Induced subgraph density. III. Cycles and subdivisions

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July 4, 2022; revised June 18, 2024

[1]Supported by AFOSR grants A9550-19-1-0187 and FA9550-22-1-0234, and by NSF grants DMS-1800053 and DMS-2154169.

[2]Supported by EPSRC grant EP/X013642/1

[3]Supported by AFOSR grants A9550-19-1-0187 and FA9550-22-1-0234, and by NSF grants DMS-1800053 and DMS-2154169.

Abstract

We show that for every two cycles C, D, there exists $c > 0$ such that if G is both C-free and $\overline{D}$-free then G has a clique or stable set of size at least $|G|^c$. ("H-free" means with no induced subgraph isomorphic to H, and $\overline{D}$ denotes the complement graph of D.) Since the five-vertex cycle C_5 is isomorphic to its complement, this extends the earlier result that C_5 satisfies the Erdős-Hajnal conjecture. It also unifies and strengthens several other results.

The results for cycles are special cases of results for subdivisions, as follows. Let H, J be obtained from smaller graphs by subdividing every edge exactly twice. We will prove that there exists $c > 0$ such that if G is both H-free and $\overline{J}$-free then G has a clique or stable set of size at least $|G|^c$. And the same holds if H and/or J is obtained from a graph by choosing a forest F and subdividing every edge not in F at least five times. Our proof uses the framework of iterative sparsification developed in other papers of this series.

Along the way, we will also give a short and simple proof strengthening a celebrated result of Fox and Sudakov, that says that for all H, every H-free graph contains either a large stable set or a large complete bipartite subgraph.

1 Introduction

A graph is H*-free* if it has no induced subgraph isomorphic to H; and if $\mathcal{H}$ is a set of graphs, then a graph G is $\mathcal{H}$*-free* if it is H-free for all $H \in \mathcal{H}$. We say a set $\mathcal{H}$ has the *Erdős-Hajnal property* if there exists $c > 0$ such that for every $\mathcal{H}$-free graph G, there is a clique or stable set of G with cardinality at least $|G|^c$; and a graph H has the property if $\{H\}$ does.

A famous old conjecture of Erdős and Hajnal [10, 11] says that every graph has the Erdős-Hajnal property; or equivalently, every non-null set of graphs has the property. Despite extensive efforts, the conjecture has only been proved for a small family of graphs H. By a theorem of Alon, Pach and Solymosi [1], graphs that are made by vertex-substitution from other graphs with the property also have the property, so we could confine our attention to *prime* graphs, those that cannot be built from smaller graphs by vertex-substitution. But until recently, there were only three prime graphs with more than two vertices that were known to have the property: the four-vertex path, the bull [5] and the five-vertex cycle [6]. (Very recently, more graphs have been added to this list, including the five-vertex path [17] and infinitely many other prime graphs [14]; see also [15, 16, 4] for other recent progress.)

There has been significant progress when more than one graph is excluded (we review this in detail in the next section). For example, it was shown in [6] that $\{H, \overline{H}\}$ has the Erdős-Hajnal property if H is a cycle of length at most seven [6], and extending this to longer cycles was mentioned as a nice open question. Here we will prove a substantially stronger result:

1.1 *Let H and J be cycles. Then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

We will also show that:

1.2 *Let H be obtained from the complete graph K_n by subdividing every edge twice. Then $\{H, \overline{H}\}$ has the Erdős-Hajnal property.*

Both these results are consequence of more general theorems, which we give below. We discuss related work in section 2 and then give our results in section 3.

2 Related work

There has been a substantial body of work proving the Erdős-Hajnal property when two or more graphs are excluded. Some of this has proceeded by proving a stronger property: we say that a set of graphs $\mathcal{H}$ has the *strong Erdős-Hajnal property* if there exists $c > 0$ such that for every $\mathcal{H}$-free graph G with at least two vertices, there are disjoint sets $A, B \subseteq V(G)$ with size at least $c|G|$ such that the pair (A, B) is either complete or anticomplete (i.e. either all possible edges between A and B are present in G, or there are no edges between A and B). It is not hard to show that the strong Erdős-Hajnal property implies the Erdős-Hajnal property.

Which families $\mathcal{H}$ satisfy the strong Erdős-Hajnal property? By considering sparse random graphs, it follows that every finite set $\mathcal{H}$ with the strong Erdős-Hajnal property must contain both a forest and the complement of a forest. It was shown in [7] that every such $\mathcal{H}$ has the strong Erdős-Hajnal property:

2.1 *Let H and J be forests. Then $\{H, \overline{J}\}$ has the strong Erdős-Hajnal property.*

This characterizes finite sets $\mathcal{H}$ with the strong Erdős-Hajnal property, and shows that excluding both a forest and the complement of a forest implies the Erdős-Hajnal property. (By contrast, much less is known if we forbid just a forest H: for example, the Erdős-Hajnal property was only recently proved in the case $H = P_5$ [17], and only the "near Erdős-Hajnal" property is known if H is a longer path [15].)

If $\mathcal{H}$ has the strong Erdős-Hajnal property but does not include both a forest and the complement of a forest, then $\mathcal{H}$ must be infinite. There has been progress here as well. Bonamy, Bousquet and Thomassé [2] showed that excluding all long holes and antiholes gives the strong Erdős-Hajnal property:

2.2 *Let $k \geq 3$, and let $\mathcal{H}$ contain all cycles of length at least k and their complements. Then $\mathcal{H}$ has the strong Erdős-Hajnal property.*

It is interesting to compare this with 1.1. One can show it is necessary to exclude infinitely many cycles and infinitely many complements of cycles to obtain the strong Erdős-Hajnal property. By contrast, 1.1 shows that excluding a *single* cycle and a *single* cycle complement is enough to give the Erdős-Hajnal property.

A strengthening of 2.2 was also known, but first we need some definitions. *Subdividing* an edge uv means deleting the edge, and adding a path between u, v whose internal vertices are new. If the path has length $k + 1$ this is called *k-subdividing* the edge; and if the path has length at least $k + 1$ it is called *$(\geq k)$-subdividing*. A graph obtained from another graph H by subdividing some of the edges of H is called a *subdivision* of H, and it is a *k-subdivision* or *$(\geq k)$-subdivision* if every edge is k-subdivided or every edge is $(\geq k)$-subdivided, respectively.

The following substantial strengthening of 2.2 was proved in [8]:

2.3 *Let H be a graph, and let $\mathcal{H}$ contain all subdivisions of H and their complements. Then $\mathcal{H}$ has the strong Erdős-Hajnal property.*

Once again, we need to exclude infinitely many induced subdivisions of H, and the complements of infinitely many induced subdivisions. By contrast, 1.2 shows that the Erdős-Hajnal property holds if we exclude a *single* subdivision and its complement. In fact, we prove a more general result (3.4), which we discuss in the next section.

3 Results

1.1 and 1.2 are both consequences of the following more general results.

3.1 *Let H, J be 2-subdivisions of multigraphs H_0, J_0 respectively. Then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

3.2 *Let H be obtained from a multigraph H_0 by choosing a forest F of H_0, and (≥ 5)-subdividing every edge of H_0 not in $E(F)$. Let J be constructed similarly. Then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

Let us mention that we can also prove a variant of 3.2 (this will appear in Tung Nguyen's thesis [18]):

3.3 *Let H, J be (≥ 4)-subdivisions of multigraphs H_0, J_0 respectively. Then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

We do not give its proof here, and the proof method is different from those used in this paper.

There is a common generalization of 3.1 and 3.2, which is our objective. Let F be a forest, and let $s, t \geq 1$ be integers. The graph F_t^s is obtained as follows. Let us:

- add a set X of t new vertices to F;
- for all distinct $u, v \in X$, add s parallel edges with ends u, v, and for each $u \in V(F)$ and $v \in X$, add s parallel edges with ends u, v, making a multigraph;
- subdivide exactly twice every edge of this multigraph with an end in X (that is, all its edges except those of F).

If H is a graph such that $H = F_t^s$ for some such F, s, t, we call H a *Swiss Army graph*.

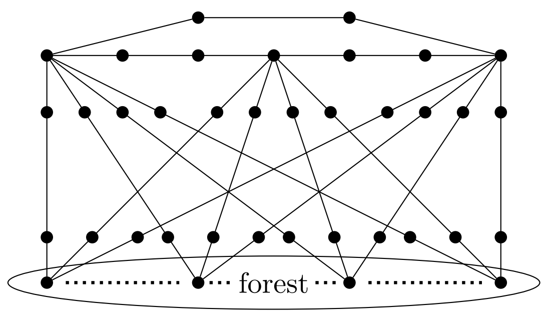

Figure 1: A Swiss Army graph with $s = 1$ and $t = 3$.

