

Polynomial bounds for chromatic number.  
I. Excluding a biclique and an induced tree

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### Abstract

Let  $H$  be a tree. It was proved by Rödl that graphs that do not contain  $H$  as an induced subgraph, and do not contain the complete bipartite graph  $K_{t,t}$  as a subgraph, have bounded chromatic number. Kierstead and Penrice strengthened this, showing that such graphs have bounded degeneracy. Here we give a further strengthening, proving that for every tree  $H$ , the degeneracy is at most polynomial in  $t$ . This answers a question of Bonamy, Pilipczuk, Rzazewski, Thomassé and Walczak.

# 1 Introduction

The Gyárfás-Sumner conjecture [4, 13] asserts:

**1.1 Conjecture:** *For every forest  $H$ , there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H$ -free graph  $G$ .*

(We use  $\chi(G)$  and  $\omega(G)$  to denote the chromatic number and the clique number of a graph  $G$ , and a graph is  $H$ -free if it has no induced subgraph isomorphic to  $H$ .) One attractive feature of this conjecture is that it is best possible in a sense: for every graph  $H$  that is not a forest, there is no function  $f$  as in 1.1 (this is easily shown with a random graph). The conjecture has been proved for some special families of trees (see, for example, [2, 5, 6, 7, 9, 10, 11]) but remains open in general.

A class  $\mathcal{C}$  of graphs is  $\chi$ -bounded if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every graph  $G$  that is an induced subgraph of a member of  $\mathcal{C}$  (see [12] for a survey). Thus the Gyárfás-Sumner conjecture asserts that, for every forest  $H$ , the class of all  $H$ -free graphs is  $\chi$ -bounded. Esperet [3] asked whether, for every  $\chi$ -bounded class,  $f$  can always be chosen to be a polynomial. Neither conjecture has been settled in general.

The complete bipartite graph with parts of cardinality  $s, t$  is denoted by  $K_{s,t}$ . Let us define  $\tau(G)$  to be the largest  $t$  such that  $G$  contains  $K_{t,t}$  as a subgraph (not necessarily induced). It was proved by Rödl (mentioned in [8], and see also [6]) that the analogue of the Gyárfás-Sumner conjecture is true if we replace  $\omega(G)$  by  $\tau(G)$ . That is:

**1.2** *For every forest  $H$ , there is a function  $f$  such that  $\chi(G) \leq f(\tau(G))$  for every  $H$ -free graph  $G$ .*

This has the same attractive feature that the result is best possible (in the same sense).

This result was strengthened by Kierstead and Penrice. Let us say a graph  $G$  is  $d$ -degenerate (where  $d \geq 0$  is an integer) if every nonnull subgraph has a vertex of degree at most  $d$ ; and the degeneracy  $\partial(G)$  of  $G$  is the smallest  $d$  such that  $G$  is  $d$ -degenerate. Then  $\chi(G) \leq \partial(G) + 1$ , and so the following result of Kierstead and Penrice [7] is a strengthening of 1.2:

**1.3** *For every forest  $H$ , there is a function  $f$  such that  $\partial(G) \leq f(\tau(G))$  for every  $H$ -free graph  $G$ .*

What about the analogue of Esperet's question: do 1.2 and 1.3 remain true if we require  $f$  to be a polynomial in  $\tau(G)$ ? This question was raised by Bonamy, Bousquet, Pilipczuk, Rzazewski, Thomassé and Walczak in [1], and they proved it when  $H$  is a path, that is:

**1.4** *For every path  $H$ , there exists  $c > 0$  such that  $\partial(G) \leq \tau(G)^c$  for every  $H$ -free graph  $G$ .*

In this paper we answer the question completely. Our main result is:

**1.5** *For every forest  $H$ , there exists  $c > 0$  such that  $\partial(G) \leq \tau(G)^c$  for every  $H$ -free graph  $G$ .*

We also look at a related question: what can we say about  $\chi(G)$  and  $\partial(G)$  if  $G$  is  $H$ -free and does not contain  $K_{s,t}$  as a subgraph? More exactly, if  $H, s$  are fixed, how do  $\chi(G)$  and  $\partial(G)$  depend on  $t$ ? We will show that the dependence is in fact linear in  $t$ :

**1.6** *For every forest  $H$  and every integer  $s > 0$ , there exists  $c > 0$  such that for every graph  $G$  and every integer  $t > 0$ , if  $G$  is  $H$ -free and does not contain  $K_{s,t}$  as a subgraph, then  $\partial(G) \leq ct$ .*

We also prove a weaker result, that under the same hypotheses,  $\chi(G) \leq ct$ , and for this the bound on  $c$  is a small function of  $s, H$ .

