z)=\emptyset$). Consequently, $S=Z$. Since $|Z| \leq |G|/2$ (because $|Z|=|B|$), we may assume that there exists $p_0 \in T$ not 2-covered by Z. Thus p_0 is pure-up, and so $P \neq \emptyset$, where P is the set of pure-up vertices. Choose $p \in P$ that 2-covers P. (Any vertex of maximum out-degree in $T[P]$ has this property.) Let Z' be the set of vertices in Z that are not adjacent from p; so $Z' \cup \{p\}$ is stable. We claim it is a 2-kernel. Certainly $Z' \cup \{p\}$ 2-covers Z; each vertex in T 1-covered by Z' is 2-covered by p; every other vertex of T 1-covered by Z is 2-covered by $Z \setminus Z'$; and each vertex of T not 1-covered by Z is in P, and hence is 2-covered by p. So $Z' \cup \{p\}$ is a 2-kernel, and therefore we may assume its size is more than $|G|/2$. Since $|Z|=|B|$, it follows that $|T \setminus B|=1$ and hence $T \setminus B = P = \{p\}$, since $P \cap B = \emptyset$; and so $p_0=p$. Since Z 1-covers B and does not 2-cover $p_0=p$, it follows that p is adjacent to every vertex in B. But then $\{p\}$ is a 2-kernel (because every vertex in $S=Z$ has an in-neighbour, since it is not a source). This proves (1).

(2) *If $v \in Z$ then the theorem holds.*

Let J be the set of vertices in $S \setminus Z$ that are 2-covered by v. (Possibly $J \cap Q \neq \emptyset$.) Let $A = S \setminus (J \cup Q \cup Z)$, and $F=(T \setminus B) \setminus N^+_G(v)$. Since $N^-_H(v)=F$, and therefore the union of the sets $\Phi(u)$ ($u \in N^-_H(v)$) includes $F \cup A$, it follows that $\phi^-(v) \geq |F|+|A|$. Moreover,

$$\phi^0(v) = \frac{1}{2}\sum_{z \in Z} |\Phi(z)| = |Q|/2.$$

Consequently, the score of v is at least $|F|+|A|+|Q|/2$, and so the latter is at most $|G|/2-|Z|$.

Choose $X \subseteq S$ minimal such that $A \cup Z \cup X$ 1-covers every vertex of T that is not pure-up. Thus $|X| \leq |F|$, since Z 1-covers $B \cup (T \cap N^+_G(v))$. Let $K = A \cup Z \cup X$. We claim that K is a 2-kernel. It certainly 2-covers S, since Z 2-covers Q, and $A \cup \{v\}$ 2-covers $S \setminus (Q \cup Z)$. It 1-covers all vertices in T that are not pure-up, from the choice of X. Suppose it does not 2-cover some $p \in T \setminus B$. Then p is pure-up, so $p \notin B$; and p is complete to all vertices in T that are not pure-up, since X 1-covers all such vertices and does not 2-cover p. Moreover, each vertex in Z is adjacent from p in H. Thus, every H-in-neighbour of p is also pure-up, and so is adjacent to v in H. Consequently

$$|\Phi(p)| + \sum_{u \in N_H^-(p)} |\Phi(u)| \leq \sum_{u \in N_H^-(v)} |\Phi(u)|;$$

and so p has smaller score than v, a contradiction.

So K is a 2-kernel. But

$$|K| \leq |X| + |A| + |Z| \leq |F| + |A| + |Z| \leq |G|/2 - |Q|/2.$$

It follows that $|K| \leq |G|/2$. This proves (2).

Henceforth we assume that $v \in V(T) \setminus B$ and, by (1), either v has score strictly less than $|G|/2 - |Z|$, or $|\Phi(v)| \geq 2$.

(3) *v extends to a strong 2-kernel.*

Suppose not. By 2.2, there exists $t \in T$, adjacent to v, such that every G-in-neighbour of t is a G-in-neighbour of v, and $t \in T \setminus B$ by 2.3. A vertex of H is a G-in-neighbour of v if and only if it is an H-in-neighbour of v, and the same is true for in-neighbours of t; so every H-in-neighbour of t is an H-in-neighbour of v. Hence $\phi^-(v) \geq \phi^-(t) + |\Phi(t)|$. Since $\phi^0(v) = |\Phi(v)|$ and $\phi^0(t) = |\Phi(t)|$, it follows that

$$\phi^-(v) + \phi^0(v)/2 \geq \phi^-(t) + |\Phi(t)| + |\Phi(v)|/2 > \phi^-(t) + \phi^0(t)/2,$$

and so the score of t is strictly less than that of v, contradicting the choice of v. This proves (3).