We will prove:

3.4 *If H, J are Swiss Army graphs, then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

Swiss Army graphs are cumbersome objects, but they contain several useful and interesting features. For instance:

- Every cycle of length at least six is an induced subgraph of a Swiss Army graph. (If the cycle has length at least seven, make the forest a path of the right length, while if the cycle has length six take $s = 2$.)
- Every 2-subdivision of a multigraph is an induced subgraph of a Swiss Army graph. (Let the forest be the null graph, and s the edge-multiplicity of the multigraph.)
- Let H be obtained from a multigraph H_0 by choosing some forest F of H_0, and (≥ 5)-subdividing every edge of H_0 not in F. Then H is an induced subgraph of a Swiss Army graph. (By adding leaves to F, we may assume every edge of H_0 not in F needs to be subdivided *exactly* five times, and then the result is clear: take t to be the number of edges of H_0 not in F.)

Since we already know the Erdős-Hajnal property for H-free graphs when H is a cycle of length five or less, the first bullet above shows that 3.4 implies 1.1. The second bullet shows that it implies 3.1, and the third that it implies 3.2. So now we need to prove 3.4, and that occupies the remainder of the paper.

James Davies (private communication) pointed out a nice application of our results, for "pivot-minors" (for definitions, see [9]). For every graph J, there is a graph H such that if G does not contain J as a pivot-minor, then it contains neither the 2-subdivision of H nor its complement as an induced subgraph (see Lemmas 2.1 and 2.3 of [9]). Consequently, 3.1 implies that for every graph J, the class of graphs not containing J as a pivot-minor has the Erdős-Hajnal property, a result recently proved by Davies in [9]. (He actually proved more, the strong Erdős-Hajnal property.)

We use standard terminology. Graphs are assumed to be finite, and to have no parallel edges or loops. Occasionally we need to allow parallel edges (but not loops), and in that case we call them *multigraphs*; $G[X]$ denotes the induced subgraph with vertex set X of a graph G; $|G|$ denotes the number of vertices of G; and $\overline{G}$ is the complement graph of G.

4 A sketch of the proof

Let us give some idea of how the proof will work. (In the proof proper, there are numerous constant factors that we will ignore here.) We have two Swiss Army graphs H, J, and we can assume that $H = J$ without loss of generality. We need to show that every $\{H, \overline{H}\}$-free graph has a stable set or clique of size $|G|^c$, where $c > 0$ is some small constant depending on H but not on G. It is just as good to show that $\alpha(G)\omega(G) \geq |G|^c$ for all such graphs G. Choose $c > 0$ very small, and suppose it does not work; then there is a minimal counterexample G, that is, G is "c-critical".

We use the strategy of iterative sparsification developed in other papers of this series (see [14, 15, 16, 17]). Since G is H-free, a theorem of Rödl implies that there is a subset $S \subseteq V(G)$ with size linear in $|G|$, inducing a subgraph that is either very sparse or very dense; and by replacing G by its complement we may assume it is very sparse. Consequently, for some $0 < y < 1$ (at most any positive constant we wish), there is a subset $S \subseteq V(G)$ with density at most y and size at least $y^a|G|$ (where a is some large positive constant that we choose for convenience). Say such a value of y is "good", and choose a good value of y such that y^2 is not good. There must be such a value, because there is no good value that is between $|G|^{-c}$ and $|G|^{-2c}$ (because if there were, we could find a large stable set contradicting that G is c-critical). From now on we just work inside the set S.

Now the proof breaks into two parts. We need to find in $G[S]$ an appropriate object, a "blown-up" relative of a Swiss Army graph that we (temporarily) call a "template", that can be described as follows. The graph H equals F_t^s for some F, s, t; let F' be a forest with no edges and about $y^{-1/16}$ vertices, and consider the graph $(F')_t^s$. Now blow up each vertex of $(F')_t^s$ that belongs to F' into a subset, a "block", pairwise disjoint, and pairwise sparse (with sparsity at most $y^{1/6}$ to each other). We want each of the blocks to be large (at least poly$(y)|G|$ for some fixed polynomial), but there is no condition on the subgraph induced on each block. That is what we mean by a template.

The first half of the proof is to show that we can find a template in $G[S]$, using the minimality of y. Then, once we have such an object, the second half of the proof is to find a copy of F within the union of the blocks, with at most one vertex in each block (that is, a "rainbow" copy of F). If we can do that, then F can be extended within the template to make a copy of H in G, a contradiction (which would prove that there is no $\{H, \overline{H}\}$-free c-critical graph G after all).

The first half of the proof, finding the template within $G[S]$, uses a refinement of a result of Fox and Sudakov, that we prove in the next section and use in sections 6 and 7. The theorem says that for any H, in any H-free graph we can find either a large sparse subset or two large sets of vertices that are very dense to each other (and "large", "sparse" and "very dense" can be tuned). We will apply this inside the graph $G[S]$, which is already y-sparse, and the first outcome is impossible because of the minimality of y. If the second happens, with two large sets A_1, B_1 say, there are only a few vertices in $S \setminus (A_1 \cup B_1)$ that have many neighbours in A_1 or in B_1 (because $G[S]$ is y-sparse), so we can put them aside and repeat. We get a sequence of about $y^{-1/16}$ pairs A_i, B_i of large sets, each pair very dense to each other, and the sets in different pairs sparse to each other. Each B_i includes a large stable set (since G is c-critical). If we can get a large induced matching within the union of these stable sets, with not many edges joining the same two sets, then we have the template; and otherwise there is a large stable set that contradicts the c-criticality of G.

The second half uses a result of [7], that says that for every forest F, in every sparse F-free graph G there are two sets of vertices of linear size that are anticomplete (that is, there are no edges between them). We need to replace the F-free hypothesis with the weaker hypothesis that there is no rainbow copy of F; and to replace the "two anticomplete sets of linear size" outcome with a "poly$(1/y)$ anticomplete sets of poly$(y)|G|$ size" outcome. Then we apply this result to the template. The "poly$(1/y)$ anticomplete sets of poly$(y)|G|$ size" outcome would contradict the c-criticality of G, so there must be a rainbow copy of F, which is what we want. This is all done in section 8.

5 Stable sets and complete bipartite subgraphs

We need to use a strengthening of a celebrated result of Fox and Sudakov [12]. Their proof was complicated, and used dependent random choice, and we begin with giving a simple proof of their theorem. Then we will modify it to prove the strengthening that we need.

The result is an asymmetric weakening of the Erdős-Hajnal conjecture in which the clique is replaced by a complete bipartite subgraph (not necessarily induced).

5.1 (Fox and Sudakov [12]) *For every graph H, there exists $c > 0$ such that every H-free graph G contains either a stable set of size at least $|G|^c$, or a complete bipartite subgraph with both parts of size at least $|G|^c$.*

We will give a simple proof of the following stronger result:

5.2 *For every graph H, and every $\delta > 0$, there exists $c > 0$ such that the following holds: every sufficiently large H-free graph G contains either a stable set of size $|G|^c$, or a complete bipartite subgraph with parts of cardinality at least $|G|^{1-\delta}$ and $|G|^c$ respectively.*

The same method, with a little more effort, also yields a second strengthening of 5.1, which is what we actually need, but we will prove that separately. If H, G are graphs, a *copy* of H in G means an isomorphism from H to an induced subgraph of G, and $\text{ind}_H(G)$ denotes the number of copies of H in G.

The proof of 5.2 relies on the following key lemma, which was proved (with different constants) by Fox and Sudakov [12]; as they showed, 5.1 follows in a few lines.

5.3 *Let H be a graph, and let $0 < \varepsilon < 1$. Let G be an H-free graph, and let $t > 0$ be an integer, with $|G| \geq t$. Then either*

- *there is a stable set of G with size t; or*

- *there are disjoint subsets W_1, W_2 of $V(G)$, with $|W_1|, |W_2| \geq (2t)^{-|H|^2}\varepsilon^{|H|^2/2}|G|$, such that every vertex in W_1 is nonadjacent to at most $2\varepsilon|W_2|$ vertices in W_2.*

Proof. We may assume that G has no stable set of size t, and so it has an edge; and therefore we may assume that $|G| > t^{|H|^2}$, since otherwise the second bullet holds (taking $|W_1| = |W_2| = 1$). So we may assume that $t, |H| \geq 2$. Let $k := |H|$ and $n := |G|$.

