Judicious partitions of bounded-degree graphs

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Abstract

We prove results on partitioning graphs G with bounded maximum degree. In particular, we provide optimal bounds for bipartitions $V(G) = V_1 \cup V_2$ in which we minimize $\max\{e(V_1), e(V_2)\}$.

1 Introduction

The Max Cut problem asks for the maximum size of a cut in a graph G. By considering random cuts, it is easy to see that every graph with m edges has a cut of size at least $m/2$ (and the obvious greedy algorithm achieves this); sharper bounds for the extremal problem were obtained by Edwards ([10], [11]), who showed that every graph G with m edges has a bipartition $V(G) = V_1 \cup V_2$ with

$$e(V_1, V_2) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}. \tag{1}$$

(For subsequent work, see [1],[2],[7], [20], and [13], [15].) Of course, maximizing $e(V_1, V_2)$ over partitions $V(G) = V_1 \cup V_2$ is equivalent to minimizing $e(V_1) + e(V_2)$; here we shall be concerned with minimizing $\max\{e(V_1), e(V_2)\}$. Problems of this type, which involve finding a bipartition in which each

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vertex class (or each subset of vertex classes) satisfies some condition simultaneously, are known as *judicious partitioning problems* (see [3], [4], [5], [6], [7], [8]).

The problem of finding a bipartition $V(G) = V_1 \cup V_2$ minimizing $\max\{e(V_1), e(V_2)\}$ was addressed in [6] (see also [16], [17], [18]), where it was shown that every graph with m edges has a bipartition in which each vertex class contains at most

$$\frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16} \tag{2}$$

edges; indeed, there is a partition that satisfies both this bound and (1). It was also shown that there is a vertex partition into r classes such that each vertex class contains at most

$$\frac{m}{r^2} + \frac{r-1}{2r^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right) \tag{3}$$

edges. These bounds are sharp for complete graphs on $rn + 1$ vertices.

In this paper, we concentrate on graphs with bounded maximal degree. In section 2 we show that if $k \geq 3$ is odd then we can improve on (2) for graphs of maximal degree at most k: every such graph has a bipartition in which each class contains at most

$$\frac{k-1}{4k}m + \frac{k-1}{4} \tag{4}$$

edges; the extremal graphs are of the form $(2t+1)K_k \cup sK_{k+1}$, for $s, t \geq 0$. As in [6], we can also demand that $e(V_1, V_2)$ is large: there is a bipartition satisfying (4) such that

$$e(V_1, V_2) \geq \frac{k+1}{2k}m. \tag{5}$$

Note that (5) is sharp for graphs of the form $tK_k \cup sK_{k+1}$. We also show that stronger results hold for k-regular graphs.

In section 3 we discuss partitions in which we seek to bound the edges contained in each vertex class both from above and from below. For instance, given a graph with m edges we would like a bipartition with close to $p^2 m$ edges in one class and close to $(1-p)^2 m$ in the other class. We prove a general result for partitions of oriented hypergraphs.

2 Bipartitions of bounded-degree graphs

Our main result in this section is the following.

Theorem 1. *Let $k \geq 3$ be an odd integer. Then every graph G with m edges and maximum degree at most k has a vertex partition $V(G) = V_1 \cup V_2$ such that, for $i = 1, 2$,*

$$e(V_i) \leq \frac{k-1}{4k}m + \frac{k-1}{4} \tag{6}$$

and

$$e(V_1, V_2) \geq \frac{k+1}{2k}m. \tag{7}$$

The extremal graphs for (6) *are of the form* $(2t+1)K_k \cup sK_{k+1}$*, for* $s, t \geq 0$.

Proof. Suppose the theorem is false, and let G be a counterexample with a minimal number of vertices. Then either G has no partition satisfying (6) and (7), or G has no partition satisfying (6) with strict inequality and G is not of the form $(2t+1)K_k \cup sK_{k+1}$.

