Polynomial bounds for chromatic number. III. Excluding a double star

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Abstract

A "double star" is a tree with two internal vertices. It is known that the Gyárfás-Sumner conjecture holds for double stars, that is, for every double star H, there is a function f_H such that if G does not contain H as an induced subgraph then $\chi(G) \leq f_H(\omega(G))$ (where χ, ω are the chromatic number and the clique number of G). Here we prove that f_H can be chosen to be a polynomial.

1 Introduction

A class of graphs is *hereditary* if it is closed under isomorphism and under taking induced subgraphs. A hereditary class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph $G \in \mathcal{C}$, where $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G. There is a large literature addressing the question of which graph classes are χ-bounded, and many open questions (see [17] for a survey).

Hereditary classes defined by excluding some fixed graph H are of particular interest. If G, H are graphs, we say G is *H-free* if no induced subgraph of G is isomorphic to H. It is easily seen that if the class of H-free graphs is χ-bounded then H must be a forest, as Erdős [4] showed that there are graphs with arbitrarily large girth and chromatic number. The famous Gyárfás-Sumner conjecture [9, 21] asserts the converse:

1.1 Conjecture: *For every forest H, there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

The Gyárfás-Sumner conjecture remains open in general, though it has been proved for some very restricted families of trees (see, for example, [3, 10, 11, 12, 13, 15, 16, 17]). In particular, it was proved by Kierstead and Penrice [12] for trees of radius two.

Louis Esperet [8] made the striking conjecture that, for every χ-bounded class $\mathcal{C}$, the function f can be chosen to be a polynomial (see the survey by Schiermeyer and Randerath [20] for results on polynomial χ-boundedness). In particular, if it had been true, this conjecture would imply a strengthening of the Gyárfás-Sumner conjecture, that the function f in 1.1 can always be chosen to be a polynomial. Esperet's conjecture has recently been shown to be false, by Briański, Davies and Walczak [2], but its specialization to the Gyárfás-Sumner conjecture is still open. Even this is a bold conjecture, as frequently, when classes are known to be χ-bounded, the best known function f grows quite rapidly, often because the proofs use multiple applications of Ramsey-type results. Nevertheless, Esperet's strengthening has been verified for some cases of the Gyárfás-Sumner conjecture, for instance when H is a star, or a four-vertex path, or a matching (see [20]); and recently it has been shown when H is obtained from a star by subdividing one edge once [14], and when H is a forest of stars [19].

A *double star* is a tree in which at most two vertices have degree more than one. Double stars have radius at most two, and so the result of Kierstead and Penrice [12] shows that the class of H-free graphs is χ-bounded whenever H is a double star. In this paper, we prove a polynomial bound. Our main result is:

1.2 *For every double star H, there is a polynomial f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

This extends a theorem of Liu, Schroeder, Wang and Yu [14], who proved the same for double stars H that have at most one vertex of degree more than two.

Our result is partially motivated by the Erdős-Hajnal conjecture. In view of the recent result for C_5-free graphs [5], the five-vertex path P_5 is the smallest open case of this conjecture. It is known that P_5 satisfies the Gyárfás-Sumner conjecture (in fact the Gyárfás-Sumner conjecture holds whever H is a path), and if P_5 satisfies Esperet's strengthening then P_5 also satisfies the Erdős-Hajnal conjecture. Thus P_5 appears likely to be a sticking point. We have not settled that; but this paper proves Esperet's strengthening for all trees that do not contain P_5.

We use standard notation throughout. When $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X. We write $\chi(X)$ for $\chi(G[X])$, and ω for $\omega(G)$, when there is no ambiguity.

2 A degeneracy variant of defective colouring

A graph G is *d-degenerate*, or has *degeneracy at most d*, if every non-null subgraph H has a vertex with degree (in H) at most d. Every d-degenerate graph has chromatic number at most $d + 1$.

Let us say a (k, d)*-colouring* of a graph is a partition $(A_1, \dots, A_k)$ of the vertex set $V(G)$, such that for $1 \le i \le k$, the subgraph induced on A_i has degeneracy at most d; and we say that G or $V(G)$ is (k, d)*-colourable* if there is such a partition. Thus, if a graph is (k, d)-colourable, its chromatic number is at most $k(d+1)$. We call this "degenerate colouring"; it is a relative of "defective colouring", where we ask that the subgraph induced on each A_i has maximum degree at most d, but it is not exactly the same (see [22] for a survey of defective colouring). Let us explain why we need to use degenerate colourings.

A standard way to bound the chromatic number of a graph G is to partition $V(G)$ into some number of parts $V_1, V_2, \dots$, and bound the chromatic numbers of the parts separately, and add to get a bound on $\chi(G)$. But we will be trying to prove that $\chi(G)$ is at most $\omega(G)^d$, for some appropriately large constant d. So for this "addition" method to work when ω is large, if the best bound we know for one of the parts is something like $(\omega(G) - 1)^d$, we would need much better bounds for all the other parts.

Fix a double star H, and choose a large constant d; and suppose that we try to prove by induction on $\omega(G)$ that every H-free graph G has chromatic number at most $\omega(G)^d$. The proof (by our method) does not work. There comes a stage where $V(G)$ is partitioned into an unbounded number of parts $V_1, V_2, \dots$. We will know, from the induction on $\omega(G)$, that each part has chromatic number at most something like $(\omega - 1)^d$ (where $\omega = \omega(G)$), but we will not know a better bound for *any* of the parts. The "addition" method given above will therefore fail miserably. But we will know something about the edges between parts, which we might hope will save us (though in fact it will not). We will know that for each part V_i, each of its vertices has only a small number of neighbours in the union of the later parts $V_{i+1} \cup \cdots \cup V_n$; say at most ω^r neighbours, where r is much less than d. Of course if there were no edges between the parts, all would be fine, and one might hope that similarly, because of the sparseness of the edges between parts, the effect of these edges could be fitted into the difference between ω^d and $(\omega - 1)^d$. But we can't do this (at least with no further information); the effect is multiplicative rather than additive. Even if for each i, each vertex of V_i has only at most one neighbour in $V_{i+1} \cup \cdots \cup V_n$, the chromatic number of the union of the parts might be $3/2$ times the maximum chromatic number of the individual parts, which is much too big.

Thus the inductive proof that every H-free graph G has chromatic number at most $\omega(G)^d$ fails; and for that reason we will instead prove by induction a stronger statement, about degenerate colourings. We will prove by induction on ω that if G is H-free, then G is (ω^d, ω^{r+1})-colourable. Then, when the situation above arises, we will know that each part admits an $((\omega - 1)^d, (\omega - 1)^{r+1})$-colouring and hence an $(\omega^d, (\omega - 1)\omega^r)$-colouring. The union of these colourings becomes an (ω^d, ω^{r+1})-colouring, which is what we want, and now it all works. Induction on $\omega(G)$ will be used to prove the statement about degenerate colouring, and then we deduce the statement about normal colouring at the end.

Let us state formally the lemma we just mentioned:

2.1 *Let $k, d, d' \ge 0$ be integers. Let $V(G)$ be partitioned into $V_1, V_2, \dots, V_n$, such that*

- *for $1 \le i \le n$, $G[V_i]$ admits a (k,d)-colouring; and*
- *for $1 \le i < n$, every vertex of V_i has at most d' neighbours in $V_{i+1} \cup \cdots \cup V_n$.*

Then G admits a $(k, d+d')$-colouring.

The proof is clear.

3 Templates

The paper by Kierstead and Penrice [12] uses the method of "templates", an idea that was introduced in [11] and has been applied in several papers to prove special cases of 1.1. We will use the same idea, but substantially modified to keep the numbers polynomial. Fix an integer s. (Eventually $s+1$ will be the maximum degree in the double star we are excluding.) We say an *s-template* in a graph G is a sequence $\mathcal{L}$ of pairwise disjoint subsets $(L_0, L_1, \ldots, L_k)$, with $k \ge 0$, such that:

- L_0 is a clique of G (possibly empty), and every vertex in L_0 is adjacent to every vertex in $L_1 \cup \cdots \cup L_k$;
- $\omega^{s+5} \le |L_i| \le 14\omega^{s+6}$ for $1 \le i \le k$; and
- for all distinct $i, j \in \{1, \ldots, k\}$, each vertex in L_i has at most ω^{s+3} non-neighbours in L_j.

