INDUCED SUBGRAPH DENSITY. VI. BOUNDED VC-DIMENSION

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Abstract. We confirm a conjecture of Fox, Pach, and Suk, that for every $d > 0$, there exists $c > 0$ such that every n-vertex graph of VC-dimension at most d has a clique or stable set of size at least n^c. This implies that, in the language of model theory, every graph definable in NIP structures has a clique or anti-clique of polynomial size, settling a conjecture of Chernikov, Starchenko, and Thomas.

Our result also implies that every two-colourable tournament satisfies the tournament version of the Erdős-Hajnal conjecture, which completes the verification of the conjecture for six-vertex tournaments. The result extends to uniform hypergraphs of bounded VC-dimension as well.

The proof method uses the ultra-strong regularity lemma for graphs of bounded VC-dimension proved by Lovász and Szegedy and the method of iterative sparsification introduced by the authors in an earlier paper.

1. Introduction

All (hyper)graphs in this paper are finite and simple. A graph H is an *induced subgraph* of a graph G if H can be obtained from G by removing vertices. A class $\mathcal{C}$ of graphs is *hereditary* if it is closed under taking induced subgraphs and under isomorphism; and a hereditary class $\mathcal{C}$ is *proper* if it is not the class of all graphs. We say that $\mathcal{C}$ has the *Erdős-Hajnal property* if there exists $c > 0$ such that every graph $G \in \mathcal{C}$ has a clique or stable set of size at least $|G|^c$, where $|G|$ denotes the number of vertices of G. A conjecture of Erdős and Hajnal [17, 18] (see [13, 29] for surveys and [8, 15, 39, 40] for some recent partial results) asserts that:

Conjecture 1.1. *Every proper hereditary class of graphs has the Erdős-Hajnal property.*

For a set $\mathcal{F}$ of subsets of a set V, a subset S of V is *shattered* by $\mathcal{F}$ if for every $A \subseteq S$ there exists $B \in \mathcal{F}$ with $B \cap S = A$. The *VC-dimension* of $\mathcal{F}$ (introduced by Vapnik and Chervonenkis in [51]) is the largest cardinality of a subset of V that is shattered by $\mathcal{F}$. Since its introduction in 1971, the notion of VC-dimension has proved to be relevant in a number of areas of pure and applied mathematics. The *VC-dimension* of a graph G is the VC-dimension of the set $\{N_G(v) : v \in V(G)\}$ of subsets of $V(G)$, where $N_G(v)$ denotes the set of all neighbours of v in G. It is not hard to see that for every $d \geq 1$, the class of graphs of VC-dimension at most d is a proper hereditary class. The aim of this paper is to confirm a conjecture of Fox, Pach, and Suk [22] that every hereditary class of graphs of bounded VC-dimension has the Erdős-Hajnal property; in their paper, they came close to settling this by proving a bound of $2^{(\log n)^{1-o(1)}}$ where the constant depending on the VC-dimension is hidden in the $o(1)$ term. (In this paper log denotes the binary logarithm.) Our result is:

Theorem 1.2. *For every $d \geq 1$, there exists $c > 0$ such that every graph G of VC-dimension at most d contains a clique or stable set of size at least $|G|^c$.*

As a result, this paper can be viewed as part of a growing body of literature [21–24, 30] that, when VC-dimension is bounded, completely settle or significantly improve bounds for well-known open problems in extremal combinatorics.

The story behind Theorem 1.2, perhaps, began in geometric graph theory with the result of Larman, Matoušek, Pach, and Törőcsik [31] that the class of intersection graphs of convex compact sets in the plane has the Erdős-Hajnal property. Alon, Pach, Pinchasi, Radoičić, and Sharir [2] generalized this result to classes of semi-algebraic graphs of bounded description complexity. Fox, Pach, and Tóth [25] provided another extension of [31] by verifying the Erdős-Hajnal property of the class of string graphs where every two curves cross a bounded number of times (and a recent result by Tomon [50] shows that the condition "every two curves cross a bounded number of times" can be dropped). In yet another direction, Sudakov and Tomon [49] recently proved Conjecture 1.1 for the classes of algebraic graphs of

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bounded description complexity, which is an analogue of the result of [2]. All of these hereditary classes (except the one in [50]) turn out to have bounded VC-dimension. Indeed,

- for the classes of semi-algebraic graphs of bounded description complexity, this is true by the classical Milnor-Thom theorem in real algebraic geometry (see [36]) and the Sauer-Shelah-Perles lemma [45, 46];
- for the classes of algebraic graphs of bounded description complexity, this is a consequence of a theorem of Rónyai, Babai, and Ganapathy [44] on the number of zero-patterns of polynomials and the Sauer-Shelah-Perles lemma; and
- for the classes of string graphs where every two curves intersect at a bounded number of points, this follows from a result of Pach and Tóth [42, Lemma 4.2] together with the standard fact that a hereditary class has bounded VC-dimension if and only if it does not contain a bipartite graph, the complement of a bipartite graph, and a split graph (see Lemma 5.1).

In a model-theoretic setting, Theorem 1.2 states that every class of graphs definable in NIP (non-independence property) structures has the Erdős-Hajnal property (see [47] for a general reference on NIP theories), which was formally stated as a conjecture by Chernikov, Starchenko, and Thomas [11]. Two notable special cases of NIP graphs include distal graphs and stable graphs.[1] Malliaris and Shelah [33, 34] implicitly proved Conjecture 1.1 for stable graphs by developing regularity lemmas for these graphs (see [9] for a short proof using pseudo-finite model theory), which in fact contains the result of [49] but does not give an explicit exponent. In the case of distal graphs, Basu [4] proved the Erdős-Hajnal property for graphs definable by o-minimal structures (which in fact extends [2]), before Chernikov and Starchenko [10] made use of the theory of Keisler measures in NIP to formulate regularity lemmas for distal graphs and settle the general problem in this direction (see also [48] for a short and pure model-theoretic proof). Recently, Fu [27] also combined tools from model theory and a result from [15] to prove the Erdős-Hajnal property of the class of graphs of VC-dimension at most two.

In the combinatorial setting, Theorem 1.2 turns out to be equivalent to the fact that every tournament not containing a fixed two-colourable subtournament has a transitive vertex subset of polynomial size; we will discuss this in more detail in Section 6. Theorem 1.2 also extends to the setting of uniform hypergraphs, allowing us to give tighter asymptotics on Ramsey numbers for k-uniform hypergraphs with bounded VC-dimension; we will discuss this in Section 4.