Let $Q' = \bigcup_{z \in Z \setminus N^-(v)} \Phi(z)$, and $Q'' = \bigcup_{z \in Z \cap N^-(v)} \Phi(z)$; so $Q'' = Q \setminus Q'$. Let J be the set of vertices in $S \setminus Q$ that are 2-covered by v in $G \setminus B$. So, J, Z are both subsets of $S \setminus Q$, but they might intersect each other. S is also partitioned into three subsets, $S \cap N_G^+(v)$, $S \cap N_G^-(v)$ and $S \setminus N_G(v)$, where we define $N_G(v) = N_G^+(v) \cup N_G^-(v)$. (See figure 3.) We intend to find a 2-kernel containing v of size at most $|G|/2$, but we must be careful only to add vertices in $S \setminus N_G(v)$, to keep the set stable.

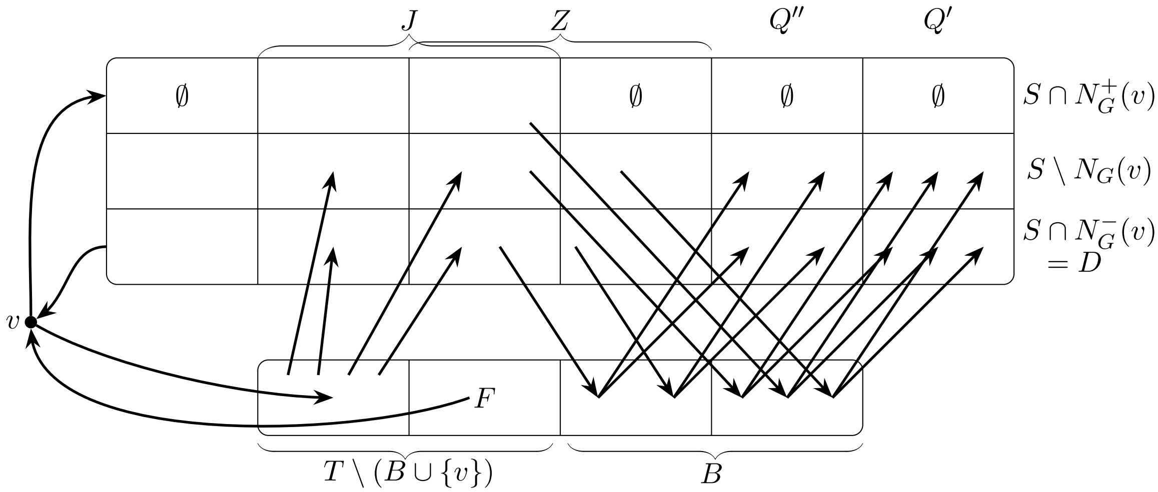

Figure 3: v is adjacent to everything in the top row of boxes, and from everything in the third. Its adjacency to B is not specified in the figure. It has no out-neighbours in Q since $v \notin B$, and so all its out-neighbours in S belong to J.

Let $D = N^-_G(v) \cap S$, and $F = (T \setminus B) \cap N^-_G(v)$. Thus

$$N^-_H(v) = F \cup (Z \cap D).$$

The union of the sets $\Phi(t)$ $(t \in F)$ includes $F \cup (S \setminus (Q \cup J \cup Z))$, and $\bigcup_{z \in Z \cap D} \Phi(z) = Q''$. Consequently

$$\phi^-(v) \geq |F| + |S \setminus (Q \cup J \cup Z)| + |Q''|,$$

and so the score of v is at least

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + \phi^0(v)/2.$$

Since $\phi^0(v) \geq 1$, and either $\phi^0(v) \geq 2$ or the score of v is strictly less than $|G|/2 - |Z|$, it follows that