We say that k is the *length* of the s-template, and define its *value* to be

$$|L_1 \cup \cdots \cup L_k| + 7\omega^{s+5}|L_0| + k\omega^{s+5}.$$

Let us define $V(\mathcal{L}) = L_0 \cup L_1 \cup \cdots \cup L_k$, and $N(\mathcal{L})$ to be the set of vertices in $V(G) \setminus V(\mathcal{L})$ that have a neighbour in $L_1 \cup \cdots \cup L_k$. (Note that we do not consider neighbours in L_0.)

The idea of the proof is as follows. Let G be an H-free graph (where H is a double star). Suppose that G has chromatic number at least some huge (but some fixed constant) power of ω. It follows from a theorem of an earlier paper [18] that G has a subgraph which is a complete bipartite graph, in which both parts have cardinality $14\omega^{s+6}$. Consequently G contains an s-template of length two and value at least $28\omega^{s+6}$. Thus we may choose an s-template $\mathcal{L}$ with maximum value, and its value will also be at least $28\omega^{s+6}$. We have tuned the various numbers in the definition of an s-template so that such s-templates have many useful properties.

The bulk of the proof is to bound the chromatic number of the set $N(\mathcal{L})$ (more exactly, to show it admits a certain degenerate colouring). The main idea for this part of the proof is: let us say $v \in N(\mathcal{L})$ is "pendant" if there exist distinct $i, j \in \{1, \ldots, k\}$, and a vertex $u \in L_j$, and a stable set S of $s+1$ vertices in L_i, all adjacent to u, such that v is not adjacent to u, and v has exactly one neighbour in S. There cannot be many pendant vertices, because there otherwise there would be many using the same i, j, u, S and we could find a copy of the excluded double star. But if a vertex in $N(\mathcal{L})$ is not pendant, its neighbour set in $L_1 \cup \cdots \cup L_k$ is highly restricted, and we can exploit this to find an appropriate degenerate colouring of all the non-pendant vertices, and hence of all of $N(\mathcal{L})$.

It remains to use this to find a degenerate colouring of the whole of G. Let Z be the union of L_0 and the set of vertices in $N(\mathcal{L})$ that have only a few non-neighbours in each of $L_1, \ldots, L_k$ (a "few" means a small constant power of ω). We can show that $|Z|$ is at most another small power of ω, from the optimality of the template; and that every vertex in $N(\mathcal{L})$ with at least a few neighbours that are not in $V(\mathcal{L}) \cup N(\mathcal{L})$ must belong to Z, because otherwise we could find a copy of the double star. So every vertex in $(V(\mathcal{L}) \cup N(\mathcal{L})) \setminus Z$ has only a few neighbours not in this set. Now delete this set and do it again, as often as possible. When we can no longer find s-templates of sufficiently large value, the part of the graph that remains has bounded chromatic number, by the theorem of [18]; so we have constructed a sequence of subsets of $V(G)$ that partition $V(G)$, each with a degenerate colouring, and we can apply 2.1 to this sequence of subsets to obtain the desired result.

We begin in this section by proving some properties of *optimal* s-templates, that is, s-templates chosen with maximum value.

3.1 *Let $(L_0, L_1, \ldots, L_k)$ be an s-template in G. There is a clique of G with one vertex in each of $L_1, \ldots, L_k$, and consequently $k + |L_0| \le \omega$.*

Proof. We may assume that $k > 0$. Choose $v_1 \in L_1$; and inductively for $2 \le i \le k$, having chosen $v_1, \ldots, v_{i-1}$, choose $v_i \in L_i$ as follows. There are at most ω^{s+3} vertices in L_i nonadjacent to v_h, for $1 \le h < i$; and since $\{v_1, \ldots, v_{i-1}\}$ is a clique and therefore $i - 1 \le \omega$, it follows that $\omega^{s+3}(i-1) \le \omega^{s+4} < \omega^{s+5} \le |L_i|$. Consequently there exists $v_i \in L_i$ adjacent to all of $v_1, \ldots, v_{i-1}$. This completes the inductive definition of $v_1, \ldots, v_k$. Hence $L_0 \cup \{v_1, \ldots, v_k\}$ is a clique, and so $k + |L_0| \le \omega$. This proves 3.1. ∎

3.2 *Let $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ be an optimal s-template in G. If $\mathcal{L}$ has value at least $28\omega^{s+6}$, then $k \ge 2$, and $|L_i| \ge 5\omega^{s+5}$ for $1 \le i \le k$.*

Proof. If $k = 0$, the s-template has value $7\omega^{s+5}|L_0| \le 7\omega^{s+6} < 28\omega^{s+6}$, a contradiction. If $k = 1$, then since $|L_1| \le 14\omega^{s+6}$, and $|L_0| + k \le \omega$ by 3.1, the s-template has value at most

$$14\omega^{s+6} + 7\omega^{s+5}|L_0| + \omega^{s+5} \le 14\omega^{s+6} + 7\omega^{s+5}(\omega - 1) + \omega^{s+5} \le 21\omega^{s+6} < 28\omega^{s+6},$$

a contradiction. So $k \ge 2$.

By reordering $L_1, \ldots, L_k$ we may assume that $|L_1|, \ldots, |L_h| \ge 5\omega^{s+5}$ and $|L_{h+1}|, \ldots, |L_k| < 5\omega^{s+5}$. We will show that $h = k$. For $h + 1 \le i \le k$, choose $v_i \in L_i$ such that $\{v_{h+1}, \ldots, v_k\}$ is a clique X (this is possible by 3.1). For $1 \le i \le h$, let L'_i be the set of all vertices in L_i that are adjacent to every vertex of X. Since each vertex of X has at most ω^{s+3} non-neighbours in L_i, it follows that

$$|L'_i| \ge |L_i| - \omega^{s+4} \ge 5\omega^{s+5} - \omega^{s+5} \ge \omega^{s+5}$$

for $1 \le i \le h$. Consequently

$$(L_0 \cup \{x_{h+1}, \ldots, x_k\}, L'_1, \ldots, L'_h)$$

is an s-template in G. Its value is that of $(L_0, L_1, \ldots, L_k)$ plus

$$7\omega^{s+5}(k-h) - \omega^{s+5}(k-h) - \left(|L_1 \cup \cdots \cup L_k| - |L'_1 \cup \cdots \cup L'_h|\right),$$

and so this is at most zero, since $(L_0, \ldots, L_k)$ is optimal. But

$$|L_1 \cup \cdots \cup L_k| - |L'_1 \cup \cdots \cup L'_h| \le \sum_{1 \le i \le h} \left(|L_i| - |L'_i|\right) + \sum_{h+1 \le i \le k} |L_i| \le h\omega^{s+4} + (k-h)(5\omega^{s+5} - 1),$$

and consequently

$$7\omega^{s+5}(k-h) - \omega^{s+5}(k-h) \leq h\omega^{s+4} + (k-h)(5\omega^{s+5} - 1),$$

that is,

$$(\omega^{s+5} + 1)(k-h) \leq h\omega^{s+4}.$$

But $\omega^{s+5} \geq h\omega^{s+4}$, and so $(\omega^{s+5}+1)(k-h) \leq \omega^{s+5}$, which implies that $h = k$. This proves 3.2. ∎

3.3 *Let $(L_0, L_1, \dots, L_k)$ be an optimal s-template in G. If its value is at least $28\omega^{s+6}$, then for each $i \in \{1, \dots, k\}$, every vertex in L_i has at least $4\omega^{s+5}$ non-neighbours in L_i.*