Note first that G contains no component C isomorphic to K_{k+1}, or applying the theorem to $G \setminus C$ yields the result for G (note that K_{k+1} can be partitioned into two vertex classes of size $(k+1)/2$). Thus Brooks Theorem implies that there is a proper colouring $c : V(G) \to [k] = \{1, \ldots, k\}$. Let $[k] = A \cup B$ be a random partition into sets A, B with $|A| = (k+1)/2$ and $|B| = (k-1)/2$, chosen with equal probability among all such partitions. Then, writing $V_S = \{x : c(x) \in S\}$ for each $S \subset [k]$, we have

$$\mathbb{E}\, e(V_A, V_B) = \frac{\frac{k+1}{2}\frac{k-1}{2}}{\binom{k}{2}}m = \frac{k+1}{2k}m. \tag{8}$$

Let $V(G) = V_1 \cup V_2$ be a cut of G with maximal size: then (8) implies $e(V_1, V_2) \geq \frac{k+1}{2k}m$. We choose such a cut with $\max\{e(V_1), e(V_2)\}$ minimal. We may assume that $e(V_1) \geq e(V_2)$ Note that since $e(V_1, V_2)$ is maximal, we have $|\Gamma(v) \cap V_2| \geq |\Gamma(v) \cap V_1|$ for $v \in V_1$ and $|\Gamma(v) \cap V_1| \geq |\Gamma(v) \cap V_2|$ for $v \in V_2$, where $\Gamma(v)$ denotes the set of neighbours of v (or else we could move v to the opposite side and increase the size fo the cut).

Suppose

$$e(V_1) = \frac{k-1}{4k}m + \alpha, \tag{9}$$

so

$$\begin{aligned} e(V_2) &\le m - e(V_1, V_2) - e(V_1) \\ &\le m - \frac{k+1}{2k}m - \frac{k-1}{4k}m - \alpha \\ &= \frac{k-1}{4k}m - \alpha. \end{aligned} \tag{10}$$

If $\alpha < \frac{k-1}{4}$ the partition $V_1 \cup V_2$ will do. Thus we may assume that $\alpha \ge \frac{k-1}{4}$.

If there is a vertex $v \in V_1$ with $|\Gamma(v) \cap V_1| = |\Gamma(v) \cap V_2|$ then $d(v) \le k-1$ (since k is odd): moving v from V_1 to V_2 gives a partition $V_1' \cup V_2'$ with $e(V_1', V_2') = e(V_1, V_2)$ and $e(V_1') < e(V_1)$, while

$$e(V_2') \le e(V_2) + \frac{k-1}{2} \le \frac{k-1}{4k}m - \alpha + \frac{k-1}{2}. \tag{11}$$

By minimality of $\max\{e(V_1), e(V_2)\}$, and since $e(V_1') < e(V_1)$, we must have $e(V_2') \ge e(V_1)$. Since $\alpha \ge (k-1)/4$ it follows from (9) and (11) that $\alpha = (k-1)/4$, and we have equality in (8), (9) and (10), so there is a partition satisfying (6) and (7).

Otherwise, no $v \in V_1$ has $|\Gamma(v) \cap V_1| = |\Gamma(v) \cap V_2|$: we shall show that then G has a partition satisfying (6) and (7) strictly. Let $W_1 \subset V_1$ be minimal such that $e(W_1) \ge e(V \setminus W_1)$ and $e(W_1) \ge \frac{k-1}{4k}m + \frac{k-1}{4}$. Let $W_2 = V \setminus W_1$; note that every $v \in W_1$ satisfies $|\Gamma(v) \cap W_2| > |\Gamma(v) \cap W_1|$. Let

$$R = \min_{v \in W_1} \frac{|\Gamma(v) \cap W_2|}{|\Gamma(v) \cap W_1|} \ge \frac{k+1}{k-1}.$$

We write

$$e(W_1) = \frac{k-1}{4k}m + \frac{k-1}{4} + \beta, \tag{12}$$

where $\beta \ge 0$. Since $|\Gamma(v) \cap W_2| \ge R|\Gamma(v) \cap W_1|$ for $v \in W_1$, summing over $v \in W_1$ yields

$$e(W_1, W_2) \ge 2Re(W_1) = R\frac{k-1}{2k}m + R\frac{k-1}{2} + 2\beta R, \tag{13}$$

and so, by (12) and (13),

$$e(W_2) \le m - \big(\frac{k-1}{4k} - R\frac{k-1}{2k}\big)m - (2R+1)\big(\frac{k-1}{4} + \beta\big). \tag{14}$$