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 + |Z| \leq |G|/2.$$

Since v extends to a strong 2-kernel, for each $u \in T \setminus B$ that is not 2-covered by v, there is an in-neighbour of u in $S \setminus N_G(v)$; choose $X \subseteq S \setminus N_G(v)$ minimal 1-covering each vertex in F that is not 2-covered by v. Thus $|X| \leq |F|$. For each $u \in D$, since v extends to a 2-kernel, there exists $t \in S \setminus N_G(v)$ that 2-covers u; let $Y \subseteq S \setminus N_G(v)$ be minimal 2-covering $D \setminus (J \cup Q')$. Thus $|Y| \leq |D \setminus (J \cup Q')|$.

Let

$$K = \{v\} \cup (Z \setminus D) \cup (S \setminus (Q \cup J \cup Z \cup D)) \cup X \cup Y \cup (Q'' \setminus D).$$

We claim that K is a 2-kernel. Certainly it is stable.

(4) *K 2-covers S.*

Let $s \in S$, and assume first that $s \notin Q$. If $s \in J$ then v 2-covers s; if $s \in D \setminus J$ then Y 2-covers s; if if $s \in Z \setminus (J \cup D)$ then $s \in K$; and if $s \notin Z \cup J \cup D$ then $s \in K$. So in this case K 2-covers s. Next assume that $s \in Q$. So $s \notin J \cup N_G^+(v)$. If $s \in Q'' \setminus D$ then $s \in K$, and if $s \in Q'' \cap D$ then Y 2-covers s, so we assume that $s \in Q'$, and so $z_{b_s} \in Z \setminus D$. If $z_{b_s} \notin N_G^+(v)$ then $z_{b_s} \in K$ and so K 2-covers s, so we assume that $z_{b_s} \in N_G^+(v)$. Then b_s is adjacent from v (because b_s does not 2-cover z_{b_s} since z_{b_s} is a problem for b_s) and so K 2-covers s. This proves (4).

(5) *K 2-covers T, and hence K is a 2-kernel.*

Let $t \in T$. We may assume that $t \in N_G^-(v)$. If $t \in T \setminus B$ then $t \in F$ and X 1-covers t, so we assume that $t \in B$. If $z_t \notin N_G(v)$ then $z_t \in K$ and 1-covers t, so we assume that $z_t \in N_G(v)$. Since t is adjacent from v and z_t is a problem for t, it follows that $z_t \notin N_G^+(v)$, so $z_t \in N_G^-(v)$. Choose $y \in Y$ such that y 2-covers z_t, and choose $u \in T$ such that y-u-z_t is a directed path. Since z_t is a problem for t, it follows that t is adjacent from u, and so y 2-covers t. This proves (5).

Now let us bound the size of K. We have

$$|K| = 1 + |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |X| + |Y| + |Q'' \setminus D|.$$

We know that

$$|F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 \le |G|/2 - |Z|,$$

and $|X| \le |F|$, and $|Y| \le |D \setminus (J \cup Q')|$. Adding, we deduce that:

$$\begin{aligned}
&|K| + |F| + |S \setminus (Q \cup J \cup Z)| + |Q''| + 1 + |X| + |Y| \\
&\le 1 + |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |X| + |Y| + |Q'' \setminus D| \\
&+ (|G| - |Z| - |B|)/2 + |F| + |D \setminus (J \cup Q')|.
\end{aligned}$$

This simplifies to:

$$\begin{aligned}
|K| + |S \setminus (Q \cup J \cup Z)| + |Q''| \le |Z \setminus D| + |S \setminus (Q \cup J \cup Z \cup D)| + |Q'' \setminus D| \\
+ |G|/2 - |Z| + |D \setminus (J \cup Q')|.
\end{aligned}$$

Since $|Z| \le |B|$, and

$$|S \setminus (Q \cup J \cup Z)| = |S \setminus (Q \cup J \cup Z \cup D)| + |D \setminus (Q \cup J \cup Z)|,$$

we deduce

$$|K| + |D \setminus (Q \cup J \cup Z)| + |Q''| \le |Q'' \setminus D| + |D \setminus (J \cup Q')| + |Z \setminus D| + |G|/2 - |Z|.$$