Proof. Let $i = 1$ say, and let $v \in L_1$. Let L_1' be the set of neighbours of v in L_1, and $M = L_1 \setminus (L_1' \cup \{v\})$. We will show that $|M| \geq 4\omega^{s+5}$. If $|L_1'| < \omega^{s+5}$, then v has at least $|L_1| - \omega^{s+5}$ non-neighbours in L_1; and since $|L_1| \geq 5\omega^{s+5}$ by 3.2, it follows that $|M| \geq 4\omega^{s+5}$ as required. Thus we may assume that $|L_1'| \geq \omega^{s+5}$. For $2 \leq i \leq k$ let L_i' be the set of vertices in L_i adjacent to v; thus by 3.2,

$$|L_i'| \geq |L_i| - \omega^{s+3} \geq 5\omega^{s+5} - \omega^{s+3} \geq \omega^{s+5}$$

for $2 \leq i \leq k$. Consequently

$$(L_0 \cup \{v\}, L_1', L_2', \dots, L_k')$$

is an s-template. From the optimality of $(L_0, \dots, L_k)$, it follows that $7\omega^{s+5} - (|M|+1) - (k-1)\omega^{s+3} \leq 0$, and so

$$|M| \geq 7\omega^{s+5} - 1 - (k-1)\omega^{s+3} \geq 4\omega^{s+5}.$$

This proves 3.3. ∎

3.4 *Let $(L_0, L_1, \dots, L_k)$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. Suppose that $\omega \geq 4$, and A, B are disjoint subsets of L_1, with $|L_1 \setminus (A \cup B)| \leq \omega^{s+3}$, such that every vertex in A has fewer than ω^s non-neighbours in B. Then either $|B| < 14\omega^{s+1}$ or $A = \emptyset$.*

Proof. Suppose that $|B| \geq 14\omega^{s+1}$.

(1) $|B| \geq 2\omega^{s+5}$.

Suppose that $|B| < 2\omega^{s+5}$. Each vertex in B has at least $4\omega^{s+5}$ non-neighbours in L_1, by 3.3, and only at most $2\omega^{s+5} + \omega^{s+3}$ of them do not belong to A; and since $2\omega^{s+5} - \omega^{s+3} \geq \omega^{s+5}$, there are at least $\omega^{s+5}|B| \geq 14\omega^{2s+6}$ nonedges between B and A. Since $|A| \leq |L_1| \leq 14\omega^{s+6}$, some vertex in A has at least ω^s non-neighbours in B, a contradiction. This proves (1).

(2) $|A| < \omega^{s+5}$.

Suppose that $|A| \geq \omega^{s+5}$. Let B' be the set of all vertices in B with at most ω^{s+3} non-neighbours in A. Since there are only at most $\omega^s|A|$ nonedges between A, B, there are at most $|A|/\omega^3$ vertices in B that have more than ω^{s+3} non-neighbours in A; and so $|B'| \geq |B| - |A|/\omega^3 \geq \omega^{s+5}$, by (1) and since $|A|/\omega^3 \leq 14\omega^{s+3} < \omega^{s+5}$ (the last because $\omega \geq 4$). Hence

$$(L_0, A, B', L_2, \ldots, L_k)$$

is an s-template, and the optimality of $(L_0, \ldots, L_k)$ implies that

$$|A| + |B'| + \omega^{s+5} \leq |L_1| \leq |A| + |B'| + \omega^{s+3} + |A|/\omega^3,$$

and so $\omega^{s+5} \leq \omega^{s+3} + 14\omega^{s+3}$, a contradiction. This proves (2).

Suppose that $A \neq \emptyset$, and choose $v \in A$. Since v has at least $4\omega^{s+5}$ non-neighbours in L_1 by 3.3, and at most ω^{s+5} of them belong to A by (2), and at most ω^{s+3} are not in $A \cup B$, it follows that v has at least $3\omega^{s+5} - \omega^{s+3} \geq \omega^s$ non-neighbours in B, a contradiction. Thus $A = \emptyset$. This proves 3.4. ∎

3.5 *Let $\mathcal{L} = (L_0, L_1, \ldots, L_k)$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. There are fewer than $14\omega^{s+6}$ vertices $v \in N(\mathcal{L})$ such that for each $i \in \{1, \ldots, k\}$, v has at most $\omega^{s+2}/4$ non-neighbours in L_i.*

Proof. Suppose that there is a set $M \subseteq N(\mathcal{L})$ with $|M| = 14\omega^{s+6}$, such that for each $v \in M$ and for each $i \in \{1, \ldots, k\}$, v has at most $\omega^{s+2}/4$ non-neighbours in L_i. For $1 \leq i \leq k$, there are at most $|M|\omega^{s+2}/4 = 7\omega^{2s+8}/2$ nonedges between M and L_i; and so there are at most $7\omega^{s+5}/2$ vertices in L_i with at least ω^{s+3} non-neighbours in M. Let L'_i be the set of vertices in L_i that have fewer than ω^{s+3} non-neighbours in M. Hence $|L'_i| \geq |L_i| - (7\omega^{s+5}/2) \geq \omega^{s+5}$, since $|L_i| \geq 5\omega^{s+5}$ by 3.2. It follows that

$$(\emptyset, L'_1, \ldots, L'_k, M)$$

is an s-template. Its value is

$$|L'_1 \cup \cdots \cup L'_k| + |M| + (k+1)\omega^{s+5} \geq |L_1 \cup \cdots \cup L_k| - k(7\omega^{s+5}/2) + 14\omega^{s+6} + (k+1)\omega^{s+5}$$

and the optimality of $(L_0, \ldots, L_k)$ implies that

$$|L_1 \cup \cdots \cup L_k| - k(7\omega^{s+5}/2) + 14\omega^{s+6} + (k+1)\omega^{s+5} \leq |L_1 \cup \cdots \cup L_k| + 7\omega^{s+5}|L_0| + k\omega^{s+5},$$

that is,

$$2\omega + 1/7 \leq |L_0| + k/2,$$

contrary to 3.1. This proves 3.5. ∎

4 Using the double star

We are concerned with graphs that do not contain some fixed double star, but so far we have not used that fact. For $s \geq 1$, let H_s be the double star with $2s+2$ vertices, with two internal vertices both of degree $s+1$. Every double star is an induced subgraph of H_s for some s, so it suffices to prove the result for H_s-free graphs.

We will need the following result of [18]:

4.1 *Let H be a forest. Then there exists $c > 0$ such that for every H-free graph G and every integer $t \geq 0$, either G contains the complete bipartite graph $K_{t,t}$ as a subgraph, or G has degeneracy less than t^c, and hence has chromatic number at most t^c.*

We will also need the following version of Ramsey's theorem (well-known, but proved for instance in [19]):

4.2 *If $s \geq 0$ is an integer, then every graph G with no stable set of cardinality s has at most*

$$\omega^{s-1} + \omega^{s-2} + \cdots + \omega$$

vertices, and hence fewer than ω^s vertices if $\omega > 1$.

Let $\mathcal{L} = (L_0, \ldots, L_k)$ be an optimal s-template in G. With respect to this template, we say a vertex $v \in N(\mathcal{L})$ is

- *pendant* if there exist distinct $i, j \in \{1, \ldots, k\}$, and a vertex $u \in L_j$, and a stable set S of $s+1$ vertices in L_i, all adjacent to u, such that v is not adjacent to u, and v has exactly one neighbour in S;
- *dense* if there exist $j \in \{1, \ldots, k\}$ and $u \in L_j$, such that for all $i \in \{1, \ldots, k\} \setminus \{j\}$, there are fewer than $\omega^{s+2}/14$ vertices in L_i that are adjacent to u and not to v;
- *pure* if there are least two values of $i \in \{1, \ldots, k\}$ such that v has no neighbour in L_i, and for $1 \leq j \leq k$, either v has no neighbour in L_j or v has at most $\omega^{s+2}/7$ non-neighbours in L_j.