Let $H_0, H_1, \ldots, H_m$ be a sequence of graphs, each with vertex set $V(H)$, where $m = k(k-1)/2$, such that for $1 \leq i \leq m$, H_i is obtained from H_{i-1} by adding an edge joining two nonadjacent vertices of H_{i-1} (and consequently H_i has i edges), and such that one of $H_0, \ldots, H_m$ equals H. The proof starts with H_m and works downwards. For each $i \geq 1$, we will show that if G contains many copies of H_i then either it contains many copies of H_{i-1} or we can find a very dense bipartite subgraph. Since, as we shall see, we have many copies of H_m and no copy of H, the bipartite outcome must occur at some point. We begin with:

(1) $\mathrm{ind}_{H_m}(G) \geq t^{-k^2}n^k$.

Every graph with at least t^k vertices has either a stable set of size t or a k-clique (that is, a clique of size k), by one of the standard forms of Ramsey's theorem. By our assumption, G has no stable set of size t, and so every subset $X \subseteq V(G)$ with $|X| = s$ includes a k-clique, where $s = t^k$. Since each k-clique is included in only $\binom{n-k}{s-k}$ subsets of size s, and there are $\binom{n}{s}$ subsets of size s altogether (because $n \geq t^k$), it follows that there are at least

$$\binom{n}{s} / \binom{n-k}{s-k} = \frac{n(n-1)\cdots(n-k+1)}{s(s-1)\cdots(s-k+1)} \geq \left(\frac{n}{s}\right)^k = t^{-k^2}n^k$$

k-cliques in G. This proves (1).

For $0 \leq i \leq m$, let $f(i) = \mathrm{ind}_{H_i}(G)/n^k$. Thus we have seen that $f(m) \geq t^{-k^2}$.

(2) *For $1 \leq i \leq m$, either $f(i-1) \geq (\varepsilon/4)f(i)$, or there are disjoint subsets W_1, W_2 of $V(G)$ with $|W_1| \cdot |W_2| \geq (f(i)/4)n^2$, such that there are fewer than $\varepsilon|W_1| \cdot |W_2|$ nonedges between W_1, W_2.*

We suppose the second outcome is false. Let $p, q \in V(H)$ be distinct, such that H_i is obtained from H_{i-1} by adding the edge pq. Let $J = H_i \setminus \{p, q\}$, and let ψ be a copy of J in G. We define $x(\psi), y(\psi)$ to be the number of copies of H_i, H_{i-1} respectively in G that extend J (a copy ϕ of H_i or H_{i-1} *extends* ψ if the restriction of ϕ to $V(J)$ equals ψ). A copy of J in G is *royal* if $x(\psi) \geq f(i)n^2/2$, and a copy of H_i in G is *noble* if it extends a royal copy of J. It follows that at least half of the copies of H_i in G are noble.

Let ψ be a royal copy of J. We claim that $y(\psi) \geq (\varepsilon/2)x(\psi)$. Let P be the set of vertices $v \in V(G)$ such that mapping p to v extends ψ to a copy of $H_i \setminus \{q\}$, and let Q be the set of vertices $v \in V(G)$ such that mapping q to v extends ψ to a copy of $H_i \setminus \{p\}$. Thus either $P \cap Q = \emptyset$ or $P = Q$.

Suppose first that $P \cap Q = \emptyset$. Since ψ is royal, it follows that $|P| \cdot |Q| \geq x(\psi) \geq f(i)n^2/2$. Hence there are at least $\varepsilon|P| \cdot |Q|$ nonedges between P, Q, from our assumption; but then $y(\psi) \geq \varepsilon|P| \cdot |Q| \geq \varepsilon x(\psi)$ as claimed.

Now suppose that $P = Q$. Since ψ is royal, there are at least $x(\psi)/2$ edges with both ends in P, and so $|P|(|P|-1)/2 \geq x(\psi)/2 \geq f(i)n^2/2$. Choose $C \subseteq P$ with size $\lfloor |P|/2 \rfloor$, and let $D = P \setminus C$. Thus

$$|C| \cdot |D| \geq |P|(|P|-1)/4 \geq x(\psi)/4 \geq f(i)n^2/4.$$

Consequently there are at least $\varepsilon |C| \cdot |D|$ nonedges between C, D, from our assumption; and since each of these gives two ways to extend J to a copy of H_{i-1}, and $|C| \cdot |D| \geq x(\psi)/4$, it follows that $y(\psi) \geq (\varepsilon/2)x(\psi)$.

This proves our claim that $y(\psi) \geq (\varepsilon/2)x(\psi)$, for each royal ψ. Summing over all royal ψ, we deduce that the number of copies of H_{i-1} in G is at least $\varepsilon/2$ times the number of noble copies of H_i in G, and hence at least $\varepsilon/4$ times the number of copies of H_i. This proves (2).

Since one of $H_0, \ldots, H_m$ equals H, and G is H-free, it is not the case that $f(i-1) \geq \varepsilon f(i)/4$ for all i with $1 \leq i \leq m$; choose $i \in \{1, \ldots, m\}$ maximum such that $f(i-1) < \varepsilon f(i)/4$. Consequently, $f(j-1) \geq \varepsilon f(j)/4$ for $i+1 \leq j \leq m$, and so by (1),

$$f(i) \geq (\varepsilon/4)^{m-i} f(m) \geq (\varepsilon/4)^{m-1} t^{-k^2}.$$

From (2), there are disjoint subsets W_1, W_2 of $V(G)$ with $|W_1| \cdot |W_2| \geq (f(i)/4)n^2$, such that there are fewer than $\varepsilon |W_1| \cdot |W_2|$ nonedges between W_1, W_2. Since $|W_1|, |W_2| \leq n$, it follows that

$$|W_1|, |W_2| \geq (f(i)/4)n \geq \frac{1}{4}(\varepsilon/4)^{m-1} t^{-k^2} n \geq 2(\varepsilon/4)^{k(k-1)/2} t^{-k^2} n.$$

Since there are fewer than $\varepsilon |W_1| \cdot |W_2|$ nonedges between W_1, W_2, fewer than half the vertices in W_1 have at least $2\varepsilon |W_2|$ neighbours in W_2, and the result follows. ∎

Let us deduce 5.2 from 5.3, in the following form (to deduce 5.1 itself, take $\delta = 1/2$, say):

5.4 *For every graph H, and every δ with $0 < \delta \leq 1/2$, let $c = \delta/(6|H|^2)$; then every H-free graph G with $2|G|^{-1} + |G|^{-\delta} \leq 1$ contains either a stable set of size $|G|^c$, or a complete bipartite subgraph with parts of cardinality at least $|G|^{1-\delta}$ and at least $|G|^c$ respectively.*

Proof. Let H, δ, c be as in the theorem, and let G be H-free. Suppose first that $|G|^c \leq 2$. If G is not complete, the first outcome holds, since $|G|^c \leq 2$; and if G is complete, the second holds, since $|G| \geq |G|^{1-\delta} + 2$. Thus we may assume that $|G|^c > 2$. Let $t := \lceil |G|^c \rceil$ and $\varepsilon := 1/(4t)$.

Since $|G|^c \geq 2$, it follows that

$$|G|^{\delta - 3c|H|^2/2} = |G|^{9c|H|^2/2} \geq 2^{9|H|^2/2} \geq 2^{1+7|H|^2/2}.$$

Consequently

$$t^{-|H|^2} \left(\frac{\varepsilon}{4}\right)^{|H|^2/2} = t^{-|H|^2} \left(\frac{1}{16t}\right)^{|H|^2/2} \geq (2|G|^c)^{-3|H|^2/2} 4^{-|H|^2} \geq 2|G|^{-\delta}$$

since $t \leq 2|G|^c$.

We may assume that G has no stable set of size t, and so by 5.3, and since $\delta \leq 1/2$ and so $1 - \delta \geq c$, there are disjoint subsets W_1, W_2 of $V(G)$, with

$$|W_1|, |W_2| \geq t^{-|H|^2} \left(\frac{\varepsilon}{4}\right)^{|H|^2/2} |G| \geq 2|G|^{1-\delta} \geq 2|G|^c \geq t,$$

such that every vertex in W_1 is nonadjacent to at most $2\varepsilon|W_2|$ vertices in W_2. Choose $A \subseteq W_1$ with cardinality t. Since each vertex in A has at most $2\varepsilon|W_2|$ non-neighbours in W_1, there are at least $|W_2|(1-2\varepsilon t) = |W_2|/2 \geq |G|^{1-\delta}$ vertices in W_2 adjacent to every vertex in X. This proves 5.4. ∎

For its application later in this paper, we need a more powerful version of 5.3, replacing the hypothesis that G is H-free by the weaker hypothesis that G does not contain many copies of H, and replacing the stable set outcome with a linear sparse set outcome. (Actually, we only need the second strengthening, but the first comes for free anyway.) After some preliminaries, its proof is much the same as that of 5.3.