As in previous papers of the series, we need to work in a more general context, and will prove a stronger result: our main result says that graphs of bounded VC-dimension satisfy the polynomial form of a theorem of Rödl. For a graph G, let $\overline{G}$ denote its complement; and for $\varepsilon \in (0, \frac{1}{2})$, we say that G is *ε-sparse* if G has maximum degree at most $\varepsilon|G|$, and *ε-restricted* if one of $G, \overline{G}$ is ε-sparse. We recall that Rödl's theorem [43] then states the following:

Theorem 1.3. *For every proper hereditary class $\mathcal{C}$ of graphs and every $\varepsilon \in (0, \frac{1}{2})$, there exists $\delta > 0$ such that every graph $G \in \mathcal{C}$ contains an ε-restricted induced subgraph on at least $\delta|G|$ vertices.*

The dependence of δ on ε here is an important issue. Rödl's proof used Szemerédi's regularity lemma, and gave tower-type bounds. A better dependence was obtained by Fox and Sudakov [26], who gave a proof with $\delta = 2^{-c\log^2(\varepsilon^{-1})}$; and this was improved to $\delta = 2^{-c\log^2(\varepsilon^{-1})/\log\log(\varepsilon^{-1})}$ in [8]. Fox and Sudakov [26] conjectured that in fact the dependence can be taken to be polynomial, which would imply the Erdős-Hajnal conjecture.

Let us say a hereditary class $\mathcal{C}$ has the *polynomial Rödl property* if there exists $C > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$, every graph $G \in \mathcal{C}$ contains an ε-restricted induced subgraph on at least $\varepsilon^C|G|$ vertices. The Fox–Sudakov conjecture is then the following strengthening of Conjecture 1.1:

Conjecture 1.4. *Every proper hereditary class of graphs has the polynomial Rödl property.*

Our main result says that every hereditary class of graphs of bounded VC-dimension does indeed satisfy Conjecture 1.4.

Theorem 1.5. *For every $d \geq 1$, the class of graphs of VC-dimension at most d has the polynomial Rödl property.*

Our proof of Theorem 3.1 uses the ultra-strong regularity lemma for graphs of bounded VC-dimension proved by Lovász and Szegedy, and builds on the method of iterative sparsification introduced in earlier

[1]This is different from the notion of stable vertex subsets in graphs; in graph-theoretic language, it means graphs not containing a fixed half-graph bipartite pattern. See also [35].

papers of the series [38, 39]. The method involves passing through a sequence of induced subgraphs that are successively more restricted (see Section 3), and naturally leads to the extra generality of Theorem 3.1.

2. Ultra-strong regularity, bigraphs and nearly-pure blockades

A natural approach to proving Erdős-Hajnal-type results is to attempt to find induced subgraphs that are highly structured; the challenge is to control the size and structure of these subgraphs. In this section, we work with large subgraphs that are "approximately" nicely structured: up to some noise, they look like a blowup of some much smaller graph. These will be helpful in the next section, where we look for subgraphs where the structure is more tightly controlled.

A *blockade* in a graph G is a sequence $\mathcal{B} = (B_1, \dots, B_k)$ of pairwise disjoint (and possibly empty) subsets of $V(G)$, called *blocks*; the *length* of $\mathcal{B}$ is k and the *width* is $\min_{i\in[k]} |B_i|$. For $\ell, w \geq 0$, we say that $\mathcal{B}$ is an (ℓ, w)-*blockade* if it has length at least ℓ and width at least w.

We will be interested in blockades where the pairs (B_i, B_j) satisfy certain density conditions. Two disjoint subsets A, B of vertices are *anticomplete* in G if G has no edge between A, B; *complete* in G if A, B are anticomplete in $\overline{G}$; and *pure* if A, B are either complete or anticomplete in G. A blockade $\mathcal{B} = (B_1, \dots, B_k)$ is *anticomplete* in G if B_i, B_j are anticomplete in G for all distinct $i, j \in [k]$, *complete* in G if $\mathcal{B}$ is anticomplete in $\overline{G}$, and *pure* if, for all distinct $i, j \in [k]$, B_i, B_j are either complete or anticomplete. Note that being a pure blockade is a far weaker condition than being a complete or anticomplete blockade.

In general, complete or anticomplete blockades will be very desirable; but first we will need to work with blockades that are only pure, and indeed, that are only approximately pure. For $\varepsilon > 0$ and a graph G, two disjoint subsets A, B are *weakly ε-sparse* in G if G has at most $\varepsilon|A||B|$ edges between A, B, and *weakly ε-pure* if they are weakly ε-sparse in G or $\overline{G}$. We say that B is *ε-sparse* to A in G if every vertex of B is adjacent in G to at most $\varepsilon|A|$ vertices in A. A blockade $\mathcal{B}$ is *weakly ε-pure* if every pair B_i, B_j is weakly ε-pure.

The first step in the proof of Theorem 3.1 is to find a large (both long and wide) weakly ε-pure blockade. For this we use the ultra-strong regularity lemma of Lovász and Szegedy [32, Section 4] which was reproved by Fox, Pach, and Suk [22, Theorem 1.3] via a short proof that gives better bounds (a weaker version for bipartite graphs was also developed by Alon, Fisher, and Newman [1]). The ultra-strong regularity lemma says:

Theorem 2.1. *For every $d \geq 1$, there exists $K \geq 2$ such that the following holds. For every $\varepsilon \in (0, \frac{1}{2})$ and every graph G with VC-dimension at most d, there exist $L \in [\varepsilon^{-1}, \varepsilon^{-K}]$ and an equipartition $V(G) = V_1 \cup \dots \cup V_L$ such that all but at most an ε fraction of the pairs (V_i, V_j) are weakly ε-pure in G.*

Here, an *equipartition* of a set S is a partition $S_1 \cup \dots \cup S_k$ of S such that $|S_i|, |S_j|$ differ by at most one for all $i, j \in [k] =: \{1, \dots, k\}$.

It will be helpful to work with bipartite patterns in G. Let us make this more formal. A *bigraph* is a graph H together with a bipartition $(V_1(H), V_2(H))$ of $V(H)$. If H is a bigraph and $v \in V(H)$, we denote by $H \setminus v$ the bigraph H' obtained by deleting v, where for $i = 1, 2$, $V_i(H') = V_i(H) \setminus \{v\}$ if $v \in V_i(H)$, and $V_i(H') = V_i(H)$ otherwise. For a bigraph H, a *bi-induced* copy of H in a graph G is an injective map $\varphi \colon V(H) \to V(G)$ such that, for all $u \in V_1(H)$ and $v \in V_2(H)$, $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$. We recall the following standard fact (see [7, Lemma 3.3]).

