# A note on infinite antichain density

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**Abstract**

We show that for any sequence $(f_n)_{n\geq n_0}$ of positive integers satisfying $\sum_{n=n_0}^{\infty} f_n/2^n \leq 1/4$ and $f_n \leq f_{n+1} \leq 2f_n$, there exists an infinite antichain $\mathcal{F}$ of finite subsets of $\mathbb{N}$ such that $|\mathcal{F}\cap 2^{[n]}| \geq f_n$ for all $n \geq n_0$. It follows that for any $\varepsilon > 0$ there exists an antichain $\mathcal{F} \subseteq 2^{\mathbb{N}}$ such that

$$\liminf_{n\to\infty} |\mathcal{F} \cap 2^{[n]}| \cdot \left(\frac{2^n}{n \log^{1+\varepsilon} n}\right)^{-1} > 0.$$

This resolves a problem of Sudakov, Tomon and Wagner in a strong form, and is essentially tight.

## 1 Introduction

A family $\mathcal{F}$ of sets is an *antichain* if $A \not\subseteq B$ for all distinct $A, B \in \mathcal{F}$. Sperner's well-known theorem [2] states that any antichain $\mathcal{F} \subseteq 2^{[n]}$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$, where the upper bound is achieved by the antichain consisting of all sets of size $\lfloor n/2 \rfloor$. Sudakov, Tomon and Wagner [3] recently studied an infinite version of Sperner's problem: for an (infinite) antichain $\mathcal{F} \subseteq 2^{\mathbb{N}}$, what is the maximum possible growth rate of $|\mathcal{F} \cap 2^{[n]}|$?

It follows immediately from Sperner's Theorem that $|\mathcal{F} \cap 2^{[n]}| \leq \binom{n}{\lfloor n/2 \rfloor} = O(2^n/\sqrt{n})$. However, Sudakov, Tomon and Wagner showed that for an infinite antichain, $|\mathcal{F} \cap 2^{[n]}|$ must grow significantly more slowly. Using Kraft's inequality [1], they gave a short proof of the following.

**Theorem 1** (Sudakov, Tomon and Wagner [3]). *Let $\mathcal{F} \subseteq 2^{\mathbb{N}}$ be an antichain. Then*

$$\sum_{n=1}^{\infty} \frac{|\mathcal{F} \cap 2^{[n]}|}{2^n} \leq 2. \tag{1.1}$$

It follows immediately that $|\mathcal{F} \cap 2^{[n]}|$ cannot grow as quickly as $2^n/n \log n$, and in particular

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$$\liminf_{n\to\infty} |\mathcal{F}\cap 2^{[n]}| \cdot \left(\frac{2^n}{n\log n}\right)^{-1} = 0. \qquad (1.2)$$

Turning to lower bounds, Sudakov, Tomon and Wagner constructed an antichain with density matching (1.2) up to a polylogarithmic term.

**Theorem 2** (Sudakov, Tomon and Wagner [3])**.** *There exists an antichain $\mathcal{F}\subseteq 2^{\mathbb{N}}$ with*

$$\liminf_{n\to\infty} |\mathcal{F}\cap 2^{[n]}| \cdot \left(\frac{2^n}{n\log^{46} n}\right)^{-1} > 0.$$

They go on to speculate that the bound in Theorem 1 is essentially correct, and the exponent 46 in Theorem 2 could be improved to $1+\varepsilon$ for any $\varepsilon > 0$. We show that this is indeed the case. In fact we prove a stronger result, giving essentially optimal bounds on the growth rate of $|\mathcal{F}\cap 2^{[n]}|$.

Our main result is the following, which matches the form taken by the first part of Theorem 1, namely that under natural additional assumptions any convergence rate of the series in (1.1) is feasible.

**Theorem 3.** *Let $(f_n)_{n\geq n_0}$ be a non-decreasing sequence of positive integers for which*

$$\sum_{n=n_0}^{\infty} \frac{f_n}{2^n} \leq \frac{1}{4}$$

*and $\frac{f_n}{2^n}$ is monotonically decreasing (so $f_n \leq f_{n+1} \leq 2f_n$). Then there exists an antichain $\mathcal{F}\subseteq 2^{\mathbb{N}}$ such that*

$$|\mathcal{F}\cap 2^{[n]}| \geq f_n$$

*for all $n \geq n_0$.*

By taking $f_n$ to be about $2^n/(n\log n(\log\log n)^2)$ for sufficiently large $n$, the following result, answering the question of Sudakov, Tomon and Wagner, is immediate.