4.3 *Let $s \geq 1$ be an integer, and let G be H_s-free, with $\omega \geq 200$. Let $\mathcal{L} = (L_0, \ldots, L_k)$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. Then every vertex in $N(\mathcal{L})$ is pendant or dense or pure with respect to $\mathcal{L}$.*

Proof. Let $v \in N(\mathcal{L})$, and suppose that v is neither dense nor pendant with respect to $\mathcal{L}$. We will prove that v is pure.

(1) *For all distinct $i, j \in \{1, \ldots, k\}$, if $u \in L_j$ is nonadjacent to v, then either u, v have no common neighbour in L_i, or fewer than $\omega^{s+2}/14$ neighbours of u in L_i are nonadjacent to v.*

Let A be the set of all vertices in L_i adjacent to both u, v, and let B be the set of vertices in L_i adjacent to u and not to v. Suppose that $A \neq \emptyset$, and $|B| \geq \omega^{s+2}/14$. Since $|L_i \setminus (A \cup B)| \leq \omega^{s+3}$ (because u has at most ω^{s+3} non-neighbours in L_i), and $|B| \geq \omega^{s+2}/14 \geq 14\omega^{s+1}$ (because $\omega \geq 200$), 3.4 implies that some vertex $w \in A$ has at least ω^s non-neighbours in B. By 4.2, this set of non-neighbours includes a stable set of size s, contradicting that v is not pendant. Thus either $A = \emptyset$ or

$|B| < \omega^{s+2}/14$. This proves (1).

(2) *For all $j \in \{1,\dots,k\}$, if v has a non-neighbour in L_j, there exists $i \in \{1,\dots,k\}$ with $i \neq j$ such that v has at most ω^{s+3} neighbours in L_i.*

Choose $u \in L_j$ nonadjacent to v. Since v is not dense, there exists $i \in \{1,\dots,k\}$ with $i \neq j$ such that there are at least $\omega^{s+2}/14$ vertices in L_i adjacent to u and not to v. By (1), u, v have no common neighbour in L_i, and hence v has at most ω^{s+3} neighbours in L_i. This proves (2).

(3) *For $1 \leq j \leq k$, if v has at most ω^{s+3} neighbours in L_j, then v has no neighbours in L_j.*

By (2), there exists $i \in \{1,\dots,k\}$ different from j, such that v has at most ω^{s+3} neighbours in L_i. Suppose that v has a neighbour $w \in L_j$. Since w has at most ω^{s+3} non-neighbours in L_i, and v has at most ω^{s+3} neighbours in L_i, and $|L_i| \geq \omega^{s+5} > 2\omega^{s+3}$, there exists $x \in L_i$ adjacent to w and not to v. Now w has at least $4\omega^{s+5}$ non-neighbours in L_j, by 3.3; at most ω^{s+3} of them are nonadjacent to x, and at most ω^{s+3} of them are adjacent to v, and so at least $4\omega^{s+5} - 2\omega^{s+3} \geq \omega^s$ of them are nonadjacent to v and adjacent to x. By 4.2, this set includes a stable set of size s, and so v is pendant, a contradiction. Thus v has no neighbour in L_j. This proves (3).

(4) *For $1 \leq i \leq k$, either v has at most $\omega^{s+2}/7$ non-neighbours in L_i, or v has no neighbours in L_i.*

Let A_i, B_i be the sets of neighbours and non-neighbours respectively of v in L_i, and suppose that $A_i \neq \emptyset$, and $|B_i| > \omega^{s+2}/7$. By (2) and (3), there exists $j \in \{1,\dots,k\}$ with $j \neq i$ such that v has no neighbours in L_j. By (1), for each $u \in L_j$, either u has no neighbours in A_i, or u has at most $\omega^{s+2}/14$ neighbours in B_i. Let X be the set of vertices in L_j with no neighbour in A_i, and $Y = L_j \setminus X$. Since $A_i \neq \emptyset$, and a vertex in A_i has at most ω^{s+3} non-neighbours in L_j, it follows that $|X| \leq \omega^{s+3}$, and so $|Y| \geq |L_j| - \omega^{s+3}$. Every vertex in Y is adjacent to at most half the vertices in B_i (since $|B_i| \geq \omega^{s+2}/7$), and so some vertex $b \in B_i$ is adjacent to at most half the vertices in Y. Since b has at most ω^{s+3} non-neighbours in L_j, it follows that $|Y|/2 \leq \omega^{s+3}$; but $|X| \leq \omega^{s+3}$, and so $|L_j| \leq 3\omega^{s+3}$, a contradiction. This proves (4).

Since v is not dense, it has a non-neighbour in one of $L_1, \dots, L_k$; and so by (2) and (3), it has no neighbours in some L_j. By (2), v has at most ω^{s+3} neighbours in L_i for some $i \neq j$; and so has no neighbours in L_i by (3). From (4) it follows that v is pure. This proves 4.3. ∎

4.4 *Let $s \geq 1$ be an integer, and let G be H_s-free. Let $(L_0, \dots, L_k)$ be an optimal s-template in G. There are at most $14^{s+2}\omega^{s^2+9s+14}$ pendant vertices.*

Proof. Let $i, j \in \{1,\dots,k\}$ be distinct, let $u \in L_j$, let $S \subseteq L_i$ be a stable set of $s+1$ neighbours of u, and let $w \in S$. Let $X(i,j,u,S,w)$ be the set of all $v \in V(G) \setminus (L_0 \cup \dots \cup L_k)$ such that v is adjacent to w and is nonadjacent to all other vertices in $S \cup \{u\}$. If $X(i,j,u,S,w)$ includes a stable set of size s, say T, then the subgraph induces on $S \cup T \cup \{u\}$ is isomorphic to H_s, a contradiction. Thus, 4.2 implies that $|X(i,j,u,S,w)| \leq \omega^s$. Since there are only $k^2(14\omega^{s+6})^{s+2}$ choices for i, j, u, S, w, and every pendant vertex belongs to $X(i,j,u,S,w)$ for some choice of i,j,u,S,w, it follows that there are at most $k^2(14\omega^{s+6})^{s+2}\omega^s$ pendant vertices. Since $k \le \omega$ by 3.1, this proves 4.4. ∎

4.5 *Let $s \ge 1$ be an integer, and let G be H_s-free. Let $\mathcal{L} = (L_0, \dots, L_k)$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. Let c satisfy 4.1 when $H = H_s$. The chromatic number of the set of all dense vertices is at most $(14\omega^{s+6})^{c+1}\omega$.*

Proof. Let $1 \le j \le k$, let $u \in L_j$, and let $X(j,u)$ be the set of all $v \in N(\mathcal{L})$ such that for all $i \in \{1,\dots,k\} \setminus \{j\}$, there are fewer than $\omega^{s+2}/14$ vertices in L_i that are adjacent to u and not to v.

Suppose that $\chi(X(k,u)) > (14\omega^{s+6})^c$ for some $u \in L_k$. Then by 4.1, $G[X(k,u)]$ contains a copy of $K_{14\omega^{s+6},14\omega^{s+6}}$ as a subgraph; let M_1, M_2 be disjoint subsets of $X(k,u)$, both of cardinality $14\omega^{s+6}$, such that every vertex of M_1 is adjacent to every vertex of M_2. For $1 \le i \le k-1$, let L_i' be the set of vertices in $L_i \cap N(u)$ that have at most ω^{s+3} non-neighbours in M_1 and at most ω^{s+3} non-neighbours in M_2. There are at most $(\omega^{s+2}/14)14\omega^{s+6}$ nonedges between M_1 and $L_i \cap N(u)$, and it follows that at most ω^{s+5} vertices in $L_i \cap N(u)$ have more than ω^{s+3} non-neighbours in M_1; and the same for M_2. Consequently

$$|L_i'| \ge |L_i \cap N(u)| - 2\omega^{s+5} \ge 5\omega^{s+5} - \omega^{s+3} - 2\omega^{s+5} \ge \omega^{s+5},$$