Lemma 2.2. *For every bigraph H, there exists $d \geq 1$ such that every graph of VC-dimension at least d contains a bi-induced copy of H. Conversely, for every $d \geq 1$, there is a bigraph H such that every graph containing a bi-induced copy of H has VC-dimension at least d.*

In view of this, Theorem 2.1 can be restated as follows.

Theorem 2.3. *For every bigraph H, there exists $K \geq 2$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every graph G with no bi-induced copy of H, there exists $L \in [\varepsilon^{-1}, \varepsilon^{-K}]$ for which there is an equipartition $V(G) = V_1 \cup \dots \cup V_L$ such that all but at most an ε fraction of the pairs (V_i, V_j) are weakly ε-pure.*

We use Theorem 2.1 to obtain a blockade that is weakly ε-pure and large (more specifically, we want the length and width to have polynomial dependence on ε).

Theorem 2.4. *For every bigraph H, there exists $b \geq 1$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every graph G with $|G| \geq \varepsilon^{-b}$ and no bi-induced copy of H, there is an $(\varepsilon^{-1}, \varepsilon^b|G|)$-blockade $(B_1, \dots, B_\ell)$ in G such that*

- *$|B_1| = \cdots = |B_\ell| \le \varepsilon^2|G|$; and*
- *for all distinct $i,j \in [\ell]$, B_i, B_j are ε-sparse to each other in G or $\overline{G}$.*

Proof. Let $K \ge 2$ be given by Theorem 2.3; we claim that $b := 5K$ satisfies the theorem. To this end, by Theorem 2.3 with ε^4 in place of ε, if G is a graph with no bi-induced copy of H, then there is an equipartition $V(G) = V_1 \cup \cdots \cup V_L$ with $L \in [\varepsilon^{-4}, \varepsilon^{-4K}]$ such that all but at most an ε^4 fraction of the pairs (V_i, V_j) are weakly ε^4-pure. By Turán's theorem, there exists $J \subseteq [L]$ with $|J| \ge \frac{1}{2}\varepsilon^{-4}$ such that (V_i, V_j) is weakly ε^4-pure for all distinct $i,j \in J$. Then there exists $I \subseteq J$ with $|I| \ge \frac{1}{2}|J| \ge \frac{1}{4}\varepsilon^{-4} \ge \varepsilon^{-1}$ such that $|V_i| = |V_j|$ for all distinct $i,j \in I$. Let $\ell := \lceil \varepsilon^{-1} \rceil$; it follows that there exists $I \subseteq J$ with $I = [\ell]$ such that $|V_i| = |V_j|$ for all distinct $i,j \in I$. For every $i \in I$, let B_i be the set of vertices v in V_i such that for every $j \in I \setminus \{i\}$,

- v has at most $\frac{1}{2}\varepsilon|V_j|$ neighbours in V_j if (V_i, V_j) is weakly ε^4-sparse in G; and
- v has at most $\frac{1}{2}\varepsilon|V_j|$ nonneighbours in V_j if (V_i, V_j) is weakly ε^4-sparse in $\overline{G}$.

Then

$$|B_i| \ge |V_i| - (\ell - 1) \cdot 2\varepsilon^3|V_i| \ge |V_i| - \varepsilon^{-1} \cdot 2\varepsilon^3|V_i| = (1 - 2\varepsilon^2)|V_i| \ge |V_i|/2$$

and by removing vertices if necessary we may assume that $|B_i| = \lceil |V_i|/2 \rceil = \lceil m/2 \rceil$ where $m = |V_i|$. It follows that for all distinct $i,j \in I$, B_i, B_j are ε-sparse to each other in G or $\overline{G}$. Also, since

$$m \ge \lfloor |G|/L \rfloor \ge |G|/(2L) \ge \varepsilon^{4K+1}|G|$$

(as $|G| \ge \varepsilon^{-b} = \varepsilon^{-5K}$), it follows that for each $i \in I$,

$$|B_i| \ge m/2 \ge \varepsilon^{4K+2}|G| \ge \varepsilon^{5K}|G| = \varepsilon^b|G|$$

and

$$|B_i| \le m \le \lceil |G|/L \rceil \le 2|G|/L \le 2\varepsilon^4|G| \le \varepsilon^2|G|.$$

This proves Theorem 2.4. ■

3. Iterative sparsification

In this section we give the proof of Theorem 1.5. In view of Lemma 2.2, Theorem 1.5 is equivalent to:

Theorem 3.1. *For every bigraph H, there exists $C > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$, every graph G with no bi-induced copy of H contains an ε-restricted induced subgraph with at least $\varepsilon^C|G|$ vertices.*

The proof of Theorem 3.1 uses the framework of *iterative sparsification*, which was introduced in earlier papers from this series [38, 39] (and will be used also in [40]). Recall that a graph is *ε-restricted* if either the graph or its complement is ε-sparse. The goal is to find an ε-restricted induced subgraph of size at least $\mathrm{poly}(\varepsilon)|G|$. Rather than doing this in one step, we will instead attempt to move through a sequence of induced subgraphs that are successively more restricted: given a y-restricted induced subgraph F, we search for an induced subgraph that is $\mathrm{poly}(y)$-restricted and is at most a $\mathrm{poly}(y)$ factor smaller. Provided we can start the process, and it does not get stuck on the way, the following lemma shows that the process gives the required subset.

Lemma 3.2. *Let $c \in (0,1)$, $a \ge 2$, and $t \ge 1$. Suppose that $x \in (0,c)$, and G is a graph satisfying:*

- *there is a c-restricted induced subgraph of G with at least $c^t|G|$ vertices; and*
- *for every $y \in [x,c]$ and every y-restricted induced subgraph F of G with $|F| \ge y^{2t}|G|$, there is a y^a-restricted induced subgraph of F with at least $y^{at}|F|$ vertices.*

Then G contains an x-restricted induced subgraph with at least $x^{2at}|G|$ vertices.