Finally, there is a second pretty theorem in the paper [1] of Bonamy, Pilipczuk, Rzazewski, Thomassé and Walczak:

**1.7** *Let  $\ell$  be an integer; then there exists  $c > 0$  such that  $\partial(G) \leq \tau(G)^c$  for every graph  $G$  with no induced cycle of length at least  $\ell$ .*

We give a new proof of this, simpler than that in [1].

In this paper, all graphs are finite and have no loops or parallel edges. We denote by  $|H|$  the number of vertices of a graph  $H$ . If  $X \subseteq V(G)$ , we denote the subgraph of  $G$  induced on  $X$  by  $G[X]$ . We use “ $G$ -adjacent” to mean adjacent in  $G$ , and “ $G$ -neighbour” to mean a neighbour in  $G$ , and so on.

## 2 Producing a path-induced rooted tree.

We will prove 1.5 in this section and the next. We need to show that if a graph  $G$  has degeneracy at least some very large polynomial in  $t$  (independent of  $G$ ), and does not contain  $K_{t,t}$  as a subgraph, then it contains any desired tree as an induced subgraph. We will show this in two stages: in this section we will show that  $G$  contains a large (with degrees a somewhat smaller polynomial in  $t$ ) “path-induced” tree, and in the next section we will convert this to the desired induced tree. “Path-induced” means that each path of the tree starting at the root is an induced path of  $G$ ; so we should be talking about rooted trees. Let us say this carefully.

A *rooted tree*  $(H, r)$  consists of a tree  $H$  and a vertex  $r$  of  $H$  called the *root*. A rooted subtree of  $(H, r)$  means a rooted tree  $(J, r)$  where  $J$  is a subtree of  $H$  and  $r \in V(J)$ . The *height* of  $(H, r)$  is the length (number of edges) of the longest path of  $H$  with one end  $r$ . If  $u, v \in V(H)$  are adjacent and  $u$  lies on the path of  $H$  between  $v, r$ , we say  $v$  is a *child* of  $u$  and  $u$  is the *parent* of  $v$ . The *spread* of  $H$  is the maximum over all vertices  $u \in V(H)$  of the number of children of  $u$ . (Thus the spread is usually one less than the maximum degree.) Let  $H$  be a subgraph of  $G$  (not necessarily induced), where  $(H, r)$  is a rooted tree. We say that  $(H, r)$  is a *path-induced rooted subgraph* of  $G$  if every path of  $H$  with one end  $r$  is an induced subgraph of  $G$ .

Let  $\zeta, \eta \geq 1$ . The rooted tree  $(H, r)$  is  $(\zeta, \eta)$ -uniform if

- every vertex with a child has exactly  $\zeta$  children;
- every vertex with no child is joined to  $r$  by a path of  $H$  of length exactly  $\eta$ .

We need two lemmas:

**2.1** *Let  $k, \zeta, \eta \geq 1$  with  $\zeta \geq 2$ , and let  $(H_1, r_1), \dots, (H_k, r_k)$  be  $(k\zeta^{\eta+1}, \eta)$ -uniform rooted trees, each a subgraph of a graph  $G$ , such that  $r_i \notin V(H_j)$  for all distinct  $i, j \in \{1, \dots, k\}$ . Then for  $1 \leq i \leq k$  there is a  $(\zeta, \eta)$ -uniform rooted subtree  $(H'_i, r_i)$  of  $(H_i, r_i)$ , such that the trees  $H'_1, \dots, H'_k$  are pairwise vertex-disjoint.*

