Defective Colouring of Hypergraphs

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Abstract

We prove that the vertices of every $(r+1)$-uniform hypergraph with maximum degree Δ may be coloured with $c(\frac{\Delta}{d+1})^{1/r}$ colours such that each vertex is in at most d monochromatic edges. This result, which is best possible up to the value of the constant c, generalises the classical result of Erdős and Lovász who proved the $d = 0$ case.

1 Introduction

Hypergraph colouring is a widely studied field with numerous deep results [4, 8–10, 15–17, 22–24, 28]. In a seminal contribution, Erdős and Lovász [13] proved that every $(r+1)$-uniform hypergraph with maximum degree Δ has a vertex-colouring with at most $c\Delta^{1/r}$ colours and with no monochromatic edge, where c is an absolute constant. The proof is a simple application of what is now called the Lovász local lemma, introduced in the same paper. Indeed, hypergraph colouring was the motivation for the development of the Lovász local lemma, which has become a staple of probabilistic combinatorics.

A vertex-colouring of a (hyper)graph is *d-defective* if each vertex is in at most d monchromatic edges (equivalently, the maximum degree of each monochromatic component is at most d). Defective colouring of graphs has been widely studied; the comprehensive survey [32] has over one hundred references to papers dedicated to defective colouring. One of the early results in the area, due to Lovász [25], is that every graph with maximum degree Δ has a d-defective colouring with $\lfloor \frac{\Delta}{d+1} \rfloor + 1$ colours. An example of one of the more recent highlights is that the defective analogue of Hadwiger's conjecture holds. In particular, Edwards et al. [11] showed that every K_t-minor-free graph has a $d(t)$-defective $(t-1)$-colouring, for some function $d(t)$. Here $t-1$ colours is best possible regardless of d. The best defect bound known [30] is $d(t) = \mathcal{O}(t)$. Very little is known about defective colouring of hypergraphs.

This paper proves the common generalisation of the results of Lovász [25] and Erdős and Lovász [13] mentioned above.

Theorem 1. *For all integers $r \geqslant 1$ and $d \geqslant 0$ and $\Delta \geqslant \max\{d+1, 50^{100r^4}\}$, every $(r+1)$-uniform hypergraph G with maximum degree at most Δ has a d-defective k-colouring, where*

$$k \leqslant 100\left(\frac{\Delta}{d+1}\right)^{1/r}.$$

Several notes on Theorem 1 are in order.

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- The bound on the number of colours in the theorem of Erdős and Lovász [13] and in Theorem 1 is best possible (up to the multiplicative constant) because of complete hypergraphs. Indeed, let G be the $(r+1)$-uniform complete hypergraph on n vertices, which has maximum degree $\Delta = \binom{n-1}{r} \leqslant (\frac{en}{r})^r$. In any d-defective k-colouring of G, at least $\frac{n}{k}$ vertices are monochromatic, implying $d \geqslant \binom{n/k-1}{r} > (\frac{n}{2kr})^r \geqslant \frac{\Delta}{(2ek)^r}$. Thus $k \geqslant \frac{1}{2e}(\frac{\Delta}{d})^{1/r}$, which is within a constant factor of the upper bound in Theorem 1. It remains tight even for $(r+1)$-uniform hypergraphs with no complete $(r+2)$-vertex subhypergraph. For example, a hypergraph construction by Cooper and Mubayi [10, § 3.2.2] has this property[1].
- The correct multiplicative constant is not known even for $d = 0$ and $r = 2$ (that is, even for proper colouring of 3-uniform hypergraphs). In this case, the best upper bound known [31] is $\lceil 2\Delta^{1/2} \rceil$ while the lower bound given by complete 3-uniform hypergraphs is $(1/\sqrt{2} + o(1))\Delta^{1/2}$.
- The assumption $\Delta \geqslant d+1$ in Theorem 1 is reasonable, since if $\Delta \leqslant d$ then one colour suffices. The assumption that $\Delta \geqslant 50^{100r^4}$ enables the uniform constant 100 in the bound on k. Of course, one could drop the assumption and replace 100 by some constant c_r depending on r.
- If G is a linear hypergraph (that is, any two edges intersect in at most one vertex), then Theorem 1 may be proved directly with the Lovász local lemma. Non-linear hypergraphs are hard because the number of neighbours of a vertex v is not precisely determined by the degree of v. See the start of Section 2 for details.
- Theorem 1 can be rephrased as saying that for any k, G has a k-colouring with maximum monochromatic degree $\mathcal{O}(\frac{\Delta}{k^r})$ for fixed r. This is similar to a result of Bollobás and Scott [6] who showed that for any k every $(r+1)$-uniform hypergraph with m edges has a k-colouring with $\mathcal{O}(\frac{m}{k^r})$ monochromatic edges of each colour. In this light, Theorem 1 is a variant on so-called judicious partitions [1, 5–7, 19, 20, 29, 33, 34].

1.1 Notation

Let G be a hypergraph, which consists of a finite vertex-set $V(G)$ and an edge-set $E(G) \subseteq 2^{V(G)}$. Let $e(G) := |E(G)|$. G is *r-uniform* if every edge has size r. The *link hypergraph* of a vertex v in G, denoted G_v, is the hypergraph with vertex-set $V(G) \setminus \{v\}$ and edge-set $\{e \subseteq V(G) \setminus \{v\} : e \cup \{v\} \in E(G)\}$. If G is $(r+1)$-uniform, then G_v is r-uniform. The *degree* of a set of vertices $S \subseteq V(G)$, denoted $\deg(S)$, is the number of edges in G that contain S. We often omit set parentheses, so $\deg(x)$ and $\deg(u,v)$ denote the number of edges containing x and the number of edges containing both u and v, respectively. Let $\Delta(G) := \max\{\deg(v) : v \in V(G)\}$.