Proof. By the first condition of the lemma, there exists $y \in [x^a, c]$ minimal such that G has a y-restricted induced subgraph F with $|F| \ge y^{2t}|G|$. If $y \ge x$, then by the second condition of the lemma and since $a \ge 2$, F has a y^a-restricted induced subgraph with at least $y^{at}|F| \ge y^{at+2t}|G| \ge y^{2at}|G|$ vertices; but this contradicts the minimality of y since $x^a \le y^a < y$. Thus $x^a \le y < x$; and so F is x-restricted, and $|F| \ge y^{2t}|G| \ge x^{2at}|G|$. This proves Lemma 3.2. ■

We will find a subgraph satisfying the first bullet by using Rödl's theorem (Theorem 1.3), with a suitable $t = t(c)$. However, finding a subgraph that satisfies the second bullet is more challenging, and we need to allow for an alternative "good" outcome. We will show in Lemma 3.6 that if we get stuck then we can instead find a large complete or anticomplete blockade (note that this is much stronger than being pure; and the blockades given by Theorem 2.4 are only weakly ε-pure). Let us show that, if we can find sufficiently large complete or anticomplete blockades, then we can obtain the Erdős-Hajnal property; we will return to the (stronger) polynomial Rödl property at the end of the section.

Lemma 3.3. *Let $\mathcal{C}$ be a hereditary class of graphs. Suppose that there exists $d \geq 2$ such that for every $x \in (0, 2^{-d})$ and every $G \in \mathcal{C}$, either:*

- *G has an x-restricted induced subgraph with at least $x^d|G|$ vertices; or*
- *there is a complete or anticomplete $(k, |G|/k^d)$-blockade in G, for some $k \in [2, 1/x]$.*

Then there exists $a \geq 2$ such that every n-vertex graph in $\mathcal{C}$ has a clique or stable set of size at least $n^{1/a}$.

Proof. A *cograph* is a graph with no induced four-vertex path; and it is well-known that every k-vertex cograph has a clique or stable set of size at least $k^{1/2}$. Thus, it suffices to prove by induction that every $G \in \mathcal{C}$ contains an induced cograph of size at least $|G|^{1/(2d^2)}$. We may assume $|G| > 2^{2d^2}$. Let $x := |G|^{-1/(2d)} \in (0, 2^{-d})$. By the hypothesis, either:

- there exists $S \subseteq V(G)$ with $|S| \geq x^d|G|$ such that $G[S]$ is x-restricted; or
- there is a complete or anticomplete $(k, |G|/k^d)$-blockade in G, for some $k \in [2, 1/x]$.

If the first bullet holds, then since $|S| \geq x^d|G| \geq x^{-1}$, Turán's theorem gives a clique or stable set in $G[S]$ of size at least $(2x)^{-1} > x^{-1/2} = |G|^{1/(4d)}$. If the second bullet holds, then by induction and since $k \geq 2$, G contains an induced cograph of size at least

$$k(|G|/k^d)^{1/(2d^2)} = k^{1-1/(2d)}|G|^{1/(2d^2)} > |G|^{1/(2d^2)}.$$

This proves Lemma 3.3. ∎

The key step in making the iterative sparsification strategy work is therefore to show that if the second bullet of 3.2 does not hold then we can find a sufficiently large complete or anticomplete blockade. We will show this in Lemma 3.5: given a y-restricted graph F with no bi-induced H, we will prove that we can either pass to the desired poly(y)-restricted subgraph or find a complete or anticomplete blockade whose length and width depend polynomially on y. We will argue by induction on $|H|$, and grow the blockade one block at a time. The first step (Lemma 3.4) is to find a complete or anticomplete pair (A, B), where A has size poly$(y)|F|$ and B contains all but a small fraction of the rest of F; we then (Lemma 3.5) repeat the argument inside B, continuing until we obtain a blockade that is long enough. Restricted graphs can either be dense or sparse: we will assume for the moment that our restricted graph is sparse, and handle the dense case later by taking complements.

Lemma 3.4. *Let H be a bigraph, and let $v \in V(H)$. Let $b \geq 1$ be given by Theorem 2.4. Assume there exists $a \geq 2$ such that every n-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $y \in (0, 1/|H|)$, and let F be a y-sparse graph with no bi-induced copy of H. Then either:*

- *F has a y^{2a}-restricted induced subgraph with at least $y^{3ba^2}|F|$ vertices; or*
- *there is an anticomplete pair (A, B) in F with $|A| \geq y^{3ba^2}|F|$ and $|B| \geq (1 - 3y)|F|$.*

Proof. We have a sparse graph, and want to find an anticomplete pair (A, B). We will do this by first using ultraregularity to find a large, nearly-pure blockade and then looking at how the rest of the graph attaches to it. So let $\varepsilon := y^{3a^2}$, and suppose that the first outcome does not hold; then $|F| > y^{-3ba^2} = \varepsilon^{-b}$. By Theorem 2.4, F has a $(\varepsilon^{-1}, \varepsilon^b|F|)$-blockade $(B_1, \ldots, B_\ell)$ with $\ell = \lceil \varepsilon^{-1} \rceil$, such that:

- $|B_1| = \cdots = |B_\ell| \leq 2\varepsilon^2|F|$; and
- for all distinct $i, j \in [\ell]$, B_i, B_j are ε-sparse to each other in F or $\overline{F}$.

Let $D := V(F) \setminus (B_1 \cup \cdots \cup B_\ell)$ and $m := |B_1|$. For $i \in [\ell]$, a vertex $v \in D$ is *mixed* on B_i if it has a neighbour and a nonneighbour in B_i.

Claim. *Every vertex in D is mixed on at most $y\ell$ of the blocks $B_1, \ldots, B_\ell$.*

Subproof. Suppose there is a vertex $w \in D$ mixed on at least $y\ell$ of the blocks $B_1, \ldots, B_\ell$, say $B_1, \ldots, B_r$ where $r \geq y\ell \geq y\varepsilon^{-1} = y^{1-3a^2}$. Let J be the graph with vertex set $[r]$ where for all distinct $i, j \in [r]$, $ij \notin E(J)$ if and only if B_i, B_j are ε-sparse to each other in F.

We claim that there is no bi-induced copy of $H \setminus v$ in J. Suppose that there is; and we may assume $V(H \setminus v) \subseteq V(J)$. We assume that $v \in V_1(H)$ without loss of generality. For each $u \in V_2(H)$, let w_u be a neighbour of w in B_u if $uv \in E(H)$ and a nonneighbour of w in B_u if $uv \notin E(H)$. For each $z \in V_1(H) \setminus \{v\}$ and $u \in V_2(H)$, since $uz \notin E(H)$ if and only if $uz \notin E(J)$ if and only if B_u, B_z are ε-sparse to each other in F, w_u is adjacent in F to at most $\varepsilon|B_z|$ vertices in B_z if $uz \notin E(H)$ and nonadjacent in G to at most $\varepsilon|B_z|$ vertices in B_z if $uz \in E(H)$. Thus, for each $z \in V_1(H) \setminus \{v\}$, there are at least (note that $\varepsilon \leq y < 1/|H|$)

$$|B_z| - |V_2(H)|\varepsilon|B_z| \geq |B_z| - |H|y|B_z| > 0$$

vertices $z' \in B_z$ such that for every $u \in Y$, $w_u z' \in E(F)$ if and only if $uz \in E(H)$; let w_z be such a vertex. It follows that $\{w\} \cup \{w_z : z \in V_1(H) \setminus \{v\}\} \cup \{w_u : u \in Y\}$ forms a bi-induced copy of H in F, a contradiction.