Since $R \geq (k+1)/(k-1)$, (13) and (14) imply that

$$e(W_1, W_2) \geq \frac{k+1}{2k}m + R\frac{k-1}{2} \tag{15}$$

and

$$e(W_2) \leq \frac{k-1}{k}\frac{m}{4} - (2R+1)\frac{k-1}{4}. \tag{16}$$

Let $v \in W_1$ be a vertex with $|\Gamma(v) \cap W_2| = R|\Gamma(v) \cap W_1|$. Then let $X_1 = W_1 \backslash \{v\}$, $X_2 = W_2 \cup \{v\}$. Clearly $e(X_1) < e(W_1)$: we claim $e(X_1, X_2) > \frac{k+1}{2k}m$ and $e(X_2) < \frac{k-1}{4k}m + \frac{k-1}{4}$; by minimality of W_1, $e(X_1) < \frac{k-1}{4k}m + \frac{k-1}{4}$. To prove the bound on $e(X_2)$, it is enough by (16) to show

$$\frac{R}{1+R}d(v) < (2R+2)\frac{k-1}{4},$$

since $e(X_2) = e(W_2) + |\Gamma(v) \cap W_2| = e(W_2) + Rd(v)/(1+R)$. Since $d(v) \leq k$, this follows if $2Rk < (k-1)(R+1)^2$, which holds since $R > 1$ and $k \geq 3$. From (15) the bound on $e(X_1, X_2)$ holds if

$$\frac{R-1}{R+1}k < R\frac{k-1}{2}$$

since $e(X_1, X_2) = e(W_1, W_2) - |\Gamma(v) \cap W_2| + |\Gamma(v) \cap W_1|$. Rearranging, we see that this is equivalent to $2/(k-1) < R + 2R/(R-1)$, which holds since $R > 1$.

We have shown that G has a partition satisfying (6) and (7). If G has no partition satisfying (6) with strict inequality then we must have $\alpha = (k-1)/4$ and equality in (9) and (10). So

$$\begin{aligned} e(V_1) &= \frac{k-1}{4k}m + \frac{k-1}{4} \\ e(V_2) &= \frac{k-1}{4k}m - \frac{k-1}{4} \end{aligned}$$

and so

$$e(V_1, V_2) = \frac{k+1}{2k}m.$$

It follows that no cut of G has size more than $\frac{k+1}{2k}m$; in particular, every partition $V_A \cup V_B$ in (8) has size $\frac{k+1}{2k}m$, and this must hold regardless of the k-colouring. It follows that no vertex v of G has degree less than $k-1$, or else there is a cut in (8) that would change size if we recoloured v. Similarly, no vertex v has degree k: otherwise, v must have neighbours in every vertex class (or else we could change the size of some cut): letting $B \subseteq [k]$ consist of $(k-1)/2$ colours in which v has exactly one neighbour, and $A = [k] \setminus B$, the cut $V_A \cup V_B$ must have size $\frac{k+1}{2k}m$, while the cut $V_A \setminus \{v\}, V_B \cup \{v\}$ is strictly larger. We deduce that G is $(k-1)$-regular. Finally, if any component of G is not isomorphic to K_k then it can be $(k-1)$-coloured, which will yield a larger cut. □

We remark that for k even the theorem above (for $k+1$) immediately gives an optimal bound; the extremal graphs are of the form $(2t+1)K_{k+1}$ for $t \geq 0$.

For k-regular graphs we can get a stronger result.

Theorem 2. *Let $k \geq 3$ be an odd integer. Then every k-regular graph G has a bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2|$ and*

$$\max\{e(V_1), e(V_2)\} \leq \frac{k-1}{4k}m. \tag{17}$$

The extremal graphs are sK_{k+1} for $s \geq 1$.