**Corollary 4.** *There exists an antichain $\mathcal{F}\subseteq 2^{\mathbb{N}}$ such that*

$$|\mathcal{F}\cap 2^{[n]}| = \frac{2^n}{n\log^{1+o(1)} n}.$$

We use standard notation throughout. In particular, for a set $X$, we write $2^X$ for the power set of $X$; and $[n] = \{1,\dots,n\}$. We identify infinite binary ($\{0,1\}$-)strings with subsets of $\mathbb{N}$ in the usual way, that is, a string $x_1x_2\cdots$ corresponds to the set $\{i\in\mathbb{N} : x_i = 1\}$.

## 2 Antichain construction

In this section, we prove Theorem 3.

The elements of our antichain will each consist of two concatenated parts where the initial segment encodes the number of 1s in the remainder of the string. By construction, these elements (in particular the initial segments) naturally occur in reverse lexicographic order and are built in blocks of elements with the same initial segment.

The set of strings that we use as initial segments have the property that no string is an initial segment of any other. Such a set is called a *prefix code.* This condition, while being much weaker than that required for an antichain, gets us 'halfway' there, as it ensures that elements with prefixes earlier in reverse lexicographic order cannot be subsets of those with later prefixes. To obtain our antichain, we will then append strings to each prefix in such a way that later elements cannot be subsets of earlier ones.

*Proof of Theorem 3.* By assumption, all $f_n$ are positive. Let $k_0$ be the smallest natural number $k$ such that $f_{n_0}/2^{n_0} \geq 1/2^{k+1}$, and for $k \geq k_0$ define

$$\ell_k = \max\left\{n : \tfrac{f_n}{2^n} \geq \tfrac{1}{2^{k+1}}\right\}.$$

We note that $\ell_k$ is well defined as $f_n/2^n \to 0$ and $f_n \geq 1$ for $n \geq n_0$, which also gives $\ell_k \geq k+1$. Also, as $f_n$ is non-decreasing, $\ell_{k+1} > \ell_k$.

Define $a_k = \ell_k - k$ for $k \geq k_0$ and note that $a_k > 0$.

**Claim 1.** $\sum_{k=k_0}^{\infty} \frac{a_k}{2^k} \leq 1$.

*Proof.* We note that for any $k \geq k_0$ by definition of $\ell_k$ and by monotonicity of $(f_n/2^n)_{n \geq n_0}$, we have $\frac{f_n}{2^n} \geq 2^{-(k+1)}$ for all $n \in (\ell_{k-1}, \ell_k]$. Setting $\ell_{k_0-1} = 0$ we thus get

$$\frac{\ell_k - \ell_{k-1}}{2^{k+1}} \leq \sum_{n=\ell_{k-1}+1}^{\ell_k} \frac{f_n}{2^n}.$$

Now as

$$\sum_{k \geq k_0} \frac{\ell_k - \ell_{k-1}}{2^{k+1}} = \left(\frac{1}{2} - \frac{1}{4}\right) \sum_{k \geq k_0} \frac{\ell_k}{2^k},$$

we have

$$\sum_{k \geq k_0} \frac{a_k}{2^k} \leq \sum_{k \geq k_0} \frac{\ell_k}{2^k} \leq 4 \sum_{n \geq n_0} \frac{f_n}{2^n} \leq 1. \qquad \square$$

We construct a prefix code $(c_{k,i})_{k \geq k_0, i \in [a_k]}$ consisting of $a_k$ many strings of length $k$ with the property that the elements are lexicographically decreasing when ordered so that their indices $(k,i)$ are lexicographically increasing. Such a sequence is given by setting $c_{k,i}$ to be the string of length $k$ with digits $c_{k,i}(1), \ldots, c_{k,i}(k)$ defined by

$$\sum_{j=1}^{k} \frac{c_{k,i}(j)}{2^j} = 1 - s_{k-1} - \frac{i}{2^k}$$

where $s_k = \sum_{i=k_0}^{k} a_i/2^i$. That is, we take $c_{k,i}$ to be the first $k$ binary digits of the binary representation of the fraction $1 - s_{k-1} - \frac{i}{2^k}$, which is guaranteed to be positive since $\sum_{k=k_0}^{\infty} a_k/2^k \leq 1$. Equivalently, this sequence may be described by starting with the string of length $k_0$ consisting of all 1s, and then each string of length $k \geq k_0$ is obtained by subtracting $1/2^k$ from the previous string considered as a binary expansion of a fraction. For example, if $k_0 = 2$, $a_2 = 1$, and $a_3 = 2$ then the first three strings would be $c_{2,1} = 11$, $c_{3,1} = 101$, and $c_{3,2} = 100$.

It is not hard to see that for two distinct strings in the sequence $(c_{k,i})$, at the first position where they differ the earlier string has a 1 and the later one a 0. It follows that the $c_{k,i}$ indeed form a lexicographically decreasing prefix code.