**Proof.** Choose  $j \leq k$  maximum such that there are  $(\zeta, \eta)$ -uniform rooted subtrees  $(H'_i, r_i)$  of  $(H_i, r_i)$  for  $1 \leq i \leq j$ , such that the trees  $H'_1, \dots, H'_j$  are pairwise vertex-disjoint. Let  $X = V(H'_1) \cup \dots \cup V(H'_j)$ . Thus  $|X| \leq j\zeta^{\eta+1}$ , since each  $H'_i$  has

$$1 + \zeta + \zeta^2 + \dots + \zeta^\eta \leq \zeta^{\eta+1}$$

vertices (here we use that  $\zeta \geq 2$ ). Suppose that  $j < k$ . Then each vertex of  $(H_{j+1}, r_{j+1})$  with a child has at least  $(k-j)\zeta^{\eta+1} \geq \zeta^{\eta+1} \geq \zeta$  children not in  $X$ ; and since  $r_{j+1} \notin X$ , it follows that there is a  $(\zeta, \eta)$ -uniform rooted subtree  $(H'_{j+1}, r_{j+1})$  of  $(H_{j+1}, r_{j+1})$  vertex-disjoint from  $X$ , contrary to the maximality of  $j$ . Thus  $j = k$ , and this proves 2.1.  $\blacksquare$

Let  $t, \eta \geq 1$  and  $\zeta \geq 2$  be integers. Let  $(T, r)$  be a  $(\zeta, \eta)$ -uniform rooted tree, where  $T$  is a subgraph of  $G$ . A vertex  $u \in V(G) \setminus V(T)$  is *t-bad* for  $(T, r)$  if there is a vertex  $w \in V(T)$ , with  $\zeta$  children in  $(T, r)$ , such that  $u$  is  $G$ -adjacent to more than  $(t-1)\zeta/t$  of these children. We will often use the following:

**2.2** *Let  $t, \eta \geq 1$  and  $\zeta \geq 2$  be integers. Let  $(T, r)$  be a  $(t\zeta, \eta)$ -uniform rooted tree, where  $T$  is a subgraph of  $G$ ; and let  $u \in V(G) \setminus V(T)$ . If  $u$  is not  $t$ -bad for  $(T, r)$ , then there is a  $(\zeta, \eta)$ -uniform rooted subtree  $(S, r)$  of  $(T, r)$  such that  $u$  has no  $G$ -neighbour in  $V(S)$  except possibly  $r$ .*

We omit the proof, which is clear. The second lemma is:

**2.3** *Let  $t, \eta \geq 1$  and  $\zeta \geq 2$  be integers, where  $t$  divides  $\zeta$ . Let  $G$  be a graph that does not contain  $K_{t,t}$  as a subgraph, and let  $(T, r)$  be a  $(\zeta, \eta)$ -uniform rooted tree, where  $T$  is a subgraph of  $G$ . Then at most  $\zeta^\eta(t-1)$  vertices in  $V(G) \setminus V(T)$  are  $t$ -bad for  $(T, r)$ .*

**Proof.** For each  $w \in V(T)$  that has  $\zeta$  children, let  $C_w$  be the set of its children in  $(T, r)$ . Suppose that there are  $t$  distinct vertices  $u_1, \dots, u_t$  in  $V(G) \setminus V(T)$  such that each is  $G$ -adjacent to more than  $|C_w|(t-1)/t$  vertices in  $C_w$ . Since  $t$  divides  $|C_w|$ , it follows that each  $u_i$  has at most  $|C_w|/t - 1$   $G$ -non-neighbours in  $C_w$ , and so at most  $t(|C_w|/t - 1)$  vertices in  $C_w$  are  $G$ -nonadjacent to one of  $u_1, \dots, u_t$ . Consequently at least  $t$  vertices in  $C_w$  are  $G$ -adjacent to all of  $u_1, \dots, u_t$ , contradicting that  $G$  does not contain  $K_{t,t}$  as a subgraph. Thus there are at most  $t-1$  vertices in  $V(G) \setminus V(T)$  with more than  $|C_w|(t-1)/t$   $G$ -neighbours in  $C_w$ . So the number of vertices in  $V(G) \setminus V(T)$  that are  $t$ -bad for  $(T, r)$  is at most  $(t-1)$  times the number of vertices of  $T$  that have children, and so at most  $\zeta^\eta(t-1)$  (since  $\zeta \geq 2$ ). This proves 2.3.  $\blacksquare$

The main result of this section is the following:

**2.4** *Let  $\eta > 0$  be an integer and let  $c = (\eta+1)!$ . Let  $\zeta \geq 2$ , and let  $(H, r)$  be a rooted tree of height at most  $\eta$ , and spread at most  $\zeta$ . Let  $t \geq 1$  be an integer, and suppose that the graph  $G$  does not contain  $K_{t,t}$  as a subgraph, and does not contain a rooted tree isomorphic to  $(H, r)$  as a path-induced rooted subgraph. Then  $\partial(G) \leq (\zeta t)^c$ .*

**Proof.** We may assume that  $t \geq 2$ . We proceed by induction on  $\eta$ . If  $\eta = 1$ , it follows that  $G$  has maximum degree at most  $\zeta - 1$ , since it does not contain  $(H, r)$  as a path-induced rooted subgraph; and so  $\partial(G) \leq \zeta - 1 \leq (\zeta t)^c$  as required. So we may assume that  $\eta \geq 2$ , and the result holds for all rooted trees with height less than  $\eta$ . Let  $c' = \eta!$  and  $\zeta' = t\zeta^{\eta+1}$ . Let us say a *limb* is a  $(\zeta', \eta-1)$ -uniform rooted tree that is a path-induced rooted subgraph of  $G$ .

(1) *For each vertex  $u$ , there are at most  $\zeta - 1$   $G$ -neighbours  $v$  of  $u$  with the property that there is a limb  $(J, v)$  of  $G$  such that  $u \notin V(J)$  and  $u$  is not  $t$ -bad for  $(J, v)$ .*

Suppose there are  $\zeta$  such vertices  $v_1, \dots, v_\zeta$ , and let the corresponding limbs be  $(J_i, v_i)$  for  $1 \leq i \leq \zeta$ .

By 2.2, for  $1 \leq i \leq \zeta$ , there is a  $(\zeta^{\eta+1}, \eta - 1)$ -uniform rooted subtree  $(J'_i, v_i)$  of  $(J_i, v_i)$ , such that  $u$  has no neighbour in  $V(J'_i)$  except  $v_i$ . By 2.1, there is a  $(\zeta, \eta - 1)$ -uniform rooted subtree  $(H'_i, r_i)$  of  $(J'_i, r_i)$  for  $1 \leq i \leq \zeta$ , such that the trees  $H'_1, \dots, H'_k$  are pairwise vertex-disjoint. But then adding  $u$  to the union of these trees gives a  $(\zeta, \eta)$ -uniform rooted tree, and it is path-induced in  $G$ , and contains a rooted induced subgraph isomorphic to  $(H, r)$ , a contradiction. This proves (2).

Let  $P$  be the set of vertices  $v$  of  $G$  such that there is a limb with root  $v$ , and let  $Q = V(G) \setminus P$ . For each  $v \in P$ , there is at least one limb with root  $v$ ; select one, and call it  $(J_v, v)$ . For each edge  $e$  with at least one end in  $P$ , select one such end, and call it the *head* of  $e$ .

- Let  $A$  be the set of all edges with both ends in  $Q$ ;
- Let  $B$  be the set of all edges  $uv$  of  $G$  with head  $v$ , such that  $u \notin V(J_v)$ , and  $u$  is not  $t$ -bad for  $(J_v, v)$ ;
- Let  $C$  be the set of all edges  $uv$  of  $G$  with head  $v$ , such that  $u \notin V(J_v)$ , and  $u$  is  $t$ -bad for  $(J_v, v)$ ;
- Let  $D$  be the set of all edges  $uv$  of  $G$  with head  $v$ , such that  $u \in V(J_v)$ .

Thus every edge of  $G$  belongs to exactly one of  $A, B, C, D$ . Since  $G[Q]$  does not contain a limb, the inductive hypothesis implies that  $\partial(G[Q]) \leq (\zeta' t)^{c'}$ . Consequently

$$|A| \leq (\zeta' t)^{c'} |Q| \leq (\zeta' t)^{c'} |G|.$$

By (1), for each vertex  $u \in V(G)$ , there are at most  $\zeta - 1$  edges  $uv \in B$  with head  $v$ ; and so

$$|B| \leq (\zeta - 1) |G|.$$

For each  $v \in P$ , there are at most  $\zeta^{\eta-1}(t - 1)$  edges  $uv \in C$  with head  $v$  by 2.3, and so

$$|C| \leq \zeta^{\eta-1}(t - 1) |P| \leq \zeta^{\eta-1}(t - 1) |G|.$$

For each  $v \in P$ , since  $(J_v, v)$  is path-induced, there are at most  $\zeta'$  edges  $uv \in D$  with head  $v$ , and so

$$|D| \leq \zeta' |P| \leq \zeta' |G|.$$

Summing, we deduce that

$$|E(G)| \leq \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right) |G|,$$

and so some vertex of  $G$  has degree at most  $2 \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right)$ . Since this also holds for every non-null induced subgraph of  $G$ , we deduce that

$$\partial(G) \leq 2 \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta' \right).$$