1.2 Probabilistic Tools

We use the following standard probabilistic tools.

Lemma 2 (Lovász local lemma [13])**.** *Let $\mathcal{A}$ be a set of events in a probability space such that each event in $\mathcal{A}$ occurs with probability at most p and for each event $A \in \mathcal{A}$ there is a collection $\mathcal{A}'$ of at most*

[1] Let $\boldsymbol{e_i}$ denote the r-dimensional vector with 1 in the i^{th} coordinate and 0 elsewhere. Let G be the $(r+1)$-uniform hypergraph with vertex set $\{1,\dots,n\}^r$ and whose edges are $\{\boldsymbol{v}, \boldsymbol{v_1}, \dots, \boldsymbol{v_r}\}$ where, for each i, $\boldsymbol{v_i} - \boldsymbol{v}$ is a positive multiple of $\boldsymbol{e_i}$. Any $r+2$ vertices induce at most two edges, so G has contains no $(r+2)$-clique. G has maximum degree $\Delta = (n-1)^r < n^r$. Suppose that $V(G)$ is coloured with $k \leqslant (\Delta/(d+1))^{1/r}/r < n/(r(d+1)^{1/r})$ colours. Then there is a monochromatic set $S \subseteq V(G)$ of size at least $(d+1)^{1/r} r n^{r-1}$. Apply the following iterative deletion procedure to S: if, for some coordinate j and integers $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_r \in \{1,\dots,n\}$, there are less than $(d+1)^{1/r}$ vertices in S whose i^{th} coordinate is a_i for all $i \neq j$, then delete all these vertices. Let S' be the set remaining after applying all such deletions. Each step deletes less than $(d+1)^{1/r}$ vertices and at most rn^{r-1} steps occur so S' is non-empty. Let $\boldsymbol{v} \in S'$ have the smallest coordinate sum. By definition of S', for each i, there are at least $(d+1)^{1/r}$ vertices $\boldsymbol{v_i} \in S'$ with $\boldsymbol{v_i} - \boldsymbol{v}$ being a positive multiple of $\boldsymbol{e_i}$. Hence, $\boldsymbol{v}$ has degree at least $d+1$ in S'. Therefore, every d-defective colouring of G uses more than $(\Delta/(d+1))^{1/r}/r$ colours.

d other events such that A is independent from the collection $(B\colon B \notin \mathcal{A}' \cup \{A\})$. *If* $4pd \leqslant 1$, *then with positive probability no event in* $\mathcal{A}$ *occurs.*

Lemma 3 (Markov's inequality)**.** *If X is a nonnegative random variable and* $a > 0$, *then*

$$\mathbb{P}(X \geqslant a) \leqslant \frac{\mathbb{E}(X)}{a}.$$

Lemma 4 (Chernoff bound)**.** *Let* $X \sim \mathrm{Bin}(n, p)$. *For any* $\varepsilon \in [0, 1]$,

$$\mathbb{P}(X \geqslant (1+\varepsilon)\mathbb{E}(X)) \leqslant \exp(-\varepsilon^2 np/3),$$
$$\mathbb{P}(X \leqslant (1-\varepsilon)\mathbb{E}(X)) \leqslant \exp(-\varepsilon^2 np/2).$$

We will need a version of Chernoff for negatively correlated random variables, for example, see [21, Thm. 1]. Boolean random variables $X_1, \dots, X_n$ are *negatively correlated* if, for all $S \subseteq \{1, \dots, n\}$,

$$\mathbb{P}(X_i = 1 \text{ for all } i \in S) \leqslant \prod_{i \in S} \mathbb{P}(X_i = 1).$$

Lemma 5 (Chernoff for negatively correlated variables)**.** *Suppose* $X_1, \dots, X_n$ *are negatively correlated Boolean random variables with* $\mathbb{P}(X_i = 1) \leqslant p$ *for all i. Then, for any* $t \geqslant 0$,

$$\mathbb{P}\big(\sum_i X_i \geqslant pn + t\big) \leqslant \exp(-2t^2/n).$$

Finally we need McDiarmid's bounded differences inequality [26].

Lemma 6 (McDiarmid's inequality)**.** *Let* $T_1, \dots, T_n$ *be n independent random variables. Let X be a random variable determined by* $T_1, \dots, T_n$, *such that changing the value of* T_j *(while fixing the other* T_i*) changes the value of X by at most* c_j. *Then, for any* $t \geqslant 0$,

$$\mathbb{P}(X \geqslant \mathbb{E}(X) + t) \leqslant \exp\Big(-\frac{2t^2}{\sum_i c_i^2}\Big).$$