Thus, there is no bi-induced copy of $H \setminus v$ in J. By the choice of a, J thus contains a clique or stable set I with $|I| \geq |J|^{1/a} \geq (y^{1-3a^2})^{1/a} = y^{-3a+1/a}$. Let $S := \bigcup_{i \in I} B_i$; then $|S| = |I|m$ since $|B_i| = m$ for all $i \in I$. If I is a stable set in J, then $F[S]$ has maximum degree at most

$$m + |I|\varepsilon m = (1/|I| + \varepsilon)|I|m \leq (y^{3a-1/a} + y^{3a})|S| \leq 2y^{3a-1}|S| \leq y^{2a}|S|;$$

and similarly, if I is a clique in J, then $\overline{F}[S]$ has maximum degree at most $y^{2a}|S|$. Thus $F[S]$ is y^{2a}-restricted which is the first outcome of the lemma, a contradiction. This proves the claim. $\square$

Now, by the claim, there exists $i \in [\ell]$ such that there are at most $y|D|$ vertices in D that are mixed on B_i. Since F is y-sparse, there are at most $y|F|$ vertices in D that are complete to B_i. Thus, since

$$|B_1| + \cdots + |B_\ell| \leq \ell\varepsilon^2|F| \leq 2\varepsilon|F| = 2y^{3a^2}|F| \leq y|F|,$$

there are at least

$$|F| - y|D| - y|F| - (|B_1| + \cdots + |B_\ell|) \geq (1 - 3y)|F|$$

vertices in F with no neighbour in B_i. Because $|B_i| \geq \varepsilon^b|F| = y^{3ba^2}|F|$, the second outcome of the lemma holds. This proves Lemma 3.4. ■

We now apply Lemma 3.4 repeatedly to move from an anticomplete pair to an anticomplete blockade.

Lemma 3.5. *Let H be a bigraph, and let $v \in V(H)$. Let $b \geq 1$ be given by Theorem 2.4. Assume there exists $a \geq 2$ such that every n-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $0 < y \leq 2^{-12|H|}$, and let F be a y-sparse graph with no bi-induced copy of H. Then either:*

- *F has a y^a-restricted induced subgraph with at least $y^{2ba^2}|F|$ vertices; or*
- *there is an anticomplete $(y^{-1/2}, y^{2ba^2}|F|)$-blockade in F.*

Proof. Suppose that the second outcome does not hold. Let $n \geq 0$ be maximal such that there is a blockade $(B_0, B_1, \ldots, B_n)$ in F with $|B_n| \geq (1 - 3y^{1/2})^n|F|$ and $|B_{i-1}| \geq y^{2ba^2}|F|$ for all $i \in [n]$. Since the second outcome does not hold, $n < y^{-1/2}$; and so, since $y \leq 2^{-12}$,

$$|B_n| \geq (1 - 3y^{1/2})^n|F| \geq 4^{-3y^{1/2}n}|F| \geq 4^{-3}|F| \geq y^{1/2}|F|.$$

Hence $F[B_n]$ has maximum degree at most $y|F| \leq y^{1/2}|B_n|$; and so Lemma 3.4 (with $y^{1/2}$ in place of y, note that $y^{1/2} \leq 2^{-6|H|} < 1/|H|$) implies that either:

- there exists $S \subseteq B_n$ with $|S| \geq y^{3ba^2/2}|B_n|$ such that $F[S]$ is y^a-restricted; or
- there is an anticomplete pair (A, B) in $F[B_n]$ with $|A| \geq y^{3ba^2/2}|B_n|$ and $|B| \geq (1 - 3y^{1/2})|B_n|$.

If the second bullet holds, then $(B_0, B_1, \ldots, B_{n-1}, A, B)$ would be a blockade contradicting the maximality of n since $|A| \geq y^{3ba^2/2}|B_n| \geq y^{3ba^2/2+1/2}|F| \geq y^{2ba^2}|F|$. Thus the first bullet holds; and so $S \subseteq V(F)$ is y^a-restricted in F and satisfies $|S| \geq y^{3ba^2/2}|B_n| \geq y^{3ba^2/2+1/2}|G| \geq y^{2ba^2}|F|$. Hence the first outcome of the lemma holds. This proves Lemma 3.5. ■

For a bigraph H, its *bicomplement* is the bigraph $\overline{\overline{H}}$ with the same bipartition and edge set $\{uv : u \in V_1(H), v \in V_2(H), uv \notin E(H)\}$. We can now prove that the conditions of Lemma 3.3 are satisfied.

Lemma 3.6. *For every bigraph H, there exists $d \geq 2$ such that for every $x \in (0, 2^{-d})$ and every graph G with no bi-induced copy of H, either:*

- *G has an x-restricted induced subgraph with at least $x^d|G|$ vertices; or*
- *there is a complete or anticomplete $(k, |G|/k^d)$-blockade in G, for some $k \in [2, 1/x]$.*

Proof. We argue by induction on $|H|$. We may assume that $|H| \geq 2$. Choose $v \in V(H)$. By Lemma 3.3 and the induction hypothesis applied to $H \setminus v$, there exists $a \geq 4$ such that every n-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. By taking complements, it follows that every n-vertex graph with no bi-induced copy of $\overline{\overline{H}} \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $c := 2^{-12|H|}$, and let $b \geq 1$ be given by Theorem 2.4. By Rödl's theorem 1.3, we can choose some $t \geq ba^2$ such that every graph G with no bi-induced copy of H contains a c-restricted induced subgraph with at least $c^t|G|$ vertices. We claim that $d := 2\max(at, |H|) \geq 8t$ satisfies the theorem.

To show this, let $x \in (0, 2^{-d}) \subseteq (0, c)$, and suppose that G has no bi-induced copy of H. Suppose that the second outcome of the lemma does not hold; that is, there is no $k \in [2, 1/x]$ such that there is a complete or anticomplete $(k, |G|/k^d)$-blockade in G.