We recall that  $\zeta' = t\zeta^{\eta+1}$ ; and so

$$\partial(G) \leq 2 \left( \zeta^{c'(\eta+1)} t^{2c'} + (\zeta - 1) + \zeta^{\eta-1}(t - 1) + \zeta^{\eta+1} t \right).$$

Of the four terms on the right side, the sum of the second and third is at most the fourth, so

$$\partial(G) \leq 2 \left( \zeta^{c'(\eta+1)} t^{2c'} + 2\zeta^{\eta+1} t \right);$$

and since  $c' \geq 2$ , the second term here is at most the first, so

$$\partial(G) \leq 4\zeta^{c'(\eta+1)} t^{2c'} \leq \zeta^{c'(\eta+1)} t^{2c'+2}$$

(since we may assume that  $t \geq 2$ , and so  $t^2 \geq 4$ ). Consequently  $\partial(G) \leq \zeta^{c'(\eta+1)} t^{2c'+2}$ . But  $c = c'(\eta+1)$  and  $2c' + 2 \leq c$ , and so  $\partial(G) \leq (\zeta t)^c$ . This proves 2.4.  $\blacksquare$

We remark that 2.4 implies 1.4, and a strengthening:

**2.5** *If  $H$  is a path, and  $t \geq 1$  is an integer, and  $G$  is  $H$ -free and does not contain  $K_{t,t}$  as a subgraph, then  $\partial(G) \leq (2t)^{|H|!}$ .*

**Proof.** Let  $\zeta = 2$ , and  $\eta = |E(H)|$ . Let  $r$  be one end of  $H$ . Then  $G$  does not contain  $(H, r)$  as a path-induced rooted subgraph, and so  $\partial(G) \leq (2t)^{|H|!}$  by 2.4. This proves 2.5.  $\blacksquare$

### 3 Growing a tree

If  $(T, r)$  is a rooted tree and  $v \in V(T)$ , the *height* of  $v$  in  $(T, r)$  is the number of edges in the path between  $v, r$ ; and so the height of  $(T, r)$  is the largest of the heights of its vertices. Let  $(T, r)$  be a rooted tree, and let  $(S, r)$  be a rooted subtree. The graph obtained from  $T$  by deleting all the edges of  $S$  is disconnected, and each of its components contains a unique vertex of  $S$ ; for each  $v \in V(S)$ , let  $T_v$  be the component that contains  $v \in V(S)$ . We call the rooted tree  $(T_v, v)$  the *decoration of  $S$  at  $v$  in  $T$* .

Let  $G$  be a graph, let  $(S, r)$  be a rooted tree, and let  $\zeta \geq 2$  and  $\eta \geq 1$ . We say that  $(S, r)$  is  $(\zeta, \eta)$ -*decorated* in  $G$  if  $S$  is an induced subgraph of  $G$  with height at most  $\eta - 1$ , and there is a rooted tree  $(T, r)$  with the following properties:

- $(S, r)$  is a rooted subtree of  $(T, r)$ , and  $(T, r)$  is a path-induced rooted subgraph of  $G$ ;
- for each  $u \in V(S)$  and  $v \in V(T) \setminus V(S)$ , if  $u, v$  are  $G$ -adjacent then they are  $T$ -adjacent;
- for each  $v \in V(S)$ , the decoration of  $S$  at  $v$  in  $T$  is  $(\zeta, \eta - h)$ -uniform, where  $h$  is the height of  $v$  in  $(S, r)$ .

Thus, informally,  $T$  is obtained from  $S$  by attaching to  $S$  uniform trees rooted at each vertex of  $S$ . Note that  $T$  is only required to be path-induced: the various uniform trees that are attached to  $S$  might have edges between them.