Proof. Consider a partition with $|V_1| = |V_2|$; suppose that $e(V_1, V_2)$ is maximal among such partitions. Note that since G is regular we have $e(V_1) = e(V_2)$. Thus we need only find a partition in which one vertex class satisfies (17). Now if $|\Gamma(v) \cap V_2| > |\Gamma(v) \cap V_1|$ for all $v \in V_1$ then

$$e(V_1) \leq \frac{1}{2}|V_1|\frac{k-1}{2} = \frac{k-1}{8}|G| = \frac{k-1}{4k}m, \tag{18}$$

and similarly for $e(V_2)$ if $|\Gamma(v) \cap V_1| > |\Gamma(v) \cap V_2|$ for all $v \in V_2$. Otherwise, we can find $v \in V_1$ with $|\Gamma(v) \cap V_1| > |\Gamma(v) \cap V_2|$ and $w \in V_2$ with $|\Gamma(w) \cap V_2| > |\Gamma(w) \cap V_1|$: exchanging v and w gives a cut with larger size.

Now if (18) holds with equality then we may assume $|\Gamma(v) \cap V_2| > |\Gamma(v) \cap V_1|$ for all $v \in V_1$, and in particular that every vertex in V_1 has $|\Gamma(v) \cap V_2| = (k+1)/2$. If there is $v \in V_2$ with $|\Gamma(v) \cap V_1| < |\Gamma(v) \cap V_2|$ then exchanging v with any vertex adjacent to v in V_1 gives a larger cut: we deduce that $|\Gamma(v) \cap V_1| > |\Gamma(v) \cap V_2|$ for all $v \in V_2$, and thus $|\Gamma(v) \cap V_1| = (k+1)/2$ for all $v \in V_2$. Thus every vertex in V_1 and V_2 has exactly $(k+1)/2$ neighbours in the opposite class, and furthermore this holds for every bipartition into sets of equal size satisfying (18). If $v \in V_1$ and $w \in V_2$ are adjacent then exchanging v and w must leave every vertex with $(k+1)/2$ neighbours in the opposite class, and so we must have $\Gamma(v) \cup \{v\} = \Gamma(w) \cup \{w\}$. It is then easy to check that every component of G must be isomorphic to K_{k+1}. □

A similar result to Theorem 2 holds when k is even.

Theorem 3. *Let $k \geq 2$ be an even integer. Then every k-regular graph G with even order has a bipartition $V(G) = V_1 \cup V_2$ with $|V_1| = |V_2|$ and*

$$\max\{e(V_1), e(V_2)\} \leq \frac{1}{4}\frac{k}{k+1}m.$$

The extremal graphs are of the form $2tK_{k+1}$, $t \geq 1$.

Every k-regular graph of odd order has a bipartition with $|V_1| = |V_2| - 1$ and

$$\max\{e(V_1), e(V_2)\} \leq \frac{1}{4}\frac{k}{k+1}m + \frac{k}{4}.$$

The extremal graphs are of the form $(2t+1)K_{k+1}$, $t \geq 0$.

Proof. If $|G|$ is even then let $V(G) = V_1 \cup V_2$ be a partition with $|V_1| = |V_2|$, and $e(V_1, V_2)$ maximal among such partitions. If there is $v \in V_1$ with $|\Gamma(v) \cap V_2| > |\Gamma(v) \cap V_1|$ and $w \in V_2$ with $|\Gamma(w) \cap V_1| \geq |\Gamma(w) \cap V_2|$ then exchanging v and w gives a bigger cut; similarly, if there is $v \in V_1$ with $|\Gamma(v) \cap V_1| \geq |\Gamma(w) \cap V_2|$ and $w \in V_2$ with $|\Gamma(w) \cap V_1| > |\Gamma(w) \cap V_2|$ we can exchange v and w. Also, if there are adjacent vertices $v \in V_1$ and $w \in V_2$ with $|\Gamma(v) \cap V_1| = |\Gamma(v) \cap V_2|$ and $|\Gamma(w) \cap V_1| = |\Gamma(w) \cap V_2|$ then exchanging v and w increases the size of the cut.

For a maximal cut, if $|\Gamma(v) \cap V_2| > |\Gamma(v) \cap V_1|$ for all $v \in V_1$, then summing degrees in V_1 gives

$$e(V_1) \leq \frac{1}{2}|V_1|\frac{k-2}{2} = \frac{k-2}{4k}m < \frac{1}{4}\frac{k}{k+1}m,$$

and similarly for $e(V_2)$ if $|\Gamma(v) \cap V_1| > |\Gamma(v) \cap V_2|$ for all $v \in V_2$. Since $e(V_1) = e(V_2)$, this proves the bound in these cases.