2 Proof

For motivation we first consider a naïve application of the Lovász local lemma. Suppose G is a linear $(r+1)$-uniform hypergraph. Colour G with $k := \lfloor 100(\frac{\Delta}{d+1})^{1/r} \rfloor > 99(\frac{\Delta}{d+1})^{1/r}$ colours uniformly at random. For each set F of $d+1$ edges all containing a common vertex, let B_F be the event that the vertex set of F is monochromatic. Then, since G is linear, $p := \mathbb{P}(B_F) = k^{-r(d+1)}$. For a fixed F, the number of F' sharing a vertex with F is at most $D := (r(d+1)+1)\Delta(r+1)\binom{\Delta}{d}$; here we have specified the vertex shared with F, the edge containing that vertex, the common vertex of the edges in F', and the remaining d edges of F'. Now $D \leqslant 3r^2 d\Delta(e\Delta/d)^d$ and so

$$\begin{aligned} 4pD &< 4 \cdot 99^{-r(d+1)}\big(\tfrac{d+1}{\Delta}\big)^{d+1} \cdot 3r^2 e^d d^{-d+1} \Delta^{d+1} \\ &= 12r^2 e^d \cdot 99^{-r(d+1)} \cdot d(d+1)\big(\tfrac{d+1}{d}\big)^d \\ &\leqslant 24r^2 d^2 e^{d+1} \cdot 99^{-r(d+1)} \leqslant 1. \end{aligned}$$

Hence, by the Lovász local lemma, there is a colouring in which no B_F occurs; that is, there is a d-defective k-colouring of G. It was crucial in this argument that G was linear so that the powers of Δ in D and p cancelled out exactly. For non-linear G, the number of neighbours of a vertex v is not determined by the degree of v and so p may be larger without a corresponding decrease in D. A more involved argument is required.

2.1 First Steps

Here we outline our colouring strategy before diving into the details. We are given an $(r+1)$-uniform hypergraph G with maximum degree Δ and wish to colour its vertices so that every vertex is in at most d monochromatic edges. For a fixed colouring ϕ, the *monochromatic degree* of a vertex v, denoted $\deg_\phi(v)$, is the number of monochromatic edges containing v (which must have colour $\phi(v)$).

First we colour the vertices of G uniformly at random with k colours where $k = \lfloor 49(\frac{\Delta}{d+1})^{1/r} \rfloor$. Since $\Delta \geqslant d+1$, we have $k > 48(\frac{\Delta}{d+1})^{1/r}$. Say a vertex is *bad* if its monochromatic degree is greater than d and *good* otherwise. We are aiming for a colouring in which every vertex is good. The expected monochromatic degree of a vertex v in such a colouring is $k^{-r}\deg(v) \leqslant k^{-r}\Delta < 48^{-r}(d+1)$. In particular, each individual vertex has small (certainly, by Markov's inequality, less than 48^{-r}) probability of being bad. However, the goodness of a vertex v depends on the colours assigned to vertices in the neighbourhood of v and so 48^{-r} is not a sufficiently small probability to conclude (by, say, the Lovász local lemma) that there is a particular colouring for which all vertices are good.

Instead of colouring all of G with a single random colouring, we do so over many rounds. After a round (where we coloured a hypergraph G), any good vertices will keep their colours and be discarded (they have been coloured appropriately). Let G' be the subhypergraph of G induced by the bad vertices. In the next round we uniformly and randomly colour the vertices of G' with a new palette of colours completely disjoint from those used in previous rounds. Using new colours ensures that monochromatic edges can only be produced within individual rounds. If the palettes all have the same size and the process runs for too many rounds, then we will end up using too many colours. However, if $\Delta(G') \leqslant 2^{-r}\Delta(G)$, then we can use half the number of colours in the next round and so use $\mathcal{O}((\frac{\Delta}{d+1})^{1/r})$ colours across all the rounds. Thus, our aim is to prove the following nibble-style lemma from which Theorem 1 easily follows.

Lemma 7. *Fix non-negative integers r, Δ, d with $r \geqslant 1$ and $\Delta \geqslant \max\{d+1, 50^{50r^3}\}$. Then every $(r+1)$-uniform hypergraph G with maximum degree at most Δ has a partial colouring with at most $49(\frac{\Delta}{d+1})^{1/r}$ colours such that every coloured vertex has monochromatic degree at most d and the subhypergraph G' of G induced by the uncoloured vertices satisfies $\Delta(G') \leqslant 2^{-r}\Delta$.*

Proof of Theorem 1 assuming Lemma 7. We start with a $(r+1)$-uniform hypergraph G with maximum degree at most $\Delta = \Delta_0$ for some $\Delta_0 \geqslant \max\{d+1, 50^{100r^4}\}$. Apply Lemma 7 to get a partial colouring of G where:

- every vertex has monochromatic degree at most d,
- at most $49(\frac{\Delta_0}{d+1})^{1/r}$ colours are used, and
- the subhypergraph G_1 of G induced by uncoloured vertices has $\Delta(G_1) \leqslant \Delta_1 = 2^{-r}\Delta_0$.

Iterate this procedure (using a palette of new colours each round) to obtain, for $i = 0, 1, \ldots$, an induced subhypergraph G_i of G with $\Delta(G_i) \leqslant \Delta_i = 2^{-ri}\Delta$ such that $G[V(G) - V(G_i)]$ has been coloured with at most

$$49\left(\frac{\Delta_0}{d+1}\right)^{1/r} + 49\left(\frac{\Delta_1}{d+1}\right)^{1/r} + \cdots + 49\left(\frac{\Delta_{i-1}}{d+1}\right)^{1/r} = 49\left(\frac{\Delta}{d+1}\right)^{1/r}(1 + 2^{-1} + \cdots + 2^{-(i-1)}) \leqslant 98\left(\frac{\Delta}{d+1}\right)^{1/r}$$

colours and every monochromatic degree is at most d. Continue carrying out rounds of colouring until $\Delta_i < d+1$ or $\Delta_i < 50^{50r^3}$.