We need a lemma:

5.5 *Let $h \geq 1$ be an integer and let $0 < c < 1$. For every graph G, either:*

- $\mathrm{ind}_{K_h}(G) \geq c^{h(h+1)/2}|G|^h$, *or*
- *there exists $S \subseteq V(G)$ with $|S| \geq c^{h(h-1)/2}|G|$ such that $G[S]$ has fewer than $c|S|^2$ edges.*

Proof. Suppose that G is a graph such that $G[S]$ has at least $c|S|^2$ edges for every $S \subseteq V(G)$ with $|S| \geq c^{h(h-1)/2}|G|$. We observe first:

(1) *For every set $S \subseteq V(G)$ with $|S| \geq c^{h(h-1)/2}|G|$, there are at least $c|S|$ vertices in S that have degree at least $c|S|$ in $G[S]$.*

Let there be $x|S|$ vertices in S that have degree at least $c|S|$. Then the sum of the degrees in $G[S]$ is at most $x|S|^2 + c|S|^2$, and since $G[S]$ has at least $c|S|^2$ edges, it follows that $x + c \geq 2c$. This proves (1).

Choose $v_1, \dots, v_h \in V(G)$ independently and uniformly at random. For $1 \leq i \leq h$, let E_i be the event that $x_1, \dots, x_i$ are distinct and pairwise adjacent, and there are at least $c^i|G|$ vertices that are distinct from $x_1, \dots, x_i$ and adjacent to all of $x_1, \dots, x_i$; and let p_i be the probability of E_i. We claim that:

(2) $p_i \geq c^{i(i+1)/2}$ *for* $1 \leq i \leq h$.

We prove this by induction on i. The result holds for $i = 1$, since at least $c|G|$ vertices of G have degree at least $c|G|$ by (1). We assume that $i \geq 2$ and the result holds for $i-1$. But E_i is the event that

- E_{i-1} holds, and
- v_i belongs to X where X is the set of vertices that are distinct from $x_1, \dots, x_{i-1}$ and adjacent to all of $x_1, \dots, x_{i-1}$, and
- v_i is adjacent to at least $c^i|G|$ members of X.

If E_{i-1} holds, then the set X above has size at least $c^{i-1}|G| \geq c^{h(h-1)/2}|G|$, and so by (1), at least $c|X|$ vertices in X are adjacent to at least $c|X|$ vertices in X. Consequently

$$p_i \geq p_{i-1}(c|X|/|G|) \geq p_{i-1}c^i \geq c^{(i-1)i/2}c^i = c^{i(i+1)/2}.$$

This proves (2).

In particular, the probability that $v_1, \dots, v_h$ are all distinct and pairwise adjacent is at least $c^{h(h+1)/2}$; and so $\mathrm{ind}_{K_h}(G) \ge c^{h(h+1)/2}|G|^h$. This proves 5.5. ∎

We deduce the following more powerful version of 5.3:

5.6 *Let H be a graph with $h \ge 1$ vertices, and let $0 < \varepsilon < 1/4$. Let G be a graph with $\mathrm{ind}_H(G) < (\varepsilon^h|G|)^h$. Then either*

- *there exists $S \subseteq V(G)$ with $|S| \ge \varepsilon^{h(h-1)/2}|G|$ such that $|E(G[S])| < \varepsilon|S|^2$, or*
- *there exist disjoint $W_1, W_2 \subseteq V(G)$ with $|W_1|, |W_2| \ge 2(\varepsilon/2)^{h^2}|G|$ such that there are fewer than $\varepsilon|W_1| \cdot |W_2|$ nonedges between W_1 and W_2.*

Proof. Let $n := |G|$. We may assume there is no S satisfying the first outcome, and so $\mathrm{ind}_{K_h}(G) \ge \varepsilon^{h(h+1)/2}n^h$ by 5.5. Define m and $H_0, H_1, \dots, H_m$ as in the proof of 5.3, and for $0 \le i \le m$, let $f(i)n^h$ be the number of copies of H_i in G. Thus we have seen that $f(m) \ge \varepsilon^{h(h+1)/2}$. As in the proof of 5.3, we have

(1) *For $1 \le i \le m$, either $f(i-1) \ge (\varepsilon/4)f(i)$, or there are disjoint subsets W_1, W_2 of $V(G)$ with $|W_1| \cdot |W_2| \ge (f(i)/4)n^2$, such that there are fewer than $\varepsilon|W_1| \cdot |W_2|$ nonedges between W_1, W_2.*

Since one of $H_0, \dots, H_m$ equals H, and $\mathrm{ind}_H(G) < (\varepsilon^h|G|)^h < (\varepsilon/4)^m$ (because $\varepsilon \le 1/4$), it is not the case that $f(i-1) \ge \varepsilon f(i)/4$ for all i with $1 \le i \le m$; choose $i \in \{1, \dots, m\}$ maximum such that $f(i-1) < \varepsilon f(i)/4$. Consequently, $f(j-1) \ge \varepsilon f(j)/4$ for $i+1 \le j \le m$, and so

$$f(i) \ge (\varepsilon/4)^{m-i}f(m) \ge (\varepsilon/4)^{m-1}\varepsilon^{h(h+1)/2} = \varepsilon^{(h(h-1)/2-1+h(h+1)/2}4^{1-m} = 2^{-h^2+h+2}\varepsilon^{h^2-1} \ge 8(\varepsilon/2)^{h^2}.$$

From (1), there are disjoint subsets W_1, W_2 of $V(G)$ with $|W_1| \cdot |W_2| \ge (f(i)/4)n^2 \ge 2(\varepsilon/2)^{h^2}n^2$, such that there are fewer than $\varepsilon|W_1| \cdot |W_2|$ nonedges between W_1, W_2. ∎

We remark that 5.6 implies 5.3, with worse constants: given t and ε as in 5.3, define $\varepsilon' = \min(\varepsilon, 1/(2t))$, and apply 5.6 with ε replaced by ε'.

6 A sparse sequence of dense pairs

Suppose we are given a sparse H-free graph G, say with density at most y. The goal of this section is to show that either we can drop to a much sparser (with density $O(y^2)$) induced subgraph that is still large (size $\mathrm{poly}(y)|G|$), or we can find a 'nice' structure: a sequence of disjoint sets $A_1, \dots, A_k, B_1, \dots, B_k$ such that pairs A_i, B_i are dense to each other, while distinct pairs are sparse to each other (and all the parameters have suitable polynomial dependence on y). We will use these structures in later sections to build the induced subgraphs (Swiss army graphs) that we are looking for.

If $A, B \subseteq V(G)$ are disjoint, and $0 \le x \le 1$, we say that B is *x-sparse* to A if every vertex in B has at most $x|A|$ neighbours in A; and B is *x-dense* to A if every vertex in B has at least $x|A|$ neighbours in A.

6.1 *Let H be a graph, let $y \in (0, 2^{-99})$, and let G be an H-free y-sparse graph. Suppose that there is no subset $S \subseteq V(G)$ with $|S| \geq y^{|H|^2}|G|$ such that $G[S]$ is y^2-sparse. Then there exist disjoint $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k \subseteq V(G)$, where $k = \lceil y^{-1/4} \rceil$, such that:*

- *$|A_i|, |B_i| = \lceil (y^2/16)^{|H|^2}|G| \rceil$, and there are at most $(y^2/2)|A_i||B_i|$ nonedges between A_i, B_i, for $1 \leq i \leq k$; and*
- *each of A_i, B_i is $y^{1/6}$-sparse to each of A_j, B_j, for all distinct $i, j \in \{1, \ldots, k\}$.*

Proof. Let $h := |H|$. Since there is no S as in the theorem, it follows that $E(G) \neq \emptyset$ and so $y|G| \geq 1$. Let $s \geq 0$ be maximum such that there are disjoint $X_1, \ldots, X_s, Y_1, \ldots, Y_s \subseteq V(G)$ satisfying:

- for all $i \in [s]$, $|X_i|, |Y_i| = \lceil 2(y^2/16)^{h^2}|G| \rceil$, and there are at most $(y^2/8)|X_i||Y_i|$ nonedges between X_i, Y_i; and
- for $1 \leq i < j \leq s$, each of X_j, Y_j is $y^{1/2}$-sparse to each of X_i, Y_i.

We claim that:

(1) $s \geq y^{-1/4}$.