and so

$$(\emptyset, L_1', \dots, L_{k-1}', M_1, M_2)$$

is an s-template. Its value is

$$|L_1' \cup \dots \cup L_{k-1}'| + |M_1| + |M_2| + (k+1)\omega^{s+5} \ge |L_1 \cup \dots \cup L_{k-1}| - (k-1)(\omega^{s+3} + 2\omega^{s+5}) + 28\omega^{s+6} + (k+1)\omega^{s+5},$$

since u has at most ω^{s+3} non-neighbours in L_i and therefore $|L_i'| \ge |L_i| - \omega^{s+3} - 2\omega^{s+5}$ for each $i \in \{1,\dots,k-1\}$. From the optimality of $\mathcal{L}$, it follows that

$$|L_1 \cup \dots \cup L_{k-1}| - (k-1)(\omega^{s+3} + 2\omega^{s+5}) + 28\omega^{s+6} + (k+1)\omega^{s+5} \le |L_1 \cup \dots \cup L_k| + 7\omega^{s+5}|L_0| + k\omega^{s+5},$$

that is,

$$-(k-1)(\omega^{s+3} + 2\omega^{s+5}) + 28\omega^{s+6} + \omega^{s+5} \le |L_k| + 7\omega^{s+5}|L_0|.$$

Consequently

$$28\omega^{s+6} \le |L_k| + k(\omega^{s+3} + 2\omega^{s+5}) + 7\omega^{s+5}|L_0| \le |L_k| + (k + |L_0|)(7\omega^{s+5}),$$

and since $|L_k| \le 14\omega^{s+6}$ and $k + |L_0| \le \omega$ by 3.1, this is impossible. It follows that $\chi(X(k,u)) \le (14\omega^{s+6})^c$. The union of the sets $X(k,u)$ over all $u \in L_k$ thus has chromatic number at most $(14\omega^{s+6})^{c+1}$; and from the symmetry between $L_1, \dots, L_k$, the set of all dense vertices has chromatic number at most $k(14\omega^{s+6})^{c+1} \le (14\omega^{s+6})^{c+1}\omega$. This proves 4.5. ∎

5 Pure vertices

In view of 4.3, 4.4 and 4.5, in order to bound the chromatic number of $N(\mathcal{L})$ it remains to bound the chromatic number of the set of pure vertices, and that is the topic of this section. But here we will need to use induction on ω, and so as we discussed earlier, we will in fact work with degenerate colouring.

Throughout this section, let $s \geq 1$ be an integer, let G be H_s-free, and let $\mathcal{L} = (L_0, \ldots, L_k)$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. Let M denote the set of all vertices that are pure with respect to this template. For each $v \in M$, let I_v be the set of all $i \in \{1, \ldots, k\}$ such that v has a neighbour in L_i (and hence has at most $\omega^{s+2}/7$ non-neighbours in L_i). For each $I \subseteq \{1, \ldots, k\}$, let M_I be the set of all $v \in M$ with $I_v = I$. Thus $M_i = \emptyset$ if $|I| \geq k-1$, from the definition of "pure".

We wish to find a degenerate colouring of the union of all the sets M_I. One problem is that the number of sets I with M_I nonempty may be superpolynomial; but we will show that there are only a linear number of sets M_I of large cardinality, and we can colour all the small ones simultaneously.

5.1 *Let $I \subseteq \{1, \ldots, k\}$ and $u \in M_I$ and $j \in \{1, \ldots, k\} \setminus I$. For each $i \in I$, there are fewer than ω^s vertices v adjacent to u such that $v \in M_J$ for some $J \subseteq \{1, \ldots, k\} \setminus \{i, j\}$. Hence there are fewer than ω^{s+1} vertices v adjacent to u such that $v \in M_J$ for some $J \subseteq \{1, \ldots, k\} \setminus \{j\}$ with $I \not\subseteq J$.*

Proof. Let $i \in I$, and let $\mathcal{A}_i$ be the set of all $J \subseteq \{1, \ldots, k\} \setminus \{i, j\}$. Let V_i be the union of all the sets M_J for $J \in \mathcal{A}_i$. Suppose that u has ω^s neighbours in V_i. By 4.2, there is a stable set $S \subseteq V_i$ of neighbours of u with $|S| = s$. Choose $x \in L_i$ adjacent to u. Thus x has no neighbour in S since for each $J \in \mathcal{A}_i$, no vertex in M_J has a neighbour in L_i. Since x has at most ω^{s+3} non-neighbours in L_j, and $|L_j| \geq \omega^{s+5} \geq \omega^{s+3} + \omega^s$, it follows that x has at least ω^s neighbours in L_j, and hence by 4.2 there is a stable set $T \subseteq L_j$ of neighbours of x with $|T| = s$. Since $j \notin I \cup J$ for $J \in \mathcal{A}_i$, no vertex in $S \cup \{u\}$ has a neighbour in T; and so the subgraph induced on $S \cup T \cup \{u, x\}$ is isomorphic to H_s, a contradiction.

Hence u has fewer than ω^s neighbours in V_i for each choice of i. Since there are only at most $|I| \leq \omega$ choices of i, this proves 5.1. ∎

For $m \geq 0$, we say $I \subseteq \{1, \ldots, k\}$ is *m-small* if $|M_I| \leq m$, and *m-large* if it is not m-small.

5.2 *For each $m \geq 0$, the union of the sets M_I over all m-small $I \subseteq \{1, \ldots, k\}$ has chromatic number at most $2\omega(m + \omega^{s+1})$.*

Proof. For all $j \in \{1, \ldots, k\}$, let $\mathcal{A}_j$ be the set of all m-small $I \subseteq \{1, \ldots, k\} \setminus \{j\}$, and let V_j be the union of the sets M_I for all $I \in \mathcal{A}_j$. If $u, v \in V_j$ are adjacent, we will direct the edge uv as follows. Let $u \in M_I$ and $v \in M_J$ where $I, J \in \mathcal{A}_j$. If $|I| > |J|$ we direct the edge uv from u to v. If $|I| = |J|$ (and in particular, if $I = J$) we direct uv arbitrarily. We claim every vertex $u \in V_j$ has outdegree less than $m + \omega^{s+1}$. Let $u \in M_I$. Certainly u has at most m out-neighbours in M_I since $|M_I| \leq m$. If v is an out-neighbour of u and $v \in J \in \mathcal{A}_j$ where $J \neq I$, then $|J| \leq |I|$, and it follows that $I \not\subseteq J$, and $I \cup J \neq \{1, \ldots, k\}$ since $j \notin I \cup J$; and so there are fewer than ω^{s+1} such vertices v by 5.1. This proves that every vertex $u \in V_j$ has outdegree less than $m + \omega^{s+1}$. Hence every subgraph of $G[V_j]$ with n vertices has at most $n(m + \omega^{s+1} - 1)$ edges, and so (if $n > 0$) has a vertex of degree at most $2(m + \omega^{s+1} - 1)$; and hence $G[V_j]$ has degeneracy at most $2(m + \omega^{s+1} - 1)$, and so has chromatic number less than $2(m + \omega^{s+1})$. Since there are at most ω choices of j, and every m-small set I with $M_i \neq \emptyset$ belongs to some $\mathcal{A}_j$, this proves 5.2. ∎

It remains to handle the m-large sets. For $t \geq 1$, let us say two disjoint subsets A, B of $V(G)$ are *s-crowded* if there is no stable set X with $|X \cap A| = |X \cap B| = s$.

5.3 *Let $m \geq (s+1)\omega^s$, and let $I, J \subseteq \{1, \dots, k\}$ be m-large, with $I \neq J$. If $\omega^3 > \omega + s/7$, then either:*

- *$J \subseteq I$ and every vertex in M_I has fewer than ω^s neighbours in M_J; or*
- *$I \subseteq J$ and every vertex in M_J has fewer than ω^s neighbours in M_I; or*
- *$I \cup J = \{1, \dots, k\}$ and M_I, M_J are s-crowded.*

Consequently there are at most $k-1$ m-large sets.