Now given a particular string $c_{k,i}$ of length $k$, let $F_{k,i}$ be the set of all binary strings of length $\ell_k$ satisfying the following conditions:

(1) The first $k$ digits are precisely $c_{k,i}$.

(2) There are precisely $i$ many 1s after the $k$th digit.

(3) If $k > k_0$, there is at least one 1 after the $\ell_{k-1}$th digit.

We then define the family

$$\mathcal{F} := \bigcup_{\substack{k \geq k_0, \\ i \in [a_k]}} F_{k,i}$$

and view this as a subset of $2^{\mathbb{N}}$ by filling out the strings with 0s in the usual way.

**Claim 2.** *$\mathcal{F}$ is an antichain.*

*Proof.* Take any distinct $x = x_1x_2x_3\ldots, y = y_1y_2y_3\ldots \in \mathcal{F}$, say with $x \in F_{k,i}$ and $y \in F_{k',i'}$. If $k = k'$ and $i = i'$, then $x$ and $y$ have the same number of 1s after the $k$th digit. Since $x$ and $y$ are distinct but agree on the first $k$ digits, this means we find $j$ and $j'$ such that $x_j = 0, y_j = 1, x_{j'} = 1$ and $y_{j'} = 0$. Hence we may assume that $k \leq k'$, and if $k = k'$ then $i < i'$.

By construction, we have that $c_{k,i}$ appears earlier than $c_{k',i'}$ in reverse lexicographic order. It follows that $x_j = 1$ and $y_j = 0$ where $j$ is the first position at which $x$ and $y$ differ, and moreover this must occur at some $j \leq k$ as the $c_{k,i}$ form a prefix code. In addition, if $k < k'$, then by Condition (3) there is some position $j > \ell_{k'-1}$ for which $y_j = 1$. But all 1s in $x$ occur within the first $\ell_k \leq \ell_{k'-1}$ places, so $x_j = 0$. Otherwise, if $k = k'$ and $i < i'$ then by Condition (2) this means that $x$ has fewer 1s after digit $k$ than $y$ does, so there is necessarily some position $j$ for which $x_j = 0$ and $y_j = 1$. Thus, $x$ is neither a subset nor superset of $y$. □

**Claim 3.** *For each $k \geq k_0$ and $n \in (\ell_{k-1}, \ell_k]$ there are at least $2^{n-k} - 1$ strings in $\mathcal{F} \cap 2^{[n]}$.*

*Proof.* We proceed by induction on $k$. For $k = k_0$, Condition (3) is void. Thus we have $2^{n-k_0} - 1$ choices of binary strings $b$ between positions $k_0 + 1$ and $n$ that have at least one 1. Denoting concatenation of strings by multiplication, for each $b$ there is precisely one corresponding string in $\mathcal{F}$ agreeing with $b$ in these positions, namely $c_{k_0,i}b$ where $i$ is the number of 1s in $b$. Note that, since $a_k = \ell_k - k$ for all $k \geq k_0$ by definition, the number of 1s in $b$ does not exceed $a_k$ which ensures that $c_{k_0,i}b$ can be found in $\mathcal{F}$.

Now suppose $k > k_0$. Applying the induction hypothesis for $k - 1$ and $n' = \ell_{k-1}$ we see we have at least $2^{\ell_{k-1}-(k-1)} - 1$ strings that have no 1 after $\ell_{k-1}$, that is, $|\mathcal{F} \cap 2^{[\ell_{k-1}]}| \geq 2^{\ell_{k-1}-(k-1)} - 1$. Now consider the number of strings that have at least one 1 after $\ell_{k-1}$. We have $2^{n-k} - 2^{\ell_{k-1}-k}$ choices of binary strings $b$ between positions $k + 1$ and $n$ such that $b$ has at least one 1 after $\ell_{k-1}$, and, as above, for each $b$ there is precisely one corresponding string $c_{k,i}b$ in $\mathcal{F}$ agreeing with $b$ in these positions. Since $\ell_k > \ell_{k-1}$, this makes a total of at least

$$2^{n-k} - 2^{\ell_{k-1}-k} + 2^{\ell_{k-1}-(k-1)} - 1 \geq 2^{n-k} - 1$$

strings in $\mathcal{F} \cap 2^{[n]}$. □

Finally, for $n \in (\ell_{k-1}, \ell_k]$ we have $f_n/2^n < 2^{-k}$ by definition of $\ell_{k-1}$. Hence $2^{n-k} > f_n$ so we have constructed an antichain $\mathcal{F}$ that contains at least $2^{n-k} - 1 \geq f_n$ strings in $\mathcal{F} \cap 2^{[n]}$. This concludes the proof of Theorem 3. □

# References


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