First suppose that $\Delta_i < d+1$ and so $\Delta(G_i) \leqslant d$. Use a single new colour on the entirety of G_i to give a d-defective colouring of G. Now suppose that $d+1 \leqslant \Delta_i < 50^{50r^3}$. Properly colour G_i

with $\Delta(G_i)+1 \leqslant 50^{50r^3}$ colours. This gives a d-defective colouring of G with at most

$$98\left(\tfrac{\Delta}{d+1}\right)^{1/r} + 50^{50r^3} \leqslant 100\left(\tfrac{\Delta}{d+1}\right)^{1/r}$$

colours. The final inequality uses the fact that $\Delta \geqslant 50^{100r^4}$ and $d+1 < 50^{50r^3}$. □

Recall that a vertex is bad for a colouring ϕ if it has monochromatic degree at least $d+1$. Say that an edge e is *bad* for a colouring ϕ if every vertex in e is bad (note that a bad edge is not necessarily monochromatic). Furthermore, say that a vertex is *terrible* for a colouring ϕ if it is incident to more than $2^{-r}\Delta$ bad edges. Lemma 7 says that there is some colouring for which no vertex is terrible. The key to the proof of Lemma 7 is to show that a vertex is terrible with low probability.

In the remainder of the paper, we use the definitions of good, bad, and terrible given above and also set $k := \lfloor 49(\tfrac{\Delta}{d+1})^{1/r} \rfloor$.

Lemma 8. *Let $\Delta \geqslant \max\{d+1, 50^{50r^3}\}$. Let G be an $(r+1)$-uniform hypergraph with maximum degree at most Δ. In a uniformly random k-colouring of $V(G)$, each vertex v of G is terrible with probability at most Δ^{-5}.*

Proof of Lemma 7 assuming Lemma 8. Randomly and independently assign each vertex of G one of k colours. For each vertex v, let A_v be the event that v is terrible. By Lemma 8, $\mathbb{P}(A_v) \leqslant \Delta^{-5}$. The event A_v depends solely on the colours assigned to vertices in the closed second neighbourhood of v. Thus if two vertices v and w are at distance at least 5 in G, then A_v and A_w are independent. Thus each event A_v is mutually independent of all but at most $2(r\Delta)^4$ other events A_w. Since $4\Delta^{-5} \cdot 2(r\Delta)^4 = 8r^4/\Delta \leqslant 1$, by the Lovász local lemma, with positive probability, no event A_v occurs. Thus, there exists a k-colouring ϕ of G such that no vertex is terrible. Let G' be the subgraph of G induced by the bad vertices. Since no vertex is terrible, $\Delta(G') \leqslant 2^{-r}\Delta$. Uncolour all the bad vertices: every coloured vertex is good and so has monochromatic degree at most d. □

It remains to prove Lemma 8, which we do in Section 2.4. We have now reduced the question to a local property of a random k-colouring.

A vertex v is terrible if it is bad and at least $2^{-r}\Delta$ edges in its link graph, G_v, are bad. Analysing the dependence between the badness of different edges in G_v is difficult. We sidestep this issue by using a sunflower decomposition. A *sunflower with p petals* is a collection $A_1, \dots, A_p$ of sets for which $A_1 \setminus K, \dots, A_p \setminus K$ are pairwise disjoint where $K := A_1 \cap \dots \cap A_p$ (that is, $A_i \cap A_j = K$ for all distinct i, j). K is the *kernel* of the sunflower and $A_1 \setminus K, \dots, A_p \setminus K$ are its *petals*.

If $A_1, \dots, A_p$ are distinct edges of a uniform hypergraph that form a sunflower, then the petals are pairwise disjoint, non-empty and have the same size. The kernel may be empty in which case the sunflower is a matching of size p. In a random colouring, the colourings on different petals of a sunflower are independent. Hence, it will be useful to partition the edges of hypergraphs into sunflowers with many petals together with a few edges left over.

Lemma 9 (Sunflower decomposition)**.** *Let H be an r-uniform hypergraph and a be a positive integer. There are edge-disjoint subhypergraphs $H_1, \dots, H_s$ of H such that:*

- *Each H_i is a sunflower with exactly a petals.*
- *$H' = H - (E(H_1) \cup \dots \cup E(H_s))$ has fewer than $(ra)^r$ edges.*

Proof. Let $H_1, \ldots, H_s$ be a maximal collection of edge-disjoint subhypergraphs of H where each H_i is a sunflower with exactly a petals. So H' contains no sunflower with a petals. By the Erdős-Rado sunflower lemma [12], $e(H') \leqslant r!(a-1)^r < (ra)^r$ (see [2, 3, 14, 27] for recent improved bounds in the sunflower lemma). □

The proof of Lemma 8 uses a sunflower decomposition to show that if a vertex is terrible, then some reasonably large set of vertices S must have at least 3^{-r} proportion of its vertices being bad. As noted above, each vertex is bad with probability at most 48^{-r} and so we expect at most $48^{-r}|S|$ bad vertices in S. We are able to show that the number of bad vertices in (a suitable) S is not much more than the expected number with very small failure probability. This is accomplished in Lemmas 11 and 13 below, which correspond respectively to the case of large and small k.