Proof. If $J \subseteq I$ then the first bullet holds by 5.1; so we may assume that $J \not\subseteq I$ and similarly $I \not\subseteq J$. Choose $i \in I \setminus J$ and $j \in J \setminus I$. Suppose that M_I, M_J are not s-crowded, and choose $S \subseteq M_I$ and $T \subseteq M_J$ with $|S| = |T| = s$ such that $S \cup T$ is stable. Since each vertex in S has at most $\omega^{s+2}/7$ non-neighbours in i, and $s\omega^{s+2}/7 < \omega^{s+5}$, there exists $u \in L_i$ adjacent to every vertex in S. Since u has at most ω^{s+3} non-neighbours in L_j, and each vertex in T has at most $\omega^{s+2}/7$ non-neighbours in L_j, and $\omega^{s+3} + s\omega^{s+2}/7 < \omega^{s+5}$, there exists $v \in L_j$ adjacent to u and to each vertex in T. But then the subgraph induced on $S \cup T \cup \{u, v\}$ is isomorphic to H_s. This proves that M_I, M_J are s-crowded.

Let $S \subseteq M_I$ be stable with $|S| = s$. (This is possible by 4.2 since $m \geq \omega^s$.) Since M_I, M_J are s-crowded, 4.2 implies that there are fewer than ω^s vertices in M_J with no neighbour in S; and so some vertex in S has at least

$$(|M_J| - \omega^s)/s > (m - \omega^s)/s \geq \omega^s$$

neighbours in M_J. By 5.1 it follows that $I \cup J = \{1, \dots, k\}$ and so the third bullet holds. This proves the first assertion of 5.3.

To show that there are at most $k-1$ m-large sets, for each $X \subseteq \{1, \dots, k\}$ with $|X| \leq k-1$, let $n(X)$ be the number of m-large sets that include X. We prove by induction on $k - |X|$ that $n(X) \leq k - |X| - 1$. Since $M_I = \emptyset$ if $|I| \geq k-1$, it follows that if $|X| = k-1$ then $n(X) = 0$ and the claim holds. Thus we assume that $|X| \leq k-2$ and the result holds for all larger subsets of $\{1, \dots, k\}$. Since $k - |X| - 1 \geq 1$, we may assume that at least two m-large sets include X, and so at least one properly includes X. Choose an m-large set J minimal such that $X \subseteq J$ and $X \neq J$. If I is an m-large set including X, then by 5.1 and the minimality of J, either $I = X$, or $J \subseteq I$, or $I \cup J = \{1, \dots, k\}$. Moreover, if $I \cup J = \{1, \dots, k\}$, then I includes K, where $K = \{1, \dots, k\} \setminus (J \setminus X)$. Consequently $n(X) \leq n(J) + n(K) + 1$. Since J, K are both proper supersets of X, and not equal to $\{1, \dots, k\}$, the inductive hypothesis implies that $n(J) \leq k - |J| - 1$ and $n(K) \leq k - |K| - 1$, and so

$$n(X) \leq (k - |J| - 1) + (k - |K| - 1) + 1 = k - |X| - 1.$$

This completes the proof that $n(X) \leq k - |X| - 1$ for each $X \subseteq \{1, \dots, k\}$ with $|X| \leq k-1$. Setting $X = \emptyset$, it follows that there are only $k-1$ m-large sets. This proves 5.3. ∎

5.4 *Suppose that $d \geq 1$ and $D \geq 0$ have the property that every H_s-free graph G' with $\omega(G') < \omega(G)$ is $(\omega(G')^d, D)$-colourable, and let $m = (s+1)\omega^s$. Assume that $\omega^3 > \omega + s/7$ and $\omega \geq 200$. Then the union of the sets M_I over all m-large I is (K, D)-colourable where*

$$K = (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1.$$

Proof. Let M^* be the union of the sets M_I for all m-large I. We may assume that $M^* \neq \emptyset$.

(1) *There is a partition $\mathcal{A}, \mathcal{B}$ of the set of m-large sets, such that*

- *M_I, M_J are s-crowded for each $I \in \mathcal{A}$ and $J \in \mathcal{B}$; and*
- *all the sets in $\mathcal{A}$ have an element in common, and so do all the sets in $\mathcal{B}$.*

There is an m-large set J since $M^* \neq \emptyset$; choose J minimal. Let $\mathcal{A}$ be the set of all m-large sets that include J, and let $\mathcal{B}$ be the set of m-large sets that do not include J. Choose $j \in J$ and $j' \in \{1, \dots, k\} \setminus J$. ($J$ is nonempty since every pure vertex has a neighbour in $L_1 \cup \dots \cup L_k$, and $J \neq \{1, \dots, k\}$ from the definition of pure.) Every set in $\mathcal{A}$ includes J, and so contains j. We claim that every set in $\mathcal{B}$ includes $\{1, \dots, k\} \setminus J$ and so contains j'. To see this, let $I' \in \mathcal{B}$. Since $I' \in \mathcal{B}$, I' does not include J. Also $J \not\supseteq I'$ from the minimality of J, and so from 5.3, $I' \cup J = \{1, \dots, k\}$. This proves that every set in $\mathcal{B}$ includes $\{1, \dots, k\} \setminus J$ and so contains j'. Finally, we must show that $M_I, M_{I'}$ are s-crowded for all $I \in \mathcal{A}$ and $I' \in \mathcal{B}$. Since $J \subseteq I$ and $J \not\subseteq I'$, it follows that $I \not\subseteq I'$; and since $\{1, \dots, k\} \setminus J \subseteq I'$ and $\{1, \dots, k\} \setminus J \not\subseteq I$, it follows that $I' \not\subseteq I$. By 5.3, $M_I, M_{I'}$ are s-crowded. This proves (1).

Choose $\mathcal{A}, \mathcal{B}$ as in (1). Let A be the union of all the sets M_I for $I \in \mathcal{A}$, and define B similarly. Thus $M^* = A \cup B$.

(2) *Every clique included in A has cardinality at most $\omega - 1$, and the same for B.*

Suppose that there is a clique $X \subseteq A$ with $|X| = \omega$. Let $j \in \{1, \dots, k\}$ belong to all the sets in $\mathcal{A}$. Since $(\omega^{s+2}/7)\omega < \omega^{s+5} \leq |L_j|$, there exists $v \in L_j$ adjacent to every vertex of X, contradicting that X is a clique of G of maximum cardinality. The same holds for B. This proves (2).

Let $n = \omega^{s+2}$. If $|A| \leq n\omega$, then by (2) it follows that $G[M^*]$ admits an $(n\omega + (\omega - 1)^d, D)$-colouring and the theorem holds. Thus we may assume that $|A| > n\omega$. Let X_1 be the largest clique contained in A, and inductively for $i \geq 2$ let X_i be the largest clique contained in $A \setminus (X_1 \cup \dots \cup X_{i-1})$. Let $|X_n| = t$. Since $|A| > n\omega$ it follows that $t > 0$. Let $X = X_1 \cup \dots \cup X_n$. Thus $|X| \leq n\omega$, and $\omega(G[A \setminus X]) \leq t$.

Let C be the set of all vertices $v \in B$ such that for some $I \in \mathcal{A}$, v has at least ω^s non-neighbours in $M_I \cap X$.

(3) $|C| \leq n^s \omega^{2s+2}$.

For each $v \in C$, there exist $I \in \mathcal{A}$ and a stable set $S \subseteq M_I \cap X$ with $|S| = s$ such that v has no neighbour in S, by 4.2. For each such I and S, and each $I' \in \mathcal{B}$, there are at most ω^s vertices in $M_{I'}$ that have no neighbours in S, by 4.2 and since $M_I, M_{I'}$ are s-crowded (because $I \not\subseteq I'$ and $I' \not\subseteq I$ by (1), and by 5.3). Consequently for each choice of I, S there are at most $k\omega^s$ vertices in C with no neighbour in S, since $|\mathcal{B}| \leq k$ by 5.3. For each choice of I there are only $|X|^s \leq n^s \omega^s$ choices of S, since $|X| \leq n\omega$; and there are only k choices of I by 5.3. Thus $|C| \leq (k\omega^s)(n^s\omega^s)k \leq n^s\omega^{2s+2}$. This proves (3).