Since

$$|D \setminus (J \cup Q')| - |D \setminus (Q \cup J \cup Z)| = |(D \setminus J) \cap (Q'' \cup (Z \setminus Q'))| \le |D \cap (Q'' \cup Z)|,$$

this further simplifies to:

$$|K| + |Q'' \cap D| \le |D \cap (Q'' \cup Z)| - |Z \cap D| + |G|/2,$$

and so $|K| \le |G|/2$. This proves 2.4. ∎

3 Large 2-kernels

In this section, we turn to a second topic, Spiro's question 1.3. While it seems to be asking for something close to the opposite of 1.1, Spiro observed that 1.1 implies 1.3. Here is his argument: to prove 1.3 for a digraph G, choose a large number n. If G has a source v, delete v and all its out-neighbours and apply induction; while if G has no sources, for each vertex v of G, add n new vertices adjacent from v and with no other neighbours. Applying 1.1 with n sufficiently large implies that G satisfies 1.3.

If G is a digraph and $X \subseteq V(G)$, let $N_G^+[X]$ denote the set of vertices that either belong to X or are adjacent from a vertex in X. The same construction (adding $nw(v)$ new out-leaves for each vertex) shows that 1.1 implies a slightly stronger statement ($\mathbb{Z}_+$ denotes the set of non-negative integers, and $f(X)$ denotes $\sum_{v \in X} f(v)$):

3.1 Conjecture: *In every digraph G, and for every map $f : V(G) \to \mathbb{Z}_+$ there is a 2-kernel K such that $f(N_G^+[K]) \geq f(V(G))/2$.*

In this section we show that 3.1 is true for split digraphs, and indeed for a somewhat more general class of graphs. If G is an oriented graph, let us say a *break* of G is a partition (S, T) of $V(G)$ such that $G[S]$ is *acyclic* (that is, has no directed cycles), and $G[T]$ is a tournament. We will show:

3.2 *In every oriented graph G that admits a break, and for every map $f : V(G) \to \mathbb{Z}_+$, there is a 2-kernel K such that $f(N_G^+[K]) \geq f(V(G))/2$.*

The greater generality given by the function f will be useful for the inductive proof, allowing us to delete vertices without changing $f(V(G))$. We need a result of von Neumann and Morgenstern [6]:

3.3 *Every acyclic digraph has a unique 1-kernel.*

In order to prove 3.2, we prove a stronger statement (by the *non-neighbourhood* of a vertex v, we mean the digraph induced on the set of vertices different from and nonadjacent with v):

3.4 *Let (S, T) be a break of an oriented graph G, and let $f : V(G) \to \mathbb{Z}_+$ be a map. Then there is a 2-kernel K such that $f(N_G^+[K]) \geq f(V(G))/2$, where either $K \subseteq S$, or K consists of some $v \in T$ together with the unique 1-kernel of its non-neighbourhood.*

Proof. We assume the result holds for all oriented graphs that admit breaks (S', T') with $2|S'| + |T'| < 2|S| + |T|$. For each $X \subseteq S$, let $A(X)$ be the unique 1-kernel of $G[X]$ (which exists by 3.3); and for each $v \in T$, let $M(v)$ be its non-neighbourhood. Let us say a 2-kernel K of G is *special for* (G, S, T) if either $K \subseteq S$, or $K = \{v\} \cup A(M(v))$ for some $v \in T$.