2.2 When k is large: $k \geqslant \Delta^{1/(6r^2)}$

Recall that $48(\frac{\Delta}{d+1})^{1/r} < k \leqslant 49(\frac{\Delta}{d+1})^{1/r}$ throughout. When k is large we expect a medium-sized vertex-set S to have close to $|S|$ different colours appearing on it (that is, to be close to rainbow). If two vertices have different colours, then the events that they are bad will be negatively correlated and hence we expect only a small proportion of S to be bad. The negative correlation is made precise in Lemma 10 and the upper tail concentration of the number of bad vertices in S is established in Lemma 11.

Lemma 10. *Let $S = \{v_1, \ldots, v_\ell\}$ be a set of at most k vertices in G and let D be the event that $v_1, \ldots, v_\ell$ are all given different colours. Let X be the number of bad vertices in S. Then, in a uniformly random k-colouring of $V(G)$, for any $t \geqslant 0$,*

$$\mathbb{P}(X \geqslant \ell \cdot 48^{-r} + t \mid D) \leqslant \exp\big(-2t^2/\ell\big).$$

Proof. Let B_j be the event $\{v_j \text{ is bad}\}$ and X_j be the indicator random variable for B_j so $X = \sum_j X_j$. For an edge e containing a vertex v, the probability e is monochromatic is k^{-r}. Hence, the expected number of monochromatic edges containing v is at most $\Delta k^{-r} < 48^{-r}(d+1)$. Thus, $\mathbb{P}(X_j = 1) \leqslant 48^{-r}$ by Markov's inequality (Lemma 3).

Fix distinct colours $c_1, \ldots, c_\ell$ and let V_j be the set of vertices given colour c_j. Conditioned on the event $C_j = \{v_j \text{ is coloured } c_j\}$, B_j is increasing in V_j, while D is non-increasing in V_j. Hence, by the Harris inequality [18], $\mathbb{P}(B_j \cap D \mid C_j) \leqslant \mathbb{P}(B_j \mid C_j)\mathbb{P}(D \mid C_j)$. Using this and the symmetry of the colours gives

$$\mathbb{P}(B_j \mid D) = \mathbb{P}(B_j \mid D \cap C_j) = \frac{\mathbb{P}(B_j \cap D \mid C_j)}{\mathbb{P}(D \mid C_j)} \leqslant \mathbb{P}(B_j \mid C_j) = \mathbb{P}(B_j).$$

But $\mathbb{P}(B_j) \leqslant 48^{-r}$, so $\mathbb{E}(X \mid D) = \sum_j \mathbb{P}(B_j \mid D) \leqslant \ell \cdot 48^{-r}$.

Let C be the event $\{\text{each } v_i \text{ is coloured } c_i\}$. Conditioned on C, B_j is increasing in V_j and non-increasing in all other V_i. We claim the B_i are negatively correlated on the event C. For $\ell = 2$ this is just the Harris inequality. Fix $\ell > 2$ and let S be a set of indices: we need to show $\mathbb{P}(\cap_{i \in S} B_i \mid C) \leqslant \prod_{i \in S} \mathbb{P}(B_i \mid C)$. If $|S| \leqslant 1$, then there is equality. Otherwise let $i_1, i_2 \in S$. Now $B_{i_1} \cap B_{i_2}$ is increasing in $V_{i_1} \cup V_{i_2}$ and non-increasing in all other V_i. By induction,

$$\mathbb{P}(\cap_{i \in S} B_i \mid C) \leqslant \mathbb{P}(B_{i_1} \cap B_{i_2} \mid C) \cdot \prod_{i \in S \setminus \{i_1, i_2\}} \mathbb{P}(B_i \mid C) \leqslant \prod_{i \in S} \mathbb{P}(B_i \mid C).$$

By symmetry of the colours, $\mathbb{P}(B_i \mid C) = \mathbb{P}(B_i \mid D)$ for all i and also $\mathbb{P}(\cap_{i \in S} B_i \mid C) = \mathbb{P}(\cap_{i \in S} B_i \mid D)$ for any set of indices S. In particular, the B_i are negatively correlated on the event D. Applying Lemma 5 to $X_1, \ldots, X_\ell$ gives the result. □

Lemma 11. *Let S be a set of vertices of G with $10^{6r} \leqslant |S| \leqslant k^{1/2}$. In a uniformly random k-colouring of $V(G)$, with failure probability at most $2(e|S|^{-1/2})^{|S|^{1/2}}$, fewer than $3^{-r}|S|$ vertices of S are bad.*

Proof. Let A be the event that the number of distinct colours on S is at most $|S| - |S|^{1/2}$. We first give an upper bound for $\mathbb{P}(A)$. The probability that a fixed vertex does not have a unique colour is at most $|S|/k$. If A does occur, then at least $|S|^{1/2}$ vertices of S do not have a unique colour. Hence,

$$\mathbb{P}(A) \leqslant \binom{|S|}{|S|^{1/2}} \left(\frac{|S|}{k}\right)^{|S|^{1/2}} \leqslant \binom{|S|}{|S|^{1/2}} |S|^{-|S|^{1/2}}.$$

If A does not occur, then there is a subset $S' \subset S$ of size $|S| - |S|^{1/2}$ where the vertices are all given different colours. Fix such an S' and let X be the number of bad vertices in S and X' be the number of bad vertices in S'. Note that if $X' < 4^{-r}|S'|$, then $X < 4^{-r}|S'| + |S|^{1/2} \leqslant 4^{-r}|S| + |S|^{1/2} \leqslant 3^{-r}|S|$.