(4) *Every clique in $B \setminus C$ has cardinality at most $\omega - t$.*

Let $Y \subseteq B \setminus C$ be a clique. Since $Y \cap C = \emptyset$, every vertex in Y has at most $k\omega^s$ non-neighbours in X, since it has at most ω^s in each $X \cap M_I$ and there are only k choices of M_I. Consequently at most $k\omega^s|Y|$ vertices in X have a non-neighbour in Y. Since $n > k\omega^s|Y|$ (since $n = \omega^{s+2}$, and $k \leq \omega$ and $|Y| < \omega$ by (2)) it follows that there exists $i \in \{1, \ldots, n\}$ such that every vertex in Y has no non-neighbour in X_i, and so $X_i \cup Y$ is a clique. But $|X| \geq t$ from the choice of t, and $|X \cup Y| \leq \omega$, and so $|Y| \leq \omega - t$. This proves (4).

Now $\chi(M^*) \leq |X| + \chi(A \setminus X) + |C| + \chi(B \setminus C)$. But $|X| \leq n\omega$; $A \setminus X$ is (t^d, D)-colourable, since $\omega(G[A \setminus X]) \leq t$ and $t < \omega$ (by (2)); $|C| \leq n^s\omega^{2s+2}$ by (3); and $B \setminus C$ is $((\omega - t)^d, D)$-colourable by (4) and since $\omega - t < \omega$ (because $t > 0$). Thus M^* is (K_1, D)-colourable where

$$K_1 = n\omega + t^d + n^s\omega^{2s+2} + (\omega - t)^d.$$

Since $1 \leq t \leq \omega - 1$, it follows that $t^d + (\omega - t)^d \leq (\omega - 1)^d + 1$ (since $d \geq 1$), and so

$$K_1 = n\omega + t^d + n^s\omega^{2s+2} + (\omega - t)^d \leq n\omega + n^s\omega^{2s+2} + (\omega - 1)^d + 1 = \omega^{s+3} + \omega^{s^2+4s+2} + (\omega - 1)^d + 1.$$

Hence M^* is (K, D)-colourable where

$$K = \omega^{s+3} + \omega^{s^2+4s+2} + (\omega - 1)^d + 1.$$

This proves 5.4. ∎

From 5.2 and 5.4, with $m = (s + 1)\omega^s$, we deduce:

5.5 *Suppose that $d \geq 1$ and $D \geq 0$ have the property that every H_s-free graph G' with $\omega(G') < \omega(G)$ is $(\omega(G')^d, D)$-colourable; and $c \geq 2s + 1$ satisfies 4.1 when $H = H_s$. Assume that $\omega^3 > \omega + s/7$ and $\omega \geq 200$. Then $V(\mathcal{L}) \cup N(\mathcal{L})$ is (K, D)-colourable where*

$$K = (\omega - 1)^d + \omega^{(c+1)(s+7)}.$$

Proof. $V(\mathcal{L})$ has cardinality at most $14\omega^{s+7}$ from the definition of an s-template and since $k + |L_0| \leq \omega$ by 3.1. Every vertex in $N(\mathcal{L})$ is pendant or dense or pure with respect to $\mathcal{L}$, by 4.3. At most $14^{s+2}\omega^{s^2+9s+14}$ are pendant, by 4.4, and the chromatic number of the set of all dense vertices is at most $(14\omega^{s+6})^{c+1}\omega$ by 4.5. By 5.2 with $m = (s + 1)\omega^s$, and 5.4, the set of all pure vertices is (K_1, D)-colourable where

$$K_1 = (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1 + (2s + 2)\omega^{s+1} + 2\omega^{s+2}.$$

Adding, we deduce that $V(\mathcal{L}) \cup N(\mathcal{L})$ is (K', D)-colourable where

$$K' = 14\omega^{s+7} + 14^{s+2}\omega^{s^2+9s+14} + (14\omega^{s+6})^{c+1}\omega + (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1 + (2s+2)\omega^{s+1} + 2\omega^{s+2}.$$

K' is the sum of nine terms, and we claim that each term is at most $\omega^{(c+1)(s+7)}/9$. For instance, for the second term, since $c \geq 2s + 1 \geq 3$ and $\omega \geq 200$,

$$14^{s+2}\omega^{s^2+9s+14} \leq \omega^{s^2+10s+16} \leq \omega^{(2s+2)(s+7)-1} \leq \omega^{(c+1)(s+7)}/9.$$

For the third term, since $c \geq 2s+1 \geq 3$ and $\omega \geq 200$, it follows that $\omega \geq 9^{1/c}14^{(c+1)/c}$, that is, $14^{c+1} \leq \omega^c/9$; and this implies that

$$(14\omega^{s+6})^{c+1}\omega \leq \omega^{(c+1)(s+7)}/9.$$

Thus our claim holds for the second and third terms in the definition of K', and similar (easier) arguments, which we omit, apply for the other seven terms. We deduce that $K' \leq (\omega-1)^d + \omega^{(c+1)(s+7)}$, and this proves 5.5. ∎

6 Proof of the main theorem

In this section we prove 1.2. If $\mathcal{L} = (L_0, \dots, L_k)$ is an optimal s-template in G, we define $Z(\mathcal{L})$ to be the union of L_0 and the set of all vertices $v \in N(\mathcal{L})$ such that for $1 \leq i \leq k$, v has at most $\omega^{s+2}/4$ non-neighbours in L_i. Let $Y(\mathcal{L}) = (V(\mathcal{L}) \cup N(\mathcal{L})) \setminus Z(\mathcal{L})$. First we need:

6.1 *Let $s \geq 1$ be an integer, and let G be H_s-free, with $\omega(G) \geq 15$ and $\omega^2 > s+1$. Let $\mathcal{L}$ be an optimal s-template in G, with value at least $28\omega^{s+6}$. Then every vertex in $Y(\mathcal{L})$ has at most ω^{s+7} neighbours in $V(G) \setminus Y(\mathcal{L})$.*

Proof. Let $v \in Y(\mathcal{L})$, and suppose that v has more than ω^{s+7} neighbours in $V(G) \setminus Y(\mathcal{L})$. By 3.1 and 3.5, $|Z(\mathcal{L})| \leq \omega + 14\omega^{s+6}$, and so v has at least ω^s neighbours in $V(G) \setminus (V(\mathcal{L}) \cup N(\mathcal{L}))$, since

$$\omega + 14\omega^{s+6} + \omega^s < \omega^{s+7}$$

(because $\omega \geq 15$). By 4.2, there is a stable set $S \subseteq V(G) \setminus (V(\mathcal{L}) \cup N(\mathcal{L}))$ of neighbours of v, with $|S| = s$. Since $v \notin Z(\mathcal{L})$, it follows that $v \in N(\mathcal{L})$ and there exists $i \in \{1, \dots, k\}$ such that v has more than $\omega^{s+2}/4$ non-neighbours in L_i. Since $k \geq 2$ by 3.2, and v has a neighbour in at least one of $L_1, \dots, L_k$ because $v \in N(\mathcal{L})$, and each $|L_j| \geq \omega^{s+5} \geq \omega^{s+2}/4$, we may choose distinct $i, j \in \{1, \dots, k\}$ such that v has a neighbour in L_j and v has more than $\omega^{s+2}/4$ non-neighbours in L_i, and choose such a pair i, j such that v has as many non-neighbours in L_i as possible. Let B be the set of non-neighbours of v in L_i. Thus $|B| > \omega^{s+2}/4$.