Claim. *For every $y \in [x, c]$ and every y-restricted induced subgraph F of G with $|F| \geq y^{2t}|G|$, there is a y^a-restricted induced subgraph of F with at least $y^{at}|F|$ vertices.*

Subproof. We claim that either:

- F has a y^a-restricted induced subgraph with at least $y^{2ba^2}|F| \geq y^{2t}|F| \geq y^{at}|F|$ vertices; or
- there is a complete or anticomplete $(y^{-1/2}, y^{2ba^2}|F|)$-blockade in F.

If F is y-sparse, then one of the bullets holds by Lemma 3.5. If not, then $\overline{F}$ is y-sparse and $\overline{\overline{H}}$-free, and contains no bi-induced copy of $\overline{\overline{H}} \setminus v$, and so one of the bullets holds by Lemma 3.5 applied to $\overline{F}$.

If the second bullet holds, then since $y^{2ba^2}|F| \geq y^{4t}|G| \geq y^{d/2}|G|$ by the choice of d, there would be a complete or anticomplete $(y^{-1/2}, y^{d/2}|G|)$ blockade in G, which contradicts that the second outcome of the lemma does not hold (note that $y^{1/2} \leq c^{1/2} \leq \frac{1}{2}$). Thus the first bullet holds, proving the claim. □

Lemma 3.2 and the claim imply that G has an x-restricted induced subgraph with at least $x^{2at}|G| \geq x^d|G|$ vertices, which is the first outcome of the theorem. This proves Lemma 3.6. ■

Lemmas 3.3 and 3.6 show the Erdős-Hajnal property for every class of graphs with no bi-induced copy of a given bipartite graph. But much of the argument so far was about density, and that allows us to go a step further, and apply Theorem 2.4 once more to prove Theorem 3.1.

Proof of Theorem 3.1. It follows from Lemmas 3.3 and 3.6 that there exists $a \geq 2$ such that every n-vertex graph with no bi-induced copy of H has a clique or stable set of size at least $n^{1/a}$; and by increasing a if necessary we may assume $2^a > |H|$. Let $b \geq 1$ be given by Theorem 2.4. We claim that $C := 2ab$ satisfies the theorem. To see this, let $\varepsilon \in (0, \frac{1}{2})$ and G be a graph with no bi-induced copy of H. It suffices to show that G has an ε-restricted induced subgraph with at least $\varepsilon^{2ab}|G|$ vertices.

We may assume that $|G| > \varepsilon^{-2ab}$. By Theorem 2.4 with ε^{2a} in place of ε, there is an $(\varepsilon^{-2a}, \varepsilon^{2ab}|G|)$-blockade $(B_1, \dots, B_\ell)$ in G where $\ell \geq \varepsilon^{-2a}$, such that $|B_1| = \cdots = |B_\ell| =: m$; and for all distinct $i, j \in [\ell]$, B_i, B_j are ε^{2a}-sparse to each other in G or $\overline{G}$. Let J be the graph with vertex set $[\ell]$ where $ij \notin E(J)$ if and only if B_i, B_j are ε^{2a}-sparse to each other in G.

Claim. *J has no bi-induced copy of H.*

Subproof. Suppose not; and we may assume $V(H) \subseteq V(J)$. For every $i \in V(H)$, let v_i be a uniformly random vertex in B_i, chosen independently. Then for (X, Y) the bipartition of H, the probability that $\{v_1, \dots, v_{|H|}\}$ forms a bi-induced copy of H in G is at least $1 - |X||Y|\varepsilon^{2a} > 1 - |H|^2 2^{-2a} > 0$ by the choice of a; and so there is a bi-induced copy of H in G, a contradiction. This proves the claim. □

Now, by the claim and the choice of a, J has a clique or stable set I with $|I| \geq |J|^{1/a} \geq \varepsilon^{-2}$. Let $S := \bigcup_{i \in I} B_i$; then $|S| = |I|m \geq m \geq \varepsilon^{2ab}|G|$. If I is stable in J, then $G[S]$ has maximum degree at most

$$m + |I|\varepsilon^a m = (1/|I| + \varepsilon^a)|I|m \leq 2\varepsilon^2|S| \leq \varepsilon|S|.$$

Similarly, if I is a clique in J, then $\overline{G}[S]$ has maximum degree at most $\varepsilon|S|$. This proves Theorem 3.1. ■

4. Ramsey numbers for hypergraphs with bounded VC-dimension

A *k-uniform hypergraph* is a hypergraph whose edges all have cardinality k. The notion of VC-dimension extends naturally to k-uniform hypergraphs $\mathcal{H}$. For a set S, let $\binom{S}{k}$ denote the family of subsets of size k of S; and for $S \in \binom{V(\mathcal{H})}{k-1}$, let $N_{\mathcal{H}}(S) := \{v \in V(\mathcal{H}) : S \cup \{v\} \in E(\mathcal{H})\}$. The *VC-dimension* of $\mathcal{H}$ is the VC-dimension of the family $\{N_{\mathcal{H}}(S) : S \in \binom{V(\mathcal{H})}{k-1}\}$ with ground set $V(\mathcal{H})$.

A *clique* in a k-uniform hypergraph $\mathcal{H}$ is a subset of $V(\mathcal{H})$ whose subsets of size k are all in $E(\mathcal{H})$, and a *stable set* is a clique in the complementary k-uniform hypergraph. The *Ramsey number* $R_k(t)$ is the smallest integer $n \geq 1$ such that every n-vertex k-uniform hypergraph contains a clique or stable set of size at least n. By Ramsey's theorem, $R_k(t)$ exists for all values of t and k; equivalently, writing $f_k(n)$ for the largest m such that every k-uniform hypergraph with n vertices has a clique or stable set of size at least m, we have $f_k(n) \to \infty$ for any fixed k. The growth rate of f_k has received considerable attention: for $k=2$, it is well known that $f_k(n)=\Theta(\log n)$; but for $k\ge 3$ the bounds are much farther apart. For $k=3$, Erdős, Hajnal and Rado [19] showed that

$$c_1 \log\log n \le f_3(n) \le c_2\sqrt{\log n};$$

and for $k\ge 4$, the best bounds are

$$c_1 \log^{(k-1)}(n) \le f_k(n) \le c_2\sqrt{\log^{(k-2)} n},$$

where $\log^{(k)} n$ denotes the k-times iterated logarithm.