Let D be the event that all vertices of S' get different colours. By Lemma 10 and the previous paragraph,

$$\begin{aligned}\mathbb{P}(X \geqslant |S| \cdot 3^{-r} \mid D) &\leqslant \mathbb{P}(X' \geqslant |S'| \cdot 4^{-r} \mid D) \\ &\leqslant \mathbb{P}(X' \geqslant |S'| \cdot 48^{-r} + |S'| \cdot 4^{-r}/\sqrt{2} \mid D) \leqslant \exp(-|S'| \cdot 4^{-2r}).\end{aligned}$$

Let $\overline{A}$ be the complement of A. Taking a union bound over all S',

$$\mathbb{P}(\{X \geqslant |S| \cdot 3^{-r}\} \cap \overline{A}) \leqslant \binom{|S|}{|S|^{1/2}} \cdot \exp(-|S'| \cdot 4^{-2r}).$$

Finally,

$$\begin{aligned}\mathbb{P}(X \geqslant |S| \cdot 3^{-r}) &\leqslant \binom{|S|}{|S|^{1/2}} \left(\exp(-|S'| \cdot 4^{-2r}) + |S|^{-|S|^{1/2}}\right) \\ &\leqslant (e|S|^{1/2})^{|S|^{1/2}} \cdot 2|S|^{-|S|^{1/2}} = 2(e|S|^{-1/2})^{|S|^{1/2}}.\end{aligned}$$

□

2.3 When k is small: $k \leqslant \Delta^{1/(6r^2)}$

Recall that $48(\frac{\Delta}{d+1})^{1/r} < k \leqslant 49(\frac{\Delta}{d+1})^{1/r}$ throughout. We need a simple max cut lemma.

Lemma 12 (Max cut). *Let G be a hypergraph whose edges have size at most $r+1$ and let ℓ be a positive integer. There is a partition $V_1 \cup \cdots \cup V_\ell$ of $V(G)$ such that, for every vertex $x \in V_i$, the number of edges containing x and at least one more vertex from V_i is at most $r \deg(x)/\ell$.*

Proof. Throughout the proof, vertices u, v, x are distinct. Choose a partition $V_1 \cup \cdots \cup V_\ell$ of $V(G)$ into ℓ parts that minimises

$$\sum_i \sum_{u,v \in V_i} \deg(u,v). \tag{1}$$

Fix a vertex x and suppose it is in some part V_a. By minimality, for all i,

$$\sum_{u \in V_a} \deg(u,x) \leqslant \sum_{u \in V_i} \deg(u,x),$$

or else we could increase (1) by moving x to V_i. But

$$\sum_i \sum_{u \in V_i} \deg(u,x) = \sum_{u \in V(G)} \deg(u,x) \leqslant r \deg(x),$$

and so $\sum_{u \in V_a} \deg(u,x) \leqslant r \deg(x)/\ell$. This last sum is at least the number of edges containing x and at least one more vertex from V_a. □

Given a large vertex-set S we aim to show that, with high probability, a small proportion of its vertices are bad. We use Lemma 12 to split S into parts so that very few edges have two vertices in the same part. Consider an arbitrary part P. We will show that, with high probability, a small proportion of the vertices in P are bad. We do this by first revealing the random k-colouring on $V(G) - P$. Since k is small, we get strong concentration on the distribution of colours on $V(G) - P$. We then reveal the colouring on P and use this concentration to show that it is unlikely that P has a high proportion of bad vertices.

Lemma 13. *Suppose $\Delta \geqslant 50^{50r^3}$, $k \leqslant \Delta^{1/r^2}$ and let S be a set of at least $(3k)^{3r}\Delta^{1/(6r)}$ vertices of G. With failure probability at most Δ^{-6}, in a uniformly random k-colouring of $V(G)$, fewer than $3^{-r}|S|$ vertices of S are bad.*

Proof. It will be helpful to partition S into multiple parts such that not too many edges meet one part in more than one vertex. We therefore apply the max cut lemma, Lemma 12, to G with $\ell = rk^r$, and restrict the resulting partition to S. We obtain a partition $\mathcal{P}$ of S into rk^r parts such that, for every vertex $x \in S$, the number of edges containing x and at least one more vertex from x's part is at most $\deg(x)/k^r$. We say a part $P \in \mathcal{P}$ is *big* if $|P| \geqslant |S|/(50r(3k)^r)$ and is *small* otherwise.

Since there are rk^r parts in $\mathcal{P}$ and small parts have less than $|S|/(50r(3k)^r)$, the number of vertices of S in small parts is less than $|S|/(50r(3k)^r) \cdot rk^r = 0.02 \cdot 3^{-r}|S|$. Hence, if $3^{-r}|S|$ vertices of S are bad, then at least $0.98 \cdot 3^{-r}$ proportion of the vertices in big parts are bad, so some big part P has at least $0.98 \cdot 3^{-r}|P|$ bad vertices. We now focus on a big part $P \in \mathcal{P}$ and show that, with failure probability at most Δ^{-8}, at most $0.98 \cdot 3^{-r}|P|$ vertices of P are bad.