(1) *v has at most $\omega^s + \omega^{s+3}$ non-neighbours in L_j.*

Let $u \in L_j$ be adjacent to v. If u has at least ω^s neighbours in B, there is a stable set T of such neighbours with $|T| = s$, by 4.2, and then the subgraph induced on $S \cup T \cup \{u, v\}$ is isomorphic to H_s, a contradiction. Thus u has fewer than ω^s neighbours in B; but it has at most ω^{s+3} non-neighbours in L_i, and hence at most that many in B, and so $|B| \leq \omega^s + \omega^{s+3}$. Since $|L_i| \geq \omega^{s+5}$, v has a neighbour in L_i; and so from the choice of the pair i, j, it follows that v has at most $\omega^s + \omega^{s+3}$ non-neighbours in L_j. This proves (1).

Since $|B| > \omega^{s+2}/4 \geq \omega^s$, it includes a stable set T of cardinality s, by 4.2. Each vertex in T has at most ω^{s+3} non-neighbours in L_j, and v has at most $\omega^s + \omega^{s+3}$ non-neighbours in L_j by (1), and since

$$s\omega^{s+3} + \omega^s + \omega^{s+3} < \omega^{s+5} \leq |L_j|,$$

(because $\omega^2 > s+1$), it follows that some vertex $u \in L_j$ is adjacent to every vertex in $T \cup \{v\}$. But then the subgraph induced on $S \cup T \cup \{u, v\}$ is isomorphic to H_s, a contradiction. This proves 6.1. ∎

Let $A \subseteq V(G)$, and let $\mathcal{L} = (L_0, \ldots, L_k)$ be an s-template of G with $V(\mathcal{L}) \subseteq A$. We make the following definitions:

- $Z_A(\mathcal{L})$ is the union of L_0 and the set of vertices in $A \setminus V(\mathcal{L})$ that have at most $\omega^{s+2}/4$ non-neighbours in L_i for each $i \in \{1, \ldots, k\}$;
- $N_A(\mathcal{L})$ is the set of vertices in $A \setminus V(\mathcal{L})$ with a neighbour in $L_1 \cup \cdots \cup L_k$; and
- $Y_A(\mathcal{L}) = (V(\mathcal{L}) \cup N_A(\mathcal{L})) \setminus Z_A(\mathcal{L})$.

6.2 *Let $s \geq 1$ be an integer, and let G be H_s-free. Let $\omega = \omega(G)$, and let $A \subseteq V(G)$, such that there is an s-template $\mathcal{L}$ of G with $V(\mathcal{L}) \subseteq A$ and with value at least $28\omega^{s+6}$. Let $\mathcal{L} = (L_0, \ldots, L_k)$ be such an s-template with maximum value.*

- *Suppose that $d \geq 1$ and $D \geq 0$ have the property that every H_s-free graph G' with $\omega(G') < \omega$ is $(\omega(G')^d, D)$-colourable; and $c \geq 2s+1$ satisfies 4.1 when $H = H_s$; and $\omega^3 > \omega + s/7$; and $\omega \geq 200$. Then $V(\mathcal{L}) \cup N_A(\mathcal{L})$ is $((\omega-1)^d + \omega^{(c+1)(s+7)}, D)$-colourable.*
- *If $\omega \geq 15$ and $\omega^2 > s+1$, then every vertex in $Y_A(\mathcal{L})$ has at most ω^{s+7} neighbours in $A \setminus Y_A(\mathcal{L})$.*

Proof. $\mathcal{L}$ is not necessarily an optimal s-template of G, since it is constrained to have vertex set included in A; and it is not necessarily an optimal s-template of $G[A]$, since perhaps $\omega(G[A]) < \omega(G)$ and then the conditions that define an s-template of G are different from those that define an s-template of $G[A]$. Nevertheless, we can apply 5.5 and 6.1 to $\mathcal{L}$ by the following trick. Let G' be the disjoint union of $G[A]$ and a complete graph $K_{\omega(G)}$ with vertex set B say. Then $\omega(G') = \omega(G)$, and since every optimal s-template of G' induces a connected subgraph with more than $\omega(G)$ vertices (because its value is at least $28\omega(G)^6$), it contains no vertex of B and so is an s-template of $G[A]$. This proves that $\mathcal{L}$ is an optimal s-template of G'. The claims of the theorem follow by applying 5.5 and 6.1 to $\mathcal{L}$ and G'. This proves 6.2. ∎

Now we prove 1.2, which we restate in a strengthened form:

6.3 *For every integer $s \geq 1$, there exists $d \geq 0$ such that if G is H_s-free, then G is (ω^d, ω^{s+8})-colourable, and hence has chromatic number at most $\omega^d(\omega^{s+8}+1)$.*

Proof. Choose $c \geq 2s+1$ satisfying 4.1 with $H = H_s$. It follows from the main theorem of [12] that there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H_s-free graph G; and so by choosing d sufficiently large we may arrange that $\chi(G) \leq \omega(G)^d$ for every H_s-graph G with $\omega(G)$ at most $\max(200, (s+1)^{1/2})$. Let us also choose d so large that $d \geq (c+1)(s+7)+1$. We claim that d satisfies 6.3. The proof is by induction on $\omega = \omega(G)$. If $\omega \leq \max(200, (s+1)^{1/2})$ the claim is true, so we may assume that $\omega > \max(200, (s+1)^{1/2})$. Consequently $\omega^3 > \omega + s/7$ and $\omega^2 > s+1$, and we can apply 6.2.

Let $V_0 = V(G)$, and choose n maximum such that there is a sequence $\mathcal{L}_i$ $(1 \leq i \leq n)$ of s-templates of G and a sequence $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n$ of subsets of $V(G)$, with the following property. For $1 \leq i \leq n$, $\mathcal{L}_i$ is an s-template of G with value at least $28\omega^{s+6}$, with $V(\mathcal{L}_i) \subseteq V_{i-1}$, chosen with maximum value among all such s-templates; and $V_i = V_{i-1} \setminus Y_{V_{i-1}}(\mathcal{L}_i)$, in the notation of 6.2. For $1 \leq i \leq n$, let $Y_i = Y_{V_{i-1}}(\mathcal{L}_i)$, and let $Y_{n+1} = V_n$. We observe:

- The sets $Y_1, \dots, Y_{n+1}$ are pairwise disjoint and have union $V(G)$.
- For $1 \leq i \leq n$, $G[Y_i]$ is $(\omega^d, \omega^{s+7}(\omega-1))$-colourable. To see this, observe that from the inductive hypothesis, every H_s-free graph G' with $\omega(G') < \omega$ is $(\omega(G')^d, \omega(G')^{s+8})$-colourable, and hence $(\omega(G')^d, D)$-colourable where $D = \omega^{s+7}(\omega - 1)$, since $\omega(G') < \omega$. From the first statement of 6.2, for $1 \leq i \leq n$, Y_i is $((\omega-1)^d + \omega^{(c+1)(s+7)}, \omega^{s+7}(\omega-1))$-colourable and hence $(\omega^d, \omega^{s+7}(\omega-1))$-colourable, since $(\omega-1)^d + \omega^{(c+1)(s+7)} \leq \omega^d$ (because $d \geq (c+1)(s+7)+1$).
- $G[Y_{n+1}]$ is $(\omega^d, \omega^{s+7}(\omega-1))$-colourable. To see this, observe that the maximality of n implies that there is no s-template $\mathcal{L}$ of G of value at least $28\omega^{s+6}$ and with $V(\mathcal{L}) \subseteq V_n$, and so $\chi(V_n) \leq (14\omega^{s+6})^c$ by 4.1. Consequently $V_n = Y_{n+1}$ is $(\omega^d, \omega^{s+7}(\omega-1))$-colourable, since $(14\omega^{s+6})^c \leq \omega^d$.
- For $1 \leq i \leq n$, every vertex in Y_i has at most ω^{s+7} neighbours in $Y_{i+1} \cup \dots \cup Y_{n+1}$. This follows from the second statement of 6.2.

By 2.1, G is $(\omega^d, \omega^{s+7}(\omega-1) + \omega^{s+7})$-colourable. This proves 6.3.

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