What can be said about hypergraphs of bounded VC-dimension? Let $f_k^d(n)$ be the largest integer m such that every k-uniform hypergraph with n vertices and VC-dimension at most d has a clique or stable set of size at least m. For $k\ge 3$, the best previous bounds are

$$2^{(\log^{(k-1)} n)^{1-o(1)}} \le f_k^d(n) \le c_2 \log^{(k-2)}(n),$$

where the upper follows from a construction of Conlon, Fox, Pach, Sudakov, and Suk [16] and the lower bound is is due to Fox, Pach, and Suk [22], who observed that it follows from their $2^{(\log n)^{1-o(1)}}$ bound for graphs of bounded VC-dimension by an adaptation of the classical Erdős-Rado greedy argument [20].

Stronger lower bounds have been obtained in more restrictive settings. Conlon, Fox, Pach, Sudakov, and Suk [16] proved that every semi-algebraic n-vertex k-uniform hypergraph of bounded description complexity admits a clique or stable set of size at least $2^{c(\log^{(k-1)} n)}$, where the constant $c>0$ depends on k and the description complexity of the hypergraph. This was later extended to k-uniform hypergraphs definable by distal structures by Chernikov, Starchenko, and Thomas [11]. Combining the Erdős-Rado argument with our Theorem 1.2 allows us to extend this further to hypergraphs with bounded VC-dimension, improving the lower bound to show that, for fixed $k\ge 3$ and $d\ge 1$,

$$f_k^d(n) = 2^{\Theta(\log^{(k-1)} n)}.$$

We give the details for completeness. For integers $d\ge 1$, $k\ge 2$, and $n\ge 1$, let $R_k^d(n)$ be the smallest integer $m\ge 1$ such that every m-vertex k-uniform hypergraph of VC-dimension at most d contains a clique or stable set of size at least n; then Theorem 1.2 says that $R_2^d(n)\le n^C$ for all $n\ge 1$, for some $C>0$ depending on d only.

Theorem 4.1. *For all integers $d\ge 1$, $k\ge 3$, and $n\ge 2$,*

$$R_k^d(n) \le 2^{\binom{R_{k-1}^d(n-1)}{k-1}} + k - 2.$$

Proof. Let $p := R_{k-1}^d(n-1)$. Let $\mathcal{H}$ be a k-uniform hypergraph with $2^{\binom{p}{k-1}}+k-2$ vertices and VC-dimension at most d. We claim that:

Claim. *For every integer q with $k-2\le q\le p$, there are disjoint $A_q, B_q \subseteq V(\mathcal{H})$ with $|A_q|=q$ and $|B_q|\ge 2^{\binom{p}{k-1}-\binom{q}{k-1}}$ such that for every $S\subseteq \binom{A_q}{k-1}$, either $S\cup\{v\}\in E(\mathcal{H})$ for all $v\in B_q$ or $S\cup\{v\}\notin E(\mathcal{H})$ for all $v\in B_q$.*

Subproof. The claim is vacuously true for $q=k-2$. For $k-2\le q<p$, we shall prove the claim of $q+1$ assuming that it is true for q. Let $u\in B_q$ be arbitrary, and let $A_{q+1}:=A_q\cup\{u\}$. For every $T\in\binom{A_q}{k-2}$ and $v\in B_q\setminus\{u\}$, let $f_T(v)$ be 0 if $T\cup\{u,v\}\in E(\mathcal{H})$ and 1 if $T\cup\{u,v\}\notin E(\mathcal{H})$. Then there exists $B_{q+1}\subseteq B_q$ with

$$\begin{aligned}|B_{q+1}| \ge \left\lceil 2^{-\binom{|A_q|}{k-2}}(|B_q|-1)\right\rceil &\ge \left\lceil 2^{-\binom{q}{k-2}}\left(2^{\binom{p}{k-1}-\binom{q}{k-1}}-1\right)\right\rceil\\ &= \left\lceil 2^{\binom{p}{k-1}-\binom{q+1}{k-1}} - 2^{-\binom{q}{k-2}}\right\rceil = 2^{\binom{p}{k-1}-\binom{q+1}{k-1}},\end{aligned}$$

such that $f_T(v)=f_T(w)$ for all $T\in\binom{A_q}{k-2}$ and $v,w\in B_{q+1}$; here the last equation holds since $\binom{q+1}{k-1}\le\binom{p}{k-1}$ and $\binom{q}{k-2}\ge 1$. Thus, by the choice of A_q,B_q, it follows that for every $S\subseteq\binom{A_{q+1}}{k-1}$, either $S\cup\{v\}\in E(\mathcal{H})$ for all $v\in B_{q+1}$ or $S\cup\{v\}\notin E(\mathcal{H})$ for all $v\in B_{q+1}$. This proves Section 4. $\square$

Now, Section 4 with $q=p$ gives disjoint $A_p,B_p\subseteq V(\mathcal{H})$ with $|A_p|=p$ and $|B_p|\ge 1$ such that for every $S\subseteq\binom{A_p}{k-1}$, either $S\cup\{v\}\in E(\mathcal{H})$ for all $v\in B_p$ or $S\cup\{v\}\notin E(\mathcal{H})$ for all $v\in B_p$. Let $v\in B_p$; and let $\mathcal{H}'$ be the $(k-1)$-uniform hypergraph with vertex set A_p where for every $S\in\binom{A_p}{k-1}$, $S\in E(\mathcal{H}')$ if and only if $S\cup\{v\}\in E(\mathcal{H})$.

Claim. *$\mathcal{H}'$ has VC-dimension at most d.*

Subproof. Suppose not; then there exists $A \subseteq A_p$ with $|A| > d$ which is shattered by the family $\{N_{\mathcal{H}'}(T) : T \in \binom{A_p}{k-2}\}$ with ground set A_p. However, by the definition of $\mathcal{H}'$, A is then shattered by the family $\{N_{\mathcal{H}}(T \cup \{v\}) : T \in \binom{A_p}{k-2}\} \subseteq \{N_{\mathcal{H}}(S) : S \in \binom{V(\mathcal{H})}{k-1}\}$ with ground set $V(\mathcal{H})$, contrary to $\mathcal{H}$ having VC-dimension at most d. This proves Section 4. □

Now, by the definition of p, Section 4 implies that $\mathcal{H}'$ has a clique or stable set S with $|S| \geq n-1$. Then $S \cup \{v\}$ is a clique or stable set in $\mathcal{H}$ of size at least n. This proves Theorem 4.1. ■

It is now not hard to iterate Theorem 4.1 and apply the bound $R_2^d(n) \leq n^C$ to get $R_k^d(n) \leq \operatorname{twr}_{k-1}(n^K)$ for some $K > 0$ depending on d, k, where twr_k is defined recursively for $k \geq 1$ by $\operatorname{twr}_1(t) := t$ and $\operatorname{twr}_k(t) := 2^{\operatorname{twr}_{k-1}(t)}$ for all $t > 0$. In other words, $f_k^d(n) \geq (\log^{(k-2)} n)^{1/K}$.