For each vertex $x \in P$, let G'_x be the r-uniform graph on $V(G) - P$, whose edges are those e with $e \cup \{x\} \in E(G)$ (that is, G'_x is the link graph of x restricted to $V(G) - P$). Define the r-uniform auxiliary (multi)hypergraph H_P to have vertex set $V(G) - P$ and edge set

$$E(H_P) = \bigcup_{x \in P} E(G'_x),$$

where edges are counted with multiplicity. Let ϕ be a uniformly random k-colouring of $V(G)$ and ϕ' be the restriction of ϕ to $V(G) - P$. Reveal ϕ' and let X be the number of monochromatic edges of H_P, again counted with multiplicity.

We now apply McDiarmid's inequality to show that X concentrates. First note that $e(H_P) \leqslant |P| \cdot \Delta$ and $\mathbb{E}(X) = e(H_P)k^{-(r-1)} \leqslant |P| \cdot \Delta k^{-(r-1)}$. For a vertex $v \in V(H_P)$, changing $\phi'(v)$ changes the value of X by at most $\deg_{H_P}(v)$. Now,

$$\sum_v \deg_{H_P}(v)^2 \leqslant \Delta \sum_v \deg_{H_P}(v) = r\Delta e(H_P) \leqslant r\Delta^2|P|.$$

By McDiarmid's inequality (Lemma 6),

$$\begin{aligned}
\mathbb{P}\Big(X \geqslant \frac{1.1 \cdot \Delta|P|}{k^{r-1}}\Big) \leqslant \mathbb{P}\Big(X \geqslant \mathbb{E}(X) + \frac{0.1 \cdot \Delta|P|}{k^{r-1}}\Big) &\leqslant \exp\Big(-\frac{|P|}{50rk^{2(r-1)}}\Big) \\
&\leqslant \exp\Big(-\frac{|S|}{2500r^2 \cdot 3^r \cdot k^{3r-2}}\Big) \\
&\leqslant \exp\big(-k^2\Delta^{1/(6r)}/(2500r^2)\big) \leqslant \Delta^{-8}/2.
\end{aligned}$$

For a vertex $x \in P$, say a colour is *x-unhelpful* if there are more than $(48^r - 1)\Delta/k^r$ monochromatic edges of G'_x of that colour. Say x is *unhelpful* if there are more than $0.45 \cdot 3^{-r}k$ x-unhelpful colours. Note that if x is unhelpful, then the number of monochromatic edges in G'_x is greater than $0.45(48^r - 1) \cdot \Delta \cdot 3^{-r}/k^{r-1}$. Hence, if more than $0.48 \cdot 3^{-r} \cdot |P|$ vertices of P are unhelpful, then the number of monochromatic edges in H_P is greater than $1.1 \cdot \Delta|P|/k^{r-1}$. We have just shown this occurs with probability less than $\Delta^{-8}/2$. Hence, with failure probability at most $\Delta^{-8}/2$, at least $(1 - 0.48 \cdot 3^{-r})|P|$ vertices of P are helpful.

Suppose that at least $(1 - 0.48 \cdot 3^{-r})|P|$ vertices of P are helpful; call the set of helpful vertices P'. Now reveal ϕ on P. For each vertex $x \in P'$, the probability that x gets given an x-unhelpful colour is at most $0.45 \cdot 3^{-r}$. Let Y be the number of $x \in P'$ coloured with an x-unhelpful colour. For different $x \in P'$, these events are independent (we have already revealed ϕ on $V(G) - P$) and so we may couple Y with a random variable $Z \sim \text{Bin}(|P'|, 0.45 \cdot 3^{-r})$ so that $Y \leqslant Z$. Hence, by the Chernoff bound (Lemma 4),

$$\begin{aligned}\mathbb{P}(Y \geqslant 0.5 \cdot 3^{-r}|P'|) \leqslant \mathbb{P}(Z \geqslant 0.5 \cdot 3^{-r}|P'|) &\leqslant \mathbb{P}(Z \geqslant 1.1 \cdot \mathbb{E}(Z)) \\ &\leqslant \exp(-0.45 \cdot 3^{-r}|P'|/300) \\ &\leqslant \exp\big(-k^{2r}\Delta^{1/(6r)}/(6000r)\big) \leqslant \Delta^{-8}/2.\end{aligned}$$

Hence, with failure probability at most $\Delta^{-8}/2 + \Delta^{-8}/2 = \Delta^{-8}$, at least $(1 - 0.5 \cdot 3^{-r})|P'| \geqslant (1 - 0.98 \cdot 3^{-r})|P|$ vertices x of P are coloured with an x-helpful colour.

We now show that if a vertex x is given an x-helpful colour, then x will be a good vertex (for ϕ). There are at most $\deg(x)/k^r \leqslant \Delta/k^r$ edges of G containing x that have at least one more vertex in P and, as x is given an x-helpful colour, there are at most $(48^r - 1)\Delta/k^r$ other monochromatic edges containing x. In particular, if x is given an x-helpful colour, then at most $48^r\Delta/k^r < d + 1$ monochromatic edges contain x and so x is good. Hence, with failure probability at most Δ^{-8}, at least $(1 - 0.98 \cdot 3^{-r})|P|$ vertices of P are good, that is, at most $0.98 \cdot 3^{-r}|P|$ vertices of P are bad.

Finally, taking a union bound over the big parts shows that the probability some big part P has at least $0.98 \cdot 3^{-r}|P|$ bad vertices is at most $rk^r\Delta^{-8} \leqslant r\Delta^{-8+1/r} \leqslant \Delta^{-6}$, as required. □

2.4 Proof of Lemma 8

To prove Lemma 8 we use the sunflower decompositions given by Lemma 9 to show that if a vertex is terrible, then some reasonably large set of vertices S must have at least 3^{-r} proportion of its vertices being bad. Lemmas 11 and 13 show that this is unlikely.