Suppose not. For $1 \leq i \leq s$,

$$|X_i| = |Y_i| = \lceil 2(y/16)^{h^2}|G| \rceil \leq \max(1, 4(y/16)^{h^2}|G|) \leq y|G|$$

(since $y|G| \geq 1$). Let $A := \bigcup_{i \in [s]}(X_i \cup Y_i)$. For each $i \in [s]$, let D_i be the set of vertices in $V(G) \setminus A$ that have either at least $y^{1/2}|X_i|$ neighbours in X_i or at least $y^{1/2}|Y_i|$ neighbours in Y_i; then $|D_i| \leq 2y^{1/2}|G|$, since there are at most $y|X_i||G|$ edges between X_i and $V(G) \setminus A$, and the same for Y_i. Let $G' := G \setminus (A \cup \bigcup_{i \in [s]} D_i)$; then since $s < y^{-1/4}$, we have

$$|G'| \geq |G| - s(2y|G| + 2y^{1/2}|G|) \geq |G| - 2y^{-1/4}(y|G| + y^{1/2}|G|) \geq (1 - 4y^{1/4})|G| \geq |G|/2.$$

By 5.6 applied to G', taking $\varepsilon = y^2/8$, either

- there exists $S \subseteq V(G')$ with $|S| \geq (y^2/8)^{h(h-1)/2}|G'| \geq y^{h^2}|G|$ such that $|E(G[S])| < (y^2/8)|S|^2$, or
- there exist disjoint $W_1, W_2 \subseteq V(G')$ with $|W_1|, |W_2| \geq 2(y^2/16)^{h^2}|G'|$ such that there are fewer than $(y^2/8)|W_1| \cdot |W_2|$ nonedges between W_1 and W_2.

Suppose that the first holds, and let S be the corresponding set. Since $|E(G[S])| < (y^2/8)|S|^2$, at most $|S|/2$ vertices in S have degree in $G[S]$ at least $y^2|S|/2$; and so there is a subset $S' \subseteq S$ that is y^2-sparse, with $|S'| \geq |S|/2 \geq (y^2/8)^{h(h-1)/2}|G'|/2 \geq y^{|H|^2}|G|$, contrary to the hypothesis.

If the second bullet holds, then by averaging, there are subsets $X_{s+1} \subseteq W_1$ and $Y_{s+1} \subseteq W_2$, both of size $\lceil 2(y^2/16)^{h^2}|G| \rceil$, such that there are fewer than $(y^2/8)|X_{s+1}| \cdot |Y_{s+1}|$ nonedges between X_{s+1} and Y_{s+1}, contrary to the maximality of s. This proves (1).

Let $k = \lceil y^{-1/4} \rceil$. For $1 \leq i \leq k$, and for each j with $i < j \leq k$, there are at most $y^{1/2}|X_i| \cdot |X_j|$ edges between X_i, X_j (since X_j is $y^{1/2}$-sparse to X_i), and so at most $2y^{1/3}|X_i|$ vertices in X_i have at least $\frac{1}{2}y^{1/6}|X_j|$ neighbours in X_j. Similarly there are at most $2y^{1/3}|X_i|$ vertices in X_i that have at least $\frac{1}{2}y^{1/6}|Y_j|$ neighbours in Y_j. Let $A_i' \subseteq X_i$ be the set of vertices $v \in X_i$ such that for each $j \in \{i+1,\dots,k\}$, v has at most $\frac{1}{2}y^{1/6}|X_j|$ neighbours in X_j and at most $\frac{1}{2}y^{1/6}|Y_j|$ neighbours in Y_j. It follows that $|A_i'| \geq (1 - 2y^{1/3}y^{-1/4})|X_i| \geq |X_i|/2$; and so there exists $A_i \subseteq A_i'$ with $|A_i| = \lceil |X_i|/2 \rceil = \lceil (y^2/16)^{h^2}|G| \rceil$. Define $B_i \subseteq Y_i$ similarly: that is, B_i is a set of vertices $v \in Y_i$ such that for each $j \in \{i+1,\dots,k\}$, v has at most $\frac{1}{2}y^{1/6}|X_j|$ neighbours in X_j and at most $\frac{1}{2}y^{1/6}|Y_j|$ neighbours in Y_j, and $|B_i| = \lceil (y^2/16)^{h^2}|G| \rceil$.

Consequently there are at most $(y^2/8)|X_i||Y_i| \leq (y^2/2)|A_i||B_i|$ nonedges between A_i, B_i. Now, for all $i, j \in \{1,\dots,k\}$ with $i < j$, each of A_i, B_i is $y^{1/6}$-sparse to each of A_j, B_j, since they are $\frac{1}{2}y^{1/6}$ sparse to X_j and to Y_j; and each of A_j, B_j is $y^{1/6}$-sparse to each of A_i, B_i since each of A_j, B_j is $y^{1/2}$-sparse to each of X_i, Y_i and $2y^{1/2} \leq y^{1/6}$. This proves 6.1. ∎

7 Blockades, and growing a hand

A *blockade* in G is a sequence $\mathcal{B} = (B_1,\dots,B_k)$ of pairwise disjoint subsets of $V(G)$, and we call $B_1,\dots,B_k$ its *blocks*. (In some earlier papers, the blocks of a blockade must be nonempty, but here it is convenient to allow empty blocks.) We define $V(\mathcal{B}) = B_1 \cup \dots \cup B_k$ The *length* of the blockade $\mathcal{B} = (B_1,\dots,B_k)$ is k, and its *width* is the minimum of the cardinalities of its blocks. If its length is at least ℓ and width at least w we call it an *(ℓ, w)-blockade*. For $\varepsilon > 0$, the blockade $\mathcal{B} = (B_1,\dots,B_k)$ is *ε-sparse* if $B_{i+1} \cup \dots \cup B_k$ is ε-sparse to B_i for all i with $1 \leq i \leq k$; and similarly $\mathcal{B}$ is *$(1-\varepsilon)$-dense* if $B_{i+1} \cup \dots \cup B_k$ is $(1-\varepsilon)$-dense to B_i for all i with $1 \leq i \leq k$.

Let $\alpha(G), \omega(G)$ be the cardinalities of the largest stable sets and cliques of G respectively; and for $c > 0$, let us say a graph G is *c-critical* if $\alpha(G)\omega(G) < |G|^c$, and $\alpha(G')\omega(G') \geq |G'|^c$ for every induced subgraph G' of G with $|G'| < |G|$. In order to prove 3.4, it suffices to show that for every two Swiss Army graphs H, J, if $c > 0$ is sufficiently small, then no $\{H, \overline{J}\}$-free graph is c-critical. If $X, Y \subseteq V(G)$ are disjoint, we say X is *anticomplete* to Y if there are no edges of G between X and Y.

As a first step, we will prove that if c is sufficiently small, then every c-critical graph contains a piece of machinery that will provide the 2-subdivision paths of a Swiss Army graph. Thus, let $\mathcal{B} = (B_1,\dots,B_k)$ be a blockade in G. A *B_i-finger* is an induced path in G of length two, with three vertices a_i-b_i-c_i in order, such that

- $a_i, b_i, c_i \notin V(\mathcal{B})$;
- c_i is complete to B_i and is anticomplete to $V(\mathcal{B}) \setminus B_i$; and
- a_i, b_i are both anticomplete to $V(\mathcal{B})$.

Let a_i-b_i-c_i be a B_i-finger, for $1 \leq i \leq k$. We call the union of these fingers a *hand* for $\mathcal{B}$ if

- $a_1, a_2, \dots, a_k$ are all the same vertex;
- $b_1, c_1, b_2, c_2, \dots, b_k, c_k$ are all distinct;
- the union of these fingers is induced; that is, for $1 \leq i < j \leq k$, $\{b_i, c_i\}$ is anticomplete to $\{b_j, c_j\}$.

We call a_1 the *palm* of the hand.

Now let $H_1, \dots, H_s$ be hands for $\mathcal{B}$, all with the same palm but otherwise pairwise vertex-disjoint and anticomplete, and take their union. We call the result an *s-thickened hand*, with *palm* the common palm of $H_1, \dots, H_s$. See figure 2.

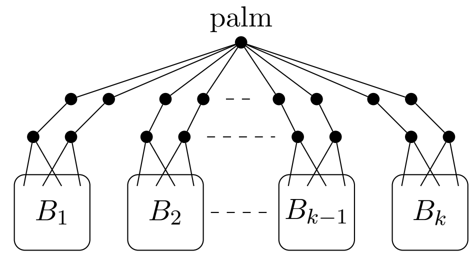

Figure 2: A 2-thickened hand for $(B_1, \dots, B_k)$.

Finally, let $T_1, \dots, T_t$ be s-thickened hands for $\mathcal{B}$, pairwise disjoint and anticomplete. Take their union, and for $1 \leq i < j \leq t$ add s paths of length three joining the palms of T_i, T_j (that is, add s edges each joining the two given palms, and then subdivide each of them twice). The graph we produce is called a *(s,t)-handset* for $\mathcal{B}$.