5. The viral property

Let us interpret Theorem 3.1 in the language of forbidden induced subgraphs. A *split* graph is a graph whose vertex set can be partitioned into a clique and a stable set. For graphs H, G, a *copy* of H in G is an injective map $\varphi\colon V(H) \to V(G)$ such that for all distinct $u, v \in V(H)$, $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$. For a finite family $\mathcal{F}$ of graphs, a graph G is *$\mathcal{F}$-free* if there is no copy of H in G for all $H \in \mathcal{F}$. The following is a well-known strengthening of Lemma 2.2 (see [7, Theorem 3.3]).

Lemma 5.1. *For every two bipartite graphs H_1, H_2 and every split graph J, there exists $d \geq 1$ such that every $\{H_1, \overline{H_2}, J\}$-free graph has VC-dimension at most d. Conversely, for every $d \geq 1$, there are bipartite graphs H_1, H_2 and a split graph J such that every graph of VC-dimension at most d is $\{H_1, \overline{H_2}, J\}$-free.*

Thus, Theorem 1.5 can be rewritten as:

Theorem 5.2. *For every two bipartite graphs H_1, H_2 and every split graph J, the class of $\{H_1, \overline{H_2}, J\}$-free graphs has the polynomial Rödl property.*

We say that a finite family $\mathcal{F}$ of graphs is *viral* if there exists $C > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and for every graph G with at most $(\varepsilon^C|G|)^{|H|}$ copies of H for all $H \in \mathcal{F}$, there is an ε-restricted induced subgraph of G with at least $\varepsilon^C|G|$ vertices. Thus, if $\mathcal{F}$ is viral then the class of $\mathcal{F}$-free graphs has the polynomial Rödl property. It is conjectured in [39] that $\{H\}$ is viral for all graphs H, which would mean that Nikiforov's theorem [41] holds with polynomial dependence. Recently, Gishboliner and Shapira [28] showed that for every two bipartite graphs H_1, H_2 and every split graph J, $\{H_1, \overline{H_2}, J\}$ is viral if and only if the class of $\{H_1, \overline{H_2}, J\}$-free graphs has the Erdős-Hajnal property, by using a strengthening of Theorem 2.1 for graphs with few copies of $H_1, \overline{H_2}, J$. Thus one can bootstrap Theorem 5.2 into the following.

Theorem 5.3. *For every two bipartite graphs H_1, H_2 and every split graph J, $\{H_1, \overline{H_2}, J\}$ is viral.*

6. Tournaments

A class $\mathcal{T}$ of tournaments has the *Erdős-Hajnal property* if there exists $c > 0$ such that every tournament $T \in \mathcal{T}$ contains a transitive subtournament with at least $|T|^c$ vertices. If Q is a tournament, a tournament is *Q-free* if it has no subtournament isomorphic to Q. We say that Q has the *Erdős-Hajnal property* if the class of Q-free tournaments has the Erdős-Hajnal property.

Alon, Pach, and Solymosi [3] proved that Conjecture 1.1 is equivalent to the following statement.

Conjecture 6.1. *Every tournament has the Erdős-Hajnal property.*

For a tournament Q and an ordering $\phi = (v_1, \dots, v_n)$ of $V(Q)$, the *backedge graph* of Q with respect to ϕ is the graph G with vertex set $V(Q)$, where for all $i, j \in [n]$ with $i < j$, $v_iv_j \in E(G)$ if and only if $(v_j, v_i) \in E(Q)$. For two tournaments Q_1, Q_2, the tournament Q obtained by *substituting* Q_2 for $v \in V(Q_1)$ has vertex set $V(Q_2) \cup (V(Q_1) \setminus \{v\})$, such that $Q[V(Q_1) \setminus \{v\}] = Q_1 \setminus v$, $Q[V(Q_2)] = Q_2$, and for all $u \in V(Q_1) \setminus \{v\}$ and $w \in V(Q_2)$, $(u, w) \in E(Q)$ if and only if $(u, v) \in E(Q_1)$. A tournament is *prime* if it cannot be obtained by substitution from tournaments with fewer vertices. By an adaptation of another argument of Alon, Pach, and Solymosi in [3], Conjecture 1.1 reduces to proving that all prime tournaments have the Erdős-Hajnal property. Partial results in this direction include [5, 12, 39, 52–54]; these show the Erdős-Hajnal property of several types of prime tournaments, that all admit a forest backedge graph except those from [39]. In another direction, Berger, Choromanski, and Chudnovsky [6] proved Conjecture 6.1 for every six-vertex tournament except for the seven-vertex Paley tournament with one vertex removed.

For an integer $k \geq 0$, a tournament is *k-colourable* if its vertex set can be partitioned into k subsets each inducing a transitive subtournament. It is known (see [14, Lemma 3.2]) that a tournament has no copy of some two-colourable tournament if and only if the set of its in-neighbourhoods has bounded VC-dimension. Moreover, a standard argument together with Lemma 2.2 proves the following.

Lemma 6.2. *For every $d \geq 1$, there exists $m \geq 1$ such that for every tournament T, if the set of its in-neighbourhoods has VC-dimension at most d then every backedge graph of T has VC-dimension at most m. Conversely, for every $m \geq 1$, there exists $d \geq 1$ such that if a backedge graph of a tournament T has VC-dimension at most m, then the set of in-neighbourhoods of T has VC-dimension at most d.*

Therefore, since every backedge graph of every n-vertex transitive tournament is a comparability graph (and so has a clique or stable set of size at least $\sqrt{n}$), Theorem 1.2 can be restated as follows.

Theorem 6.3. *Every two-colourable tournament has the Erdős-Hajnal property.*

Tournaments that admit forest backedge graphs are two-colourable, so this result contains all of the aforementioned results on prime tournaments with the Erdős-Hajnal property, except those from [39]; and since all six-vertex tournaments are two-colourable (see [37]), Theorem 6.3 implies that they all satisfy Conjecture 6.1 as well. One can also formulate a viral version for tournaments similar to the one for graphs and prove that every two-colourable tournament is viral; we omit the details.

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