(1) *We may assume that $\{v\} \cup A(M(v))$ is a 2-kernel for each $v \in T$.*

Suppose not. Certainly $\{v\} \cup A(M(v))$ is stable, so there is a vertex $w \neq v$ such that $\{v\} \cup A(M(v))$ does not 2-cover w. We claim that $N_G^-(w) \subseteq N_G^-(v)$. For suppose that $s \in N_G^-(w) \setminus N_G^-(v)$. Since $s \notin \{v\} \cup N_G^+(v)$ (because $\{v\} \cup A(M(v))$ does not 2-cover w, it follows that v, s are nonadjacent, and so $s \in M(v) \subseteq S$. But then s is 1-covered by $A(M(v)$, and so w is 2-covered by $\{v\} \cup A(M(v))$, a contradiction. This proves that $N_G^-(w) \subseteq N_G^-(v)$. Thus every 2-kernel of $G' = G \setminus v$ is also a 2-kernel of G. Define $f'(w) = f(w) + f(v)$, and $f'(x) = f(x)$ for all $x \in V(G) \setminus \{v, w\}$. Applying the inductive hypothesis to G' and f', we deduce there is a 2-kernel K of G' (and hence of G), special for $(G', S, T \setminus \{v\})$ (and hence special for (G, S, T)), such that $f'(N^+_{G'}[K]) \geq f'(V(G))/2 = f(G)/2$. But $N^+_{G'}[K] \subseteq N^+_G[K]$, and if $w \in N^+_{G'}[K]$ then $v, w \in N^+_G[K]$, and so $f'(N^+_{G'}[K]) \leq f(N^+_G[K])$. Hence $f(N^+_G[K]) \geq f(G)/2$. This proves (1).

A *sink* of G is a vertex that has no out-neighbours.

(2) *Let $s \in S$ be a sink of $G[S]$. We may assume that s is a neighbour of every vertex in T.*

For each $t \in T$, if s, t are nonadjacent, let us add the edge ts, forming an oriented graph G'. Suppose the theorem holds for G', with the same function f, and let K' be a 2-kernel of G', special for (G', S, T), with $f(N^+_{G'}[K']) \geq f(V(G'))/2 = f(V(G))/2$. For each $v \in T$, let $M'(v)$ be the non-neighbourhood of v in G'. There are four cases:

- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ adjacent from s in G;
- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ adjacent to s in G;
- $K' = \{v\} \cup A(M'(v))$ for some $v \in T$ nonadjacent with s in G;
- $K' \subseteq S$.

In the first two cases, $M'(v) = M(v)$, and $\{v\} \cup A(M(v))$ is a 2-kernel of G by (1); and $N^+_{G'}[K'] = N^+_G[K']$, and so K' satisfies the theorem. In the third case, $M'(v) = M(v) \setminus \{s\}$. If $A(M'(v))$ 1-covers s, then $A(M'(v)) = A(M(v))$ and so K' satisfies the theorem. If $A(M'(v))$ does not 1-cover s, then $A(M(v)) = A(M'(v)) \cup \{s\}$ (because s is a sink of $G[S]$), and so $K = \{v\} \cup A(M(v))$ satisfies the theorem. Finally, in the fourth case, $K' \subseteq S$. If K' is a 2-kernel of G then it satisfies the theorem, so we assume it is not; and since K' is a 2-kernel of G', it follows that K' does not 2-cover s. But then $K' \cup \{s\}$ satisfies the theorem. This proves (2).

If $S = \emptyset$, then G is a tournament and the result holds, so we assume that $S \neq \emptyset$, and hence contains a sink of $G[S]$. By (2), then $(S \setminus \{s\}, T \cup \{s\})$ is also a break of G, and from the inductive hypothesis, there is a 2-kernel K of G such that $f(N^+_G[K]) \geq f(V(G))/2$, and K is special for $(G, S \setminus \{s\}, T \cup \{s\})$. But then K is also special for (G, S, T). This proves 3.4. ∎

What happens to 3.1 if we assume that $V(G)$ can be partitioned into two sets S, T where T is a tournament and S is small? By 3.4, the conjecture holds if $|S| \leq 2$, and in hope of finding a counterexample, we worked on the case when $|S| = 3$. But the conjecture is also true in this case (by an *ad hoc* argument that does not seem capable of any generalization, and we omit the details).

There is a natural refinement of the conjectures 1.1 and 1.3, equivalent to 1.1 and implying 1.3, that:

3.5 Conjecture: *In every digraph G, and for every map $f : V(G) \to \mathbb{Z}_+$ there is a 2-kernel K such that $|K| + f(V(G))/2 \leq |G|/2 + f(N^+_G(K))$.*

To deduce this from 1.1, add $f(v)$ out-leaves to each vertex v. It implies 1.1 by taking $f(v) = 0$ for all v, and it implies 1.3 by scaling f to be very large. Perhaps the proof of 2.1 can be modified to show that split graphs satisfy 3.5, but we have not seriously attempted this.