We need the following easy lemma:

7.1 *Let F be a graph and let $t, n \geq 0$ and $m \geq 1$ be integers, with $|F| \geq m^t n$. Then either:*

- *F has a stable set of size m; or*
- *there are disjoint subsets X, Y of $V(F)$, such that X is a clique and $|X| = t$, and X is complete to Y and $|Y| \geq n$.*

Proof. We proceed by induction on t. If $t = 0$ the result is true (take $X = \emptyset$ and $Y = V(F)$), so we assume that $t > 0$ and the result holds for $t - 1$. We may assume that F has no stable set of size m, and so its chromatic number is more than $|F|/m$, and therefore it has a vertex with more than $|F|/m - 1 \geq m^{t-1}n - 1$ neighbours, and hence with at least $m^{t-1}n$ neighbours. The result follows from the inductive hypothesis applied to the subgraph induced on this set of neighbours. This proves 7.1. ∎

A blockade $(B_1, \dots, B_k)$ is *equicardinal* if all its blocks have the same size; and *symmetrically x-sparse* if B_i is x-sparse to B_j for all distinct $i, j \in \{1, \dots, k\}$. We will prove the following.

7.2 *Let $s, t \geq 1$ be integers, let $\rho > 0$, and let H be a graph. Then there exist $d, \delta, \eta > 0$ with the following property. Let $0 \leq c \leq \delta$, and let G be a c-critical H-free graph. Let $Z \subseteq V(G)$ with $|Z| \geq y^\rho |G|$ such that $G[Z]$ is y-sparse for some y with $0 < y < \eta$. Then either:*

- *there is a subset $S \subseteq Z$ with $|S| \geq y^{|H|^2}|Z|$ that is y^2-sparse; or*
- *there is a symmetrically $2y^{1/6}$-sparse equicardinal blockade $\mathcal{B}$ in G of length at least $y^{-1/64}$ and width at least $y^d|Z|/2$, and there is an (s,t)-handset for $\mathcal{B}$.*

Proof.

Let $h := |H|$. Choose $\eta > 0$ such that

$$\eta \le \min\left(2^{-99}, \left(16st^2\right)^{-6}, (8s)^{-24}, 2^{-96t}\right)$$

and

$$\eta s\left(t + 2\eta^{-1/64}\right) + \left(2st\left(t + 2\eta^{-1/64}\right) + t\right)\eta^{1/6} < 1/2.$$

Choose $d > 1$ such that $(y^2/16)^{h^2} \ge y^d$ for all $y \le \eta$, and let $\delta = 1/(64(d+\rho)t)$. We claim that d, δ, η satisfy the theorem.

Let $0 < c \le \delta$, let G be a c-critical H-free graph, and let $Z \subseteq V(G)$ with $|Z| \ge y^\rho$, such that $G[Z]$ is y-sparse, where $0 < y < \eta$. We may assume that the first outcome of the theorem, applied to $G[Z]$, is false. So by 6.1, there exist disjoint

$$A_1, A_2, \ldots, A_{k'}, B_1, B_2, \ldots, B_{k'} \subseteq Z,$$

where $k' = \lceil y^{-1/4} \rceil$, such that:

- $|A_i|, |B_i| = \lceil (y^2/16)^{h^2}|Z| \rceil$, and there are at most $(y^2/2)|A_i| \cdot |B_i|$ nonedges between A_i, B_i, for $1 \le i \le k'$; and
- each of A_i, B_i is $y^{1/6}$-sparse to each of A_j, B_j, for all distinct $i, j \in \{1, \ldots, k'\}$.

Let $k := \lceil y^{-1/16} \rceil$; we will only need the sets A_i, B_i for $1 \le i \le k$.

Let $1 \le i \le k$. Since there are at most $(y^2/2)|A_i| \cdot |B_i|$ nonedges between A_i, B_i, there exists $X \subseteq B_i$ with $|X| \ge |B_i|/2$ that is $(1-y^2)$-dense to A_i. Since G is c-critical, the graph $G[X]$ satisfies $\alpha(G[X])\omega(G[X]) \ge |X|^c$; and since $\omega(G[X]) \le \omega(G)$, we deduce that there is a stable subset $C_i \subseteq B_i$ of size at least $(|B_i|/2)^c/\omega(G)$ that is $(1-y^2)$-dense to A_i.

Let $C := C_1 \cup \cdots \cup C_k$. An *induced matching* of $G[C]$ means a set of edges of $G[C]$, pairwise vertex-disjoint and anticomplete. Choose an induced matching M of $G[C]$, maximal with the property that for all distinct $i, j \in \{1, \ldots, k\}$, at most s members of M have an end in C_i and an end in C_j. It follows that $|M| \le sk^2/2$. Let F be the graph with vertex set $\{1, \ldots, k\}$ in which i, j are adjacent if there are s members of M with an end in C_i and an end in C_j. Let $m := \lceil (y^2/16)^{-ch^2} y^{-c\rho} 2^c \rceil$, and $n := \lceil y^{-1/64} \rceil$. Thus

$$m \le 2(y^2/16)^{-ch^2} y^{-c\rho} 2^c \le y^{-cd-c\rho} 2^{c+1}.$$

By 7.1, either:

- $k < m^t n$; or
- F has a stable set of size m; or
- there are disjoint subsets X, Y of $V(F)$, such that X is a clique of F and $|X| = t$, and X is complete to Y in F, and $|Y| \ge n$.

Suppose the first bullet holds. Since $m < y^{-cd-c\rho} 2^{c+1}$, and $n \le 2y^{-1/64}$, it follows that

$$y^{-1/16} \le k < 2^{t(c+1)} y^{-c(d+\rho)t}\left(2y^{-1/64}\right).$$

Since $c(d+\rho)t \le 1/64$, we deduce that $y^{-1/16} < 2^{t(c+1)+1}y^{-1/32}$, that is, $y^{-1/32} < 2^{t(c+1)+1} < 2^{3t}$, a contradiction since $y < 2^{-96t}$.

Suppose the second bullet holds; then we may assume that I is a stable set of F and $|I| = m$. Let P be the set of vertices in $\bigcup_{i\in I} C_i$ that either belong to a member of M or are adjacent to a member of M. Thus for each $i \in I$,

$$|P \cap C_i| \le 2y^{1/6}|M| \cdot |C_i| \le sk^2y^{1/6}|C_i| \le |C_i|/2,$$

since $k \le 2y^{-1/16}$, and $y \le (8s)^{-24}$. But from the maximality of M, the union of the sets $C_i \setminus P$ $(i \in I)$ is stable in G, and so has cardinality at most $|G|^c/\omega(G)$. Consequently

$$\sum_{i\in I} |C_i|/2 < |G|^c/\omega(G).$$

But $|C_i| \ge (|B_i|/2)^c/\omega(G)$ for each $i \in I$, and so

$$\sum_{i\in I} (|B_i|/2)^c/\omega(G) < |G|^c/\omega(G).$$

Each $|B_i| \ge (y^2/16)^{h^2}|Z|$, and so $m(y^2/16)^{ch^2}(|Z|/2)^c < |G|^c$. Since $|Z| \ge y^\rho|G|$, it follows that $m(y^2/16)^{ch^2}y^{c\rho}2^{-c} < 1$, a contradiction, from the choice of m.

Suppose the third bullet holds, and let X, Y be the corresponding subsets of $V(F)$. For each edge ij of F with one end in X and the other end in $X \cup Y$, there are s edges in M between C_i and C_j; let M' be the set of all these edges. Thus $|M'| \le st(t+n)$. Let N be the set of ends of all the edges in M'. For each $i \in X$, choose $a_i \in A_i$ as follows: a_i is adjacent to each vertex in $C_i \cap N$, and nonadjacent to every vertex in $N \setminus C_i$, and the vertices a_i $(i \in I)$ are pairwise nonadjacent. To see that this is possible, observe that, since C_i is $(1-y)$-dense to A_i, and $|N \cap C_i| \le s(t+n)$, there are at most $y|A_i|s(t+n)$ vertices in A_i that have a non-neighbour in $C_i \cap N$; and since $C \setminus C_i$ is $y^{1/6}$-sparse to A_i, there are at most $(2st(t+n)+t)y^{1/6}|A_i|$ vertices in A_i that have a neighbour in $N \setminus C_i$ or are adjacent to some already-selected a_j. Since

$$y|A_i|s(t+n) + (2st(t+n)+t)y^{1/6}|A_i| < |A_i|/2$$

from the definition of η, such a choice of a_i is possible.