Otherwise, let $S_i = \{v \in V_i : |\Gamma(v) \cap V_2| = |\Gamma(v) \cap V_1|\}$, for $i = 1, 2$. Note that there are no edges between S_1 and S_2, or we can obtain a larger cut by exchanging adjacent vertices in S_1 and S_2. Let $T_i = V_i \setminus S_i$ for $i = 1, 2$. Now,

$$e(S_1, V_2) = \sum_{v \in S_1} |\Gamma(v) \cap V_2| = \sum_{v \in S_1} |\Gamma(v) \cap V_1|$$

and

$$e(S_2, V_1) = \sum_{v \in S_2} |\Gamma(v) \cap V_1| = \sum_{v \in S_2} |\Gamma(v) \cap V_2|.$$

Since $|\Gamma(v) \cap V_2| \geq (k+2)/2$ and $|\Gamma(v) \cap V_1| \leq (k-2)/2$ for $v \in T_1$, we have

$$\sum_{v \in T_1} |\Gamma(v) \cap V_1| \leq \frac{k-2}{k+2} \sum_{v \in T_1} |\Gamma(v) \cap V_2| = \frac{k-2}{k+2} e(T_1, V_2).$$

Now

$$\begin{aligned} 2e(V_1) &= \sum_{v \in T_1} |\Gamma(v) \cap V_1| + \sum_{v \in S_1} |\Gamma(v) \cap V_1| && (19) \\ &\leq \frac{k-2}{k+2} e(T_1, V_2) + e(S_1, V_2) \\ &= \frac{k-2}{k+2} e(V_1, V_2) + \frac{4}{k+2} e(S_1, V_2) && (20) \end{aligned}$$

and similarly,

$$2e(V_2) \leq \frac{k-2}{k+2} e(V_1, V_2) + \frac{4}{k+2} e(S_2, V_1). \qquad (21)$$

Since $e(S_1, S_2) = 0$, we have $e(S_1, V_2) + e(S_2, V_1) \leq e(V_1, V_2)$, and so (20) and (21) imply

$$\begin{aligned} e(V_1) + e(V_2) &\leq \frac{k-2}{k+2} e(V_1, V_2) + \frac{2}{k+2} (e(S_1, V_2) + e(S_2, V_1)) \\ &\leq \frac{k}{k+2} e(V_1, V_2) \\ &= \frac{k}{k+2} e(G) - \frac{k}{k+2} (e(V_1) + e(V_2)). \end{aligned}$$

Thus

$$e(V_1) + e(V_2) \leq \frac{1}{2} \frac{k}{k+1} e(G) \qquad (22)$$

and since $e(V_1) = e(V_2)$, (22) implies that, for $i = 1, 2$,

$$e(V_i) \leq \frac{1}{4} \frac{k}{k+1} e(G). \qquad (23)$$

Now if (23) holds with equality then we have equality in (20) and (21), so $|\Gamma(v) \cap V_2| = (k+2)/2$ for $v \in T_1$ and $|\Gamma(v) \cap V_1| = (k+2)/2$ for $v \in T_2$.

We also have equality in (22), so $e(S_1,V_2)+e(S_2,V_1)=e(V_1,V_2)$, and hence $e(T_1,T_2)=0$. Furthermore, this must hold for every partition with $|V_1|=|V_2|$ that satisfies (23). Now if a vertex $v\in S_1$ is adjacent to $w\in T_1$ then pick a vertex $x\in\Gamma(v)\cap V_2$, so $x\in T_2$. Exchanging v and x does not change the size of the cut, so x must also be adjacent to w (or else exchanging v and x would leave w with more than $(k+2)/2$ neighbours on the other side of the partition). But $x\in T_2$, so x has no neighbours in T_1, hence we must have $e(S_1,T_1)=0$. In particular, we see that every component of $G[V_1]$ and $G[V_2]$ must be a regular graph. Finally, if $v\in V_1$ is adjacent to $w\in V_2$ then either $v\in S_1$ and $w\in T_2$ or $v\in T_1$ and $w\in S_2$, so exchanging v and w does not change the size of the cut. Without loss of generality, we may assume $v\in S_1$ and $w\in T_2$, and let $V_1'\cup V_2'$ be the resulting bipartition: then every component of $G[V_1']$ and $G[V_2']$ is regular. Since $|\Gamma(w)\cap V_1'|=k/2$ and $|\Gamma(v)\cap V_2'|=(k-2)/2$ (and $\Gamma(w)\cap V_1'\subset S_1$ and $\Gamma(v)\cap V_2'\subset T_2$), it follows that $\Gamma(v)\cup\{v\}=\Gamma(w)\cup\{w\}$. It follows that all components of G are copies of K_{k+1}.