4 k-kernels

Now we turn to the proof of our third result, 1.5. We begin with:

4.1 *For all integers $k \geq 0$, if G is an acyclic digraph with only one source, then there exists $X \subseteq V(G)$ with $|X| \leq 1 + (|G|-1)/(k+1)$ that k-covers $V(G)$. Moreover, either $|G| = 1$ or $|X| \leq 1 + (|G|-2)/(k+1)$ or X is not stable.*

Proof. Let r be the unique source. If $|G| \leq k$, we may take $X = \{r\}$; then $|X| \leq 1+(|G|-2)/(k+1)$ unless $|G| = 1$, so the result holds. We assume then that $|G| > k$, and proceed by induction on G. For each $v \in V(G)$, let A_v be the set of vertices that are joined by a directed path (of any length) from v; and choose v with $|A_v|$ minimal such that $|A_v| \geq k+1$. (This is possible since $|A_r| \geq k+1$.) For each $w \in A_v$, there is a directed path P from v to w, and if P has length more than k then we may replace v by its outneighbour in P, contradicting the minimality of A_v. Thus every vertex in A_v is joined from v by a path of length at most k. If $v = r$ then we may take $X = \{r\}$ and win as before, so we assume that $v \neq r$. Let G' be the digraph obtained by deleting A_v. Every vertex of G' has an in-neighbour in G' except r, so G' has a unique source; and from the inductive hypothesis, there exists $X' \subseteq V(G')$ such that $|X'| \leq 1 + (|G'|-1)/(k+1)$ and X' k-covers $V(G')$. Moreover, either $|G'| = 1$ or $|X'| \leq 1 + (|G'|-2)/(k+1)$ or X' is not stable. Let $X = X' \cup \{v\}$. Thus X k-covers $V(G)$. Moreover, since $|A_v| \geq k+1$, it follows that $|X| \leq 1+(|G|-1)/(k+1)$, and if either $|X'| \leq 1+(|G'|-2)/(k+1)$ or X' is not stable, then correspondingly either $|X| \leq 1+(|G|-2)/(k+1)$ or X is not stable. So we assume that $|G'| = 1$, and so $V(G') = \{r\}$. Since G has a unique source, it follows that v is adjacent from r, and so X is not stable. This proves 4.1. ∎

We deduce:

4.2 *For every integer $k \geq 1$, if G is an acyclic digraph with $|G| > 1$ and with only one source, then G has a k-kernel of size at most $1 + (|G|-2)/k$.*

Proof. By 4.1 applied to G with k replaced by $k-1$, there exists $X \subseteq V(G)$ with $|X| \leq 1+(|G|-1)/k$ that $(k-1)$-covers $V(G)$. The digraph $G[X]$ is acyclic and hence has a 1-kernel Y, by 3.3. Hence Y is a k-kernel in G. Moreover, since $|G| \geq 2$, either $|X| \leq 1 + (|G|-2)/k$ (when $|Y| \leq |X|$ and the result is true), or X is not stable (when $|Y| \leq |X| - 1 \leq (|G|-1)/k$ and again the result is true). This proves 4.2. ∎

As we said before, this result is tight (see figure 1). Now let us deduce 1.5, which we restate:

4.3 *For all integers $k \geq 2$, every digraph G with $|G| > 1$ and with a spanning arborescence has a k-kernel of size at most $1 + (|G|-2)/(k-1)$.*

Since G has a spanning arborescence, its vertex set can be numbered $\{v_1, \ldots, v_n\}$ in such a way that for $2 \leq j \leq n$ there exists $i \in \{1, \ldots, j-1\}$ such that v_iv_j is an edge. Let A be the set of all edges v_iv_j of G with $i < j$, and let $B = E(G) \setminus A$. Let G_A be the subgraph with vertex set $V(G)$ and edge set A, and define G_B similarly. Both G_A, G_B are acyclic, and G_A has a unique source. By 4.2 applied to G_A with k replaced by $k-1$, G_A has a $(k-1)$-kernel X of size at most $1+(|G|-2)/(k-1)$. Now X is stable in G_A, and $G_B[X]$ is acyclic, and so has a 1-kernel Y, by 3.3. But then Y is a k-kernel in G, and $|Y| \leq |X| \leq 1 + (|G|-2)/(k-1)$. This proves 4.3. ∎

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