Proof of Lemma 8. Recall that $\Delta \geqslant 50^{50r^3}$. Fix a vertex v of G and consider the link graph G_v, which is an r-uniform hypergraph. Recall that an edge of G_v is *bad* if all its vertices are bad and is *good* otherwise. If v is terrible, then at least $2^{-r}\Delta$ edges of G_v are bad.

First suppose that $k \geqslant \Delta^{1/(6r^2)}$. By Lemma 9, there are edge-disjoint subgraphs $G_1, \dots, G_s$ of G_v each of which is a sunflower with exactly $\lfloor \Delta^{1/(12r^2)} \rfloor$ petals and such that $e(G_v - E(G_1 \cup \dots \cup G_s)) < r^r \cdot \Delta^{1/(12r)} \leqslant 6^{-r}\Delta$. Let $G' = G_1 \cup \dots \cup G_s$. For each G_i, choose a vertex from each petal to form a vertex-set S_i. If v is terrible, then the number of bad edges in G' is at least

$$\big(2^{-r} - 6^{-r}\big)\Delta \geqslant 3^{-r}\Delta \geqslant 3^{-r}e(G').$$

Hence, if v is terrible, then there is some i for which at least $3^{-r}e(G_i)$ edges of G_i are bad. But, since S_i contains exactly one vertex from each petal of G_i, at least $3^{-r}|S_i|$ vertices of S_i are bad. Also, each S_i has size $\lfloor \Delta^{1/(12r^2)} \rfloor \geqslant \Delta^{2/(25r^2)} \geqslant 50^{4r} \geqslant 10^{6r}$ and $\lfloor \Delta^{1/(12r^2)} \rfloor \leqslant k^{1/2}$. Hence, by Lemma 11, at least $3^{-r}|S_i|$ vertices of S_i are bad with probability at most

$$2(e|S|^{-1/2})^{|S|^{1/2}} \leqslant 2(e\Delta^{-1/(25r^2)})^{\Delta^{1/(25r^2)}} \leqslant 2(\Delta^{-1/(50r^2)})^{50^{2r}} \leqslant 2(\Delta^{-1/(50r^2)})^{400r^2} = 2\Delta^{-8}.$$

Taking a union bound over i shows that v is terrible with probability at most $2s\Delta^{-8} \leqslant \Delta^{-5}$.

Now suppose that $k \leqslant \Delta^{1/(6r^2)}$. By Lemma 9, there are edge-disjoint subgraphs $G_1, \dots, G_s$ of G_v each of which is a sunflower with at least $\Delta^{1/r}/(6r)$ petals and such that $e(G_v - E(G_1 \cup \dots \cup G_s)) < 6^{-r}\Delta$. Let $G' = G_1 \cup \dots \cup G_s$. For each G_i, choose a vertex from each petal to form a vertex-set S_i. If v is terrible, then the number of bad edges in G' is at least

$$(2^{-r} - 6^{-r})\Delta \geqslant 3^{-r}\Delta \geqslant 3^{-r}e(G').$$

Hence, if v is terrible, then there is some i for which at least $3^{-r}e(G_i)$ edges of G_i are bad and so at least $3^{-r}|S_i|$ vertices of S_i are bad. Now, $(3k)^{3r}\Delta^{1/(6r)} \leqslant 3^{3r}\Delta^{1/(2r)}\Delta^{1/(6r)} \leqslant \Delta^{1/r}/(6r) \leqslant |S_i|$. Hence, by Lemma 13, at least $3^{-r}|S_i|$ vertices of S_i are bad with probability at most Δ^{-6}. Taking a union bound over i shows that v is terrible with probability at most $s\Delta^{-6} \leqslant \Delta^{-5}$. □

3 Open problems

As noted in the introduction, Erdős and Lovász proved that every $(r+1)$-uniform hypergraph G with maximum degree at most Δ has chromatic number $\chi(G) = \mathcal{O}(\Delta^{1/r})$. Frieze and Mubayi [17] improved this to $\mathcal{O}((\Delta/\log\Delta)^{1/r})$ when G is a linear hypergraph and there have been similar improvements [8, 9, 24] when G satisfies other sparsity conditions (such as being triangle-free[2]).

It would be interesting to know whether logarithmic improvements occur for defective colourings of sparse hypergraphs. Frieze and Mubayi [16] showed that there exist $(r+1)$-uniform linear hypergraphs G with maximum degree Δ and $\chi(G) = \Omega((\Delta/\log\Delta)^{1/r})$. Consider a d-defective k-colouring of G (where $d \geqslant 2$). Each colour class induces a linear $(r+1)$-uniform hypergraph with maximum degree d and so is $\mathcal{O}((d/\log d)^{1/r})$-colourable. In particular,

$$k = \Omega\Big(\big(\tfrac{\Delta}{\log\Delta} \cdot \tfrac{\log d}{d}\big)^{1/r}\Big).$$

We conjecture this is tight.

Conjecture 14. *Every $(r+1)$-uniform linear hypergraph is k-colourable with defect $d \geqslant 2$, where*

$$k = \mathcal{O}\Big(\big(\tfrac{\Delta}{\log\Delta} \cdot \tfrac{\log d}{d}\big)^{1/r}\Big).$$

Finally, it would be interesting to extend Theorem 1 to the list colouring setting.

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