If $|G|$ is odd, say $|G|=2l+1$, then consider partitions $V_1\cup V_2$ with $|V_1|=l$ and $|V_2|=l+1$. Note that $e(G)=(2l+1)k/2$ and $e(V_2)=e(V_1)+k/2$. Applying the same argument as above, we get three alternatives. If $|\Gamma(v)\cap V_2|>|\Gamma(v)\cap V_1|$ for all $v\in V_1$, we obtain

$$e(V_1)\le\frac{1}{2}l\frac{k-2}{2}=\frac{k-2}{4k}m-\frac{k-2}{8}$$

and so, since $m\le\binom{k+1}{2}$,

$$e(V_2)=e(V_1)+\frac{k}{2}\le\frac{k-2}{4k}m+\frac{3k+2}{8}\le\frac{1}{4}\frac{k}{k+1}m+\frac{k}{4},$$

with equality only if $m=\binom{k+1}{2}$ and hence $G\equiv K_{k+1}$.

Alternatively, $|\Gamma(v)\cap V_1|>|\Gamma(v)\cap V_2|$ for all $v\in V_2$, and so

$$e(V_2)\le\frac{1}{2}(l+1)\frac{k-2}{2}=\frac{k-2}{4k}m+\frac{k-2}{8}<\frac{1}{4}\frac{k}{k+1}m+\frac{k}{4}.$$

Otherwise, define S_i and T_i as in the even case; the argument runs in the same way as in the even case, except $e(V_2)=e(V_1)+k/2=\frac{1}{2}(e(V_1)+e(V_2))+k/4$, so (23) becomes

$$e(V_i)\le\frac{1}{4}\frac{k}{k+1}m+\frac{k}{4}.$$

The argument for extremal graphs is identical. □

We remark that, in the case $k = 3$, a stronger result than Theorem 2 follows from a result of Locke [14], who showed that every cubic K_4-free graph G has a partition $V(G) = V_1 \cup V_2$ with $|V_1| = |V_2|$ and $e(V_1, V_2) \geq 11e(G)/15$; since $e(V_1) = e(V_2)$ for a partition into classes of equal size, we have $\max\{e(V_1), e(V_2)\} \leq 2e(G)/15$. It would be interesting to determine the optimal constants for k-regular graphs containing no K_{k+1}. For graphs with large girth it should be possible to get even stronger results (see [9], [12], [14], [19], [8]).

We remark that a random k-regular graph, or a random graph in $\mathcal{G}(n, p)$ with $p = O(1/n)$, contains (with high probability) only a few short cycles. What are the best constants we can get in the theorems above for random graphs? We shall consider this question elsewhere.

3 Judicious partitions of hypergraphs

In section 2 we showed that every graph G with maximal degree bounded by a constant has a partition $V(G) = V_1 \cup V_2$ in which $\max\{e(V_1), e(V_2)\}$ is not very large. However, this does not give us any information about $\min\{e(V_1), e(V_2)\}$. In a random bipartition, we have $\mathbb{E}\, e(V_1) = \mathbb{E}\, e(V_2) = m/4$ and $\mathbb{E}\, e(V_1, V_2) = m/2$: in this section we show that, for bounded-degree graphs, we can get quite close to this. More generally, we prove a result for imbalanced partitions into $k \geq 2$ sets.