For each $j \in J$, let D'_j be the set of vertices in A_j that are adjacent to every vertex in $N \cap C_j$, and nonadjacent to every vertex in $N \setminus C_j$, and nonadjacent to the vertices a_i $(i \in X)$. By the same argument, $|D'_j| \ge |A_j|/2$ for each $j \in Y$. Choose $D_j \subseteq D'_J$ of size $\lceil |A_j|/2 \rceil$. But then $(D_j : j \in Y)$ is an equicardinal $(n, y^d|G|/2)$-blockade, that is symmetrically $2y^{1/6}$-sparse; and there is an (s,t)-handset for it. This proves 7.2. ∎

8 From a sparse blockade to an anticomplete blockade

A blockade is *anticomplete* if every two of its blocks are anticomplete. If $\mathcal{B} = (B_1, \dots, B_k)$ is a blockade in G, we say an induced subgraph H of G is *$\mathcal{B}$-rainbow* if $V(H) \subseteq V(\mathcal{B})$ and $|B_i \cap V(H)| \le 1$ for $1 \le i \le k$. In this section we prove the following (and then we apply it to complete to the proof of 3.4):

8.1 *Let F be a forest, and let $\alpha, \beta > 0$. Then there exist $\alpha', \beta' > 0$ with the following property. Suppose that $0 < y \leq 1$, and G is a graph, and $\mathcal{B}$ is a $(y^{-\alpha}, y^{\beta}|G|)$-blockade in G that is equicardinal and symmetrically y^{α}-sparse. If there is no $\mathcal{B}$-rainbow copy of F, then G admits an anticomplete $(y^{-\alpha'}, y^{\beta'}|G|)$-blockade.*

We need the following theorem of [7]:

8.2 *For every forest F, there is an integer $d > 0$ with the following property. Let G be a graph with a blockade $\mathcal{B}$ of length at least d, and let w be the width of $\mathcal{B}$. If every vertex of G has degree less than w/d, and there is no anticomplete pair $A, B \subseteq V(G)$ with $|A|, |B| \geq w/d$, then there is a $\mathcal{B}$-rainbow copy of F in G.*

This implies:

8.3 *For every forest F, let d be as in 8.2. Let G be a graph with a blockade $\mathcal{B}$ of length at least $3d^2$, and let w be the width of $\mathcal{B}$. If $\mathcal{B}$ is equicardinal and symmetrically $1/d^2$-sparse, and there is no $\mathcal{B}$-rainbow copy of F, then there is an anticomplete pair $A, B \subseteq V(G)$ with $|A|, |B| \geq w$.*

Proof. Let G be a graph with a symmetrically $1/d^2$-sparse equicardinal blockade $\mathcal{B} = (B_1, \ldots, B_D)$ of width w, where $D = 3d^2$. Let G' be the subgraph with vertex set $V(\mathcal{B})$ and edge set the edges of G that have ends that belong to different blocks of $\mathcal{B}$. Thus G' has maximum degree at most $(D-1)w/d^2 \leq 3w$. Partition $\{1, \ldots, D\}$ into d sets of cardinality $3d$, say $I_1, \ldots, I_d$. Let $B'_h = \bigcup_{i \in I_h} B_i$ for $1 \leq i \leq d$; then $\mathcal{B}' = (B'_1, \ldots, B'_d)$ is a $(d, 3wd)$-blockade. We may assume there is no $\mathcal{B}'$-rainbow copy of F; and so by 8.2 applied to $\mathcal{B}'$, there is an anticomplete pair (A, B) with $|A|, |B| \geq 3w$. Choose $i \in \{1, \ldots, k\}$ minimum such that one of

$$A \cap (B_1 \cup \cdots \cup B_i), B \cap (B_1 \cup \cdots \cup B_i)$$

has cardinality at least w, and we may assume that $|A \cap (B_1 \cup \cdots \cup B_i)| \geq w$. From the minimality of i, $|B \cap (B_1 \cup \cdots \cup B_{i-1})| < w$, and since $|B_i| = w$ and $|B| \geq 3w$, it follows that $|B \cap (B_{i+1} \cup \cdots \cup B_k)| \geq w$. But then $A \cap (B_1 \cup \cdots \cup B_i), B \cap (B_{i+1} \cup \cdots \cup B_k)$ is a pair of subsets of $V(G)$ that are anticomplete in G (not just in G'), and both have size at least w. This proves 8.3. ∎

8.4 *Let F be a forest, and let d be as in 8.2; then for every integer $s \geq 1$ and every graph G, the following holds. Let $D = 2(2d^2)^s$, and let $\mathcal{B}$ be a blockade in G of length D, that is equicardinal and symmetrically $2/(2d^2)^s$-sparse, such that there is no $\mathcal{B}$-rainbow copy of F. Then G admits an anticomplete $(2^s, w/(2d^2)^{s-1})$-blockade, where w is the width of $\mathcal{B}$.*

Proof. This is true if $s = 1$, from 8.3, and so we assume it is true for some $s - 1 \geq 1$ and prove it for s. Let $D = 2(2d^2)^s$, and let G be a graph with an equicardinal, symmetrically $2/(2d^2)^s$-sparse blockade $\mathcal{B} = (B_1, \ldots, B_D)$ of width w, and there is no $\mathcal{B}$-rainbow copy of F. Partition $\{1, \ldots, D\}$ into d sets of cardinality D/d, say $I_1, \ldots, I_d$. Let $B'_h = \bigcup_{i \in I_h} B_i$ for $1 \leq i \leq d$; then $\mathcal{B}' = (B'_1, \ldots, B'_d)$ is an equicardinal, symmetrically $2/(2d^2)^s$-sparse $(d, wD/d)$-blockade. Let $G' = G[B_1 \cup \cdots \cup B_D]$. It follows that there is no $\mathcal{B}'$-rainbow copy of F; so from 8.2, there is an anticomplete pair (A, B) of G' with $|A|, |B| \geq wD/d^2$.

Let $D' = 2(2d^2)^{s-1}$, and let I be the set of all $i \in \{1,\dots,D\}$ such that $|A \cap B_i| \geq w/(2d^2)$. Then $|I|w + Dw/(2d^2) \geq |A| \geq wD/d^2$, and so $|I| \geq D/(2d^2) = D'$. For each $i \in I$ choose $C_i \subseteq A \cap B_i$ of cardinality $\lceil w' \rceil$, and let $\mathcal{C}$ be the blockade $(C_i : i \in I)$. Then $\mathcal{C}$ is equicardinal, of width at least $w/(2d^2)$, and it is symmetrically $2/(2d^2)^{s-1}$-sparse, and there is no $\mathcal{C}$-rainbow copy of F. Thus the inductive hypothesis, applied to $\mathcal{C}$, implies that $G[A]$ admits an anticomplete blockade, of width at least $(w/(2d^2))/(2d^2)^{s-2} = w/(2d^2)^{s-1}$ and length 2^{s-1}; and similarly so does $G[B]$. But then combining these gives an anticomplete $(2^s, w/(2d^2)^{s-1})$-blockade in G. This proves 8.4. ∎

We apply this to prove the following strengthened form of 8.1 (in which logarithms are to base two):

8.5 *Let F be a forest, and let $\alpha, \beta > 0$. Let d be as in 8.2, with $d \geq 8$. Define $\alpha' = \alpha/(5\log d)$ and $\beta' = \alpha + \beta$. Suppose that $0 < y \leq 1$, and there is a $(y^{-\alpha}, y^{\beta}|G|)$-blockade $\mathcal{B}$ in a graph G that is equicardinal and symmetrically y^{α}-sparse. If there is no $\mathcal{B}$-rainbow copy of F, then G admits an anticomplete $(y^{-\alpha'}, y^{\beta'}|G|)$-blockade.*

Proof. Choose an integer s maximal such that $y^{-\alpha} \geq 2(2d^2)^s$. It follows that $s \geq 1$, and

$$y^{-\alpha} \leq 2(2d^2)^{s+1} \leq d^{5s}$$

(since $d \geq 8$ and therefore $2^{s+2} \leq d^s$). Thus $\mathcal{B}$ has length at least $2(2d^2)^s$ and is equicardinal and symmetrically $2/(2d^2)^2$-sparse. By 8.4, G admits an anticomplete $(2^s, w/(2d^2)^{s-1})$-blockade, where w is the width of $\mathcal{B}$. But

$$2^s = d^{5s/(5\log d)} \geq y^{-\alpha/(5\log d)} = y^{-\alpha'},$$

and

$$w/(2d^2)^{s-1} \geq w/(2(2d^2)^s) \geq y^{\alpha}w \geq y^{\alpha'}|G|.$$

This proves 8.5. ∎

8.6 *Let F be a forest, and let $\alpha, \beta, \gamma > 0$. Then there exists $\delta' > 0$ such that for all c with $0 < c \leq \delta'$, if G is c-critical, and $\mathcal{B}$ is a $(y^{-\alpha}, y^{\beta}|G|)$-blockade in G that is equicardinal and symmetrically y^{γ}-sparse, then there is a $\mathcal{B}$-rainbow copy of F.*