Theorem 4. *For every integer $D \geq 1$ there is a constant K such that for every graph G with maximum degree at most D and every sequence of nonnegative real numbers $p_1, \ldots, p_k$ with $\sum_{i=1}^k p_i = 1$ there is a partition $V(G) = \bigcup_{i=1}^k V_i$ with, for $1 \leq i \leq k$,*

$$|e(V_i) - p_i^2 e(G)| \leq K$$

and, for $1 \leq i < j \leq k$,

$$|e(V_i, V_j) - 2p_i p_j e(G)| \leq K.$$

Furthermore, we may also demand $|\,|V_i| - p_i|G|\,| < K$, for $1 \leq i \leq k$.

Theorem 4 is a special case of a rather general result about partitioning oriented hypergraphs.

An *oriented hypergraph* H is given by a set V and a collection $E(H)$ of ordered tuples of (distinct) elements of V. For instance, if all tuples have size 2 then we obtain a digraph (note that we allow tuples of different sizes). Given oriented hypergraphs $H_1, \ldots, H_s$ with common vertex set V, sets $V_1, \ldots, V_t \subset V$, and integers $1 \leq k_1, \ldots, k_u \leq t$, we define

$$d^{(i)}_{k_1,\ldots,k_u}(V_1, \ldots, V_t) = \{\langle v_1, \ldots, v_u \rangle \in E(H_i) : v_i \in V_{k_i} \forall i\}.$$

We write $e_t(H_i)$ for the number of edges with t vertices (ie the number of t-tuples) in $E(H)$.

Theorem 5. *For every triple r, s, D of positive integers there is a constant $K = K(r, s, D)$ such that the following assertion holds for every $k \geq 1$. For every sequence of hypergraphs $H_1, \ldots, H_s$ with common vertex set V such that each H_i has maximum edge size at most r and maximum vertex degree at most D, and every sequence of nonnegative reals $p_1, \ldots, p_k$ with $\sum_{i=1}^k p_i = 1$, there is a partition $V = \bigcup_{i=1}^k V_i$ such that*

$$|\,|V_i| - p_i|V|\,| \leq K,$$

for $1 \leq i \leq k$, and, for $1 \leq i \leq k$, $1 \leq t \leq r$, and $1 \leq k_1, \ldots, k_t \leq k$,

$$|d^{(i)}_{k_1,\ldots,k_t}(V_1, \ldots, V_k) - e_t(H_i) \prod_{i=1}^{t} p_i| < K.$$

We shall need the following lemma in the proof of Theorem 5.

Lemma 6. *Let $t, D \geq 1$ be integers. There is a constant $K = K(t, D)$ such that for every finite set S, every sequence $(f_i)_{i=1}^t$ of functions from S to $\{0, \ldots, D\}$ and every positive integer u and nonnegative reals $p_1, \ldots, p_u$ with $\sum_{i=1}^u p_i = 1$, there is a partition $S = \bigcup_{i=1}^u S_i$ such that, for $1 \leq i \leq u$,*

$$|\,|S_i| - p_i|S|\,| < K$$

and, for $1 \leq i \leq u$ and $1 \leq j \leq t$,

$$|\sum_{x \in S_i} f_j(x) - p_i \sum_{x \in S} f_j(x)| < K.$$

Proof. Let $K = (D+1)^{t+1}$. Define an equivalence relation on S by setting $x \sim y$ if $f_i(x) = f_i(y)$ for $i = 1, \ldots, t$. Let the equivalence classes be $R_1, \ldots, R_T$, where $T \leq (D+1)^t$. For $1 \leq l \leq T$, let $R_l = \bigcup_{i=1}^{u} Q_l^{(i)}$ be an arbitrary partition with $|Q_l^{(i)}| = \lfloor p_i |R_j| \rfloor$ or $|Q_l^{(i)}| = \lceil p_i |R_j| \rceil$ for each i. Note that $|\,|Q_l^{(i)}| - p_i |R_l|\,| \leq 1$. Now let $S_i = \bigcup_{l=1}^{T} Q_l^{(i)}$ for each i. Note that $|\,|S_i| - p_i|V|\,| < T$. An easy calculation shows that, for $1 \leq i \leq u$ and $1 \leq j \leq t$,

$$\begin{aligned}
|\sum_{x \in S_i} f_j(x) - p_i \sum_{x \in S} f_j(x)| &\leq \sum_{l=1}^{T} |\sum_{x \in Q_l^{(i)}} f_j(x) - p_i \sum_{x \in R_l} f_j(x)| \\
&= \sum_{l=1}^{T} |y_l|\,|Q_l^{(i)}| - p_i |R_l|\,| \\
&\leq DT \\
&< K,
\end{aligned}$$

wheer y_l is the common value of $f_j(x)$ for $x \in R_l$. □

We can now prove the theorem.

Proof of Theorem 5. We begin by considering the graph H^* with vertex set V and edges all pairs $\{x, y\}$ that are contained in some edge in some H_i. Note that H^* has maximal degree less than rsD, so there is some partition $V = \bigcup_{i=1}^{u} V_i$ with $u \leq rsD$ such that each V_i is independent in H^*.

We shall obtain the desired partition of V by applying the lemma to each V_i in turn. Suppose we wish to partition V_i, and we have partitioned V_j as $\bigcup_{i=1}^{k} V_j^{(i)}$ for each $j < i$. We define an equivalence relation on the edges of $H_1, \ldots, H_s$ by setting edges $\langle v_1, \ldots, v_a \rangle$ and $\langle w_1, \ldots, w_b \rangle$ to be equivalent if (1) they belong to the same oriented hypergraph H_t and (2) $a = b$ and (3) v_j and w_j are in the same vertex class V_l for each j and (4) if $l < i$ and $v_j, w_j \in V_l$ then v_j and w_j are in the same vertex class $V_l^{(h)}$. For each equivalence class X we define a function f_X on V_i by setting $f_X(v)$ to be the number of edges in X that contain v. Finally, we apply the lemma to this collection of functions.

An easy check shows that this yields the desired partition. (Note that for $1 \leq a \leq r$ and (distinct) $k_1, \ldots, k_a \in [u]$, the set of edges $\langle v_1, \ldots, v_a \rangle$ with $v_i \in V_{k_i}$ for $1 \leq i \leq a$ is essentially partitioned one vertex class V_{k_i} at a time.) □

As an application of the theorem, we obtain the following result. For a graph G, a partition $V(G) = \bigcup_{i=1}^{k} V_i$, and integers $1 \leq k_0, \ldots, k_t \leq k$, let $P_{k_0,\ldots,k_t}$ be the number of paths $v_0 \cdots v_t$ with $v_i \in V_{k_i}$ for each i. Let $P_t(G)$ be the number of paths of length t in G.

Theorem 7. *For every $T, D \geq 1$ there is a constant $K = K(T, D)$ such that, for $k \geq 1$, every graph G with maximum degree at most D has a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that, for every sequence $k_0, \ldots, k_t$ with $t \leq T$,*

$$|P_{k_1,\ldots,k_t}(G) - 2P_t(G)/k^t| \leq K.$$

Similar results follow for embeddings of other subgraphs, and for imbalanced partitions.

4 Conclusion

In Theorem 5, we gave a result concerning simultaneous partitions of several hypergraphs with bounded degrees and the same vertex set. There are many related problems: for instance, what is the correct analogue of Theorem 1 for simultaneous bipartitions of more than one graph with the same vertex set and maximum degree at most k? A very natural question is whether bounds similar to (1) and (2) can be proved for simultaneous bipartitions of two graphs.

Problem 8. *Find the largest integer $f^{(2)}(m)$ such that for every pair of graphs G_1, G_2 with m edges and common vertex set V there is a bipartition $V = V_1 \cup V_2$ with*

$$\min\{e_{G_1}(V_1, V_2), e_{G_2}(V_1, V_2)\} \geq f^{(2)}(m).$$

Perhaps it is even possible to find a bipartition that gives a cut of size at least $(1 + o(1))m/2$ in each graph. Note that in a random partition we have $\mathbb{E}e_{G_1}(V_1, V_2) = e(G_1)/2$ and $\mathbb{E}e_{G_2}(V_1, V_2) = e(G_2)/2$. But these two quantities are not independent, so we face a similar problem to judicious partitions, where we seek to maximize more than one quantity simultaneously. There are many possible extensions and related problems. For instance, what about simultaneous bisections, or cuts into more than two vertex classes?

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