Proof. By reducing α or γ, we may assume that $\alpha = \gamma$ without loss of generality. Choose α', β' as in 8.1. Choose $\delta' > 0$ such that $\delta' < \alpha'/\beta'$. We claim that δ' satisfies the theorem. Let $0 < c \leq \delta'$, and let $\mathcal{B}$ be a $(y^{-\alpha}, y^{\beta}|G|)$-blockade in a c-critical graph G, that is equicardinal and symmetrically y^{α}-sparse, where $0 < y \leq 1$. Suppose there is no $\mathcal{B}$-rainbow copy of H. By 8.1, G admits an anticomplete $(y^{-\alpha'}, y^{\beta'}|G|)$-blockade $\mathcal{A}$. Let $\mathcal{A} = (A_1, \dots, A_k)$ say. Since G is c-critical, for $1 \leq i \leq k$ there exists a stable subset $C_i \subseteq A_i$ with $|C_i|\omega(G[A_i]) \geq |A_i|^c$, and hence with $|C_i| \geq y^{\beta' c}|G|^c/\omega(G)$. The the union $C_1 \cup \dots \cup C_k$ is stable, and hence has cardinality less than $|G|^c/\omega(G)$; and so, since $k \geq y^{-\alpha'}$, it follows that

$$y^{-\alpha'}y^{\beta' c}|G|^c/\omega(G) \leq |G|^c/\omega(G),$$

that is, $y^{\beta' c - \alpha'} \leq 1$, a contradiction, since $\beta' c - \alpha' < 0$. This proves 8.6. ∎

We use this to complete the proof of 3.4, which we restate:

8.7 *If H, J are Swiss Army graphs, then $\{H, \overline{J}\}$ has the Erdős-Hajnal property.*

Proof. Choose s, t, F such that H, J are both induced subgraphs of F_s^s for some forest F. If $\{F_t^s, \overline{F_t^s}\}$ has the Erdős-Hajnal property, then so does $\{H, J\}$, so we may assume that $H = J = F_t^s$. Let d, δ, η satisfy 7.2. By a theorem of Rödl [19], there exists $\zeta > 0$ such that for every H-free graph G, there exists $S \subseteq V(G)$ with $|S| \geq \zeta|G|$ such that one of $G[S], \overline{G}[S]$ is $\eta/2$-sparse. Choose $\rho > 0$ such that $\zeta \geq \eta^\rho$ and $\rho \geq |H|^2$. Choose $\alpha = 1/128$, $\beta = 2d + \rho + 1$, $\gamma = 1/12$, and choose δ' to satisfy 8.6. Choose c with $0 < c \leq \delta'$ such that $4(\rho + 1)c \leq 1$; we will show that every c-critical graph contains one of $H, \overline{H}$ as an induced subgraph, and hence the theorem holds.

Suppose that G is a c-critical graph that is $\{H, \overline{H}\}$-free. It follows that $|G|^c \geq 2$, and so $|G| \geq 2^{1/c} \geq 2^{4(\rho+1)}$. By Rödl's theorem and replacing G by its complement if necessary, we may assume that there exists $S \subseteq V(G)$ with $|S| \geq \zeta|G|$ such that $G[S]$ is η-sparse. Consequently there exists y with $0 < y \leq \eta$ such that there is a subset $S \subseteq V(G)$ with $|S| \geq y^\rho|G|$, such that $G[S]$ is $y/2$-sparse. Call such a value of y *good*. Suppose there is a good value of y with $|G|^{-2c}/2 \leq y \leq |G|^{-c}$, and let S be the corresponding subset. Since

$$y|S| > y^{\rho+1}|G| \geq |G|^{1-2c(\rho+1)}2^{-\rho-1} \geq |G|^{1/2}|G|^{-1/4} \geq 2$$

and $G[S]$ has maximum degree at most $y|S|/2$, S includes a stable set of size at least $|S|/(y|S|/2+1) \geq 1/y$ (since $y|S| \geq 2$). But $1/y \geq |G|^c$, contradicting that G is c-critical.

On the other hand, η is good, and $\eta > |G|^{-2c}$. Choose a good value of y with $|G|^{-2c} \leq y \leq \eta$ and minimal with this property. It follows that $y > |G|^{-c}$, and $y^2/2$ is not good.

From 7.2, taking $Z = S$ and with y replaced by $y/2$, we deduce that either:

- there is a subset $S' \subseteq S$ with $|S'| \geq (y/2)^{|H|^2}|S|$ that is $(y/2)^2$-sparse; or
- there is an equicardinal, symmetrically $2(y/2)^{1/6}$-sparse $((y/2)^{-1/64}, (y/2)^d|S|/2)$-blockade $\mathcal{B}$ in G, and there is an (s, t)-handset for $\mathcal{B}$.

Suppose the first holds. Then $|S'| \geq (y/2)^{|H|^2}|S| \geq (y/2)^{|H|^2}y^\rho|G| \geq (y^2/2)^\rho|G|$, and so $y^2/2$ is good, a contradiction. Thus the second holds. Since $(y/2)^{-1/64} \geq y^{-1/128} = y^{-\alpha}$, and $(y/2)^d|S|/2 \geq y^{2d+\rho}|G|/2 \geq y^\beta|G|$, and $2(y/2)^{1/6} \leq y^\gamma$, there is a symmetrically y^γ-sparse equicardinal $(y^{-\alpha}, y^\beta|G|)$-blockade $\mathcal{B}$ in G, and there is an (s, t)-handset for $\mathcal{B}$. By 8.6, there is a $\mathcal{B}$-rainbow copy of F; and combining this with the handset gives a copy of F_t^s, a contradiction. This proves 8.7. ∎

References

[1] N. Alon, J. Pach and J. Solymosi, "Ramsey-type theorems with forbidden subgraphs", *Combinatorica* **21** (2001), 155–170.

[2] M. Bonamy, N. Bousquet, and S. Thomassé, "The Erdős-Hajnal conjecture for long holes and antiholes", *SIAM J. Discrete Math.*, **30** (2016), 1159–1164.

[3] M. Bucić, T. Nguyen, A. Scott, and P. Seymour, "Induced subgraph density. I. A loglog step towards Erdős-Hajnal", *IMRN*, to appear, `arXiv:2301.10147`.

[4] M. Bucić, J. Fox, H. T. Pham, "Equivalence between Erdős-Hajnal and polynomial Rödl and Nikiforov conjectures", arxiv:2403.08303, 2024

[5] M. Chudnovsky and M. Safra, "The Erdős-Hajnal conjecture for bull-free graphs", *J. Combinatorial Theory, Ser. B*, **98** (2008), 1301–1310.

[6] M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl, "Erdős-Hajnal for graphs with no five-hole", *Proceedings of the London Math. Soc.*, **126** (2023), 997–1014, `arXiv:2102.04994`.

[7] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs", *Advances in Math.*, **375** (2020), 107396, `arXiv:1809.00919`.

[8] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. II. Excluding all subdivisions of a graph", *Combinatorica* **41** (2021), 279–405, `arXiv:1804.01060`.

[9] J. Davies, "Pivot-minors and the Erdős-Hajnal conjecture", `arXiv:2305.09133`.

[10] P. Erdős and A. Hajnal, "On spanned subgraphs of graphs", *Graphentheorie und Ihre Anwendungen* (Oberhof, 1977), https://old.renyi.hu/~p_erdos/1977-19.pdf.

[11] P. Erdős and A. Hajnal, "Ramsey-type theorems", *Discrete Applied Mathematics* **25** (1989), 37–52.

[12] J. Fox and B. Sudakov, "Induced Ramsey-type theorems", *Advances in Mathematics* **219** (2008), 1771–1800.

[13] J. Fox, T. Nguyen, A. Scott and P. Seymour, "Induced subgraph density. II. Sparse and dense sets in cographs", submitted for publication, `arXiv:2307.00801`.

[14] T. Nguyen, A. Scott and P. Seymour, "Induced subgraph density. IV. New graphs with the Erdős-Hajnal property", submitted for publication, `arXiv:2307.06455`.

[15] T. Nguyen, A. Scott, and P. Seymour, "Induced subgraph density. V. All paths approach Erdős–Hajnal", submitted for publication, `arXiv:2307.15032`.

[16] T. Nguyen, A. Scott, and P. Seymour, "Induced subgraph density. VI. Bounded VC-dimension", submitted for publication, `arXiv:2312.15572`.

[17] T. Nguyen, A. Scott and P. Seymour, "Induced subgraph density. VII. The five-vertex path", submitted for publication, `arXiv:2312.15333`.

[18] T. Nguyen, *Induced Subgraph Density*, Ph.D. thesis, Princeton University, May 2025, in preparation.

[19] V. Rödl, "On universality of graphs with uniformly distributed edges", *Discrete Mathematics* **59** (1986), 125–134.