In view of 2.4, if we have a graph  $G$  with huge degeneracy that does not contain  $K_{t,t}$ , then it contains a  $(\zeta, \eta)$ -uniform rooted tree  $(T, r)$  as a path-induced rooted subgraph; and consequently there is a one-vertex rooted tree  $(S, r)$  that is  $(\zeta, \eta)$ -decorated in  $G$ . The next result shows that if we start with  $\zeta$  large enough, then by reducing  $\zeta$  we can grow  $S$  into any larger tree that we wish, and that will prove 1.5.

**3.1** Let  $\eta, t \geq 1$  and  $\zeta \geq 2$  be integers, let  $G$  be a graph that does not contain  $K_{t,t}$  as a subgraph, and let  $(S', r)$  be a  $(\zeta', \eta)$ -decorated rooted tree in  $G$ , where  $\zeta' \geq \zeta^\eta |S'| t^{\eta+1}$ . Let  $p \in V(S')$  with height in  $(S', r)$  less than  $\eta$ . Then there is a  $G$ -neighbour  $q$  of  $p$ , with  $q \in V(G) \setminus V(S')$ , and with no other  $G$ -neighbour in  $V(S')$ , such that, if  $S$  denotes the tree obtained from  $S'$  by adding  $q$  and the edge  $pq$ , then  $(S, r)$  is a  $(\zeta, \eta)$ -decorated rooted tree in  $G$ .

**Proof.** For each  $v \in V(S')$ , let  $h(v)$  denote the height of  $v$  in  $(S', r)$ . Since  $(S', r)$  is  $(\zeta', \eta)$ -decorated in  $G$ , it follows that  $S'$  is an induced subgraph of  $G$ , and there is a rooted tree  $(T', r)$  such that

- $(S', r)$  is a rooted subtree of  $(T', r)$ , and  $(T', r)$  is a path-induced rooted subgraph of  $G$ ;
- for each  $u \in V(S')$  and  $v \in V(T') \setminus V(S')$ , if  $u, v$  are  $G$ -adjacent then they are  $T'$ -adjacent;
- for each  $v \in V(S')$ , the decoration of  $S'$  at  $v$  in  $T'$  is  $(\zeta', \eta - h(v))$ -uniform.

For each  $v \in V(S')$ , let  $(T_v, v)$  be the decoration of  $S'$  at  $v$  in  $T'$ . Since  $T_p$  is  $(\zeta', \eta - h(p))$ -uniform, and  $h(p) < \eta$ , it follows that  $p$  has  $\zeta'$  children in  $(T_p, p)$ . We need to select one of these children, say  $q$ , to add to  $S'$ , forming  $S$ . Any one of them would make a larger induced tree when added to  $S'$ , since  $(S', r)$  is a  $(\zeta, \eta)$ -decorated. But in order to make the new rooted tree  $(S, r)$   $(\zeta, \eta)$ -decorated, we will delete from  $T'$  all vertices of  $T'$  that are  $G$ -adjacent and not  $T'$ -adjacent to  $q$ ; and doing so must not destroy too much of  $T'$ .

For each  $v \in V(S')$ , let  $(S_v, v)$  be a  $(t\zeta, \eta - h(v))$ -uniform rooted subtree of  $(T_v, v)$ . By 2.3, there are at most  $(t\zeta)^{\eta-h(v)}(t-1) < t^{\eta+1}\zeta^\eta$  vertices not in  $V(S_v)$  that are  $t$ -bad for  $(S_v, v)$ , and so there fewer than  $\zeta^\eta |S'| t^{\eta+1} \leq \zeta'$  children of  $p$  in  $(T_p, p)$  that are  $t$ -bad for one of the rooted trees  $(S_v, v)$  ( $v \in V(S')$ ). Hence there is at least one child  $q$  of  $p$  in  $(T_p, p)$  that is  $t$ -bad for none of the trees  $(S_v, v)$  ( $v \in V(S')$ ). Moreover we claim that we can choose  $q$  such that  $q \notin V(S_p)$ . This is automatic if  $(S_p, p)$  has height at least two, since then every child of  $p$  in  $(S_p, p)$  is bad for  $(S_p, p)$ , so we may assume that  $(S_p, p)$  has height one, that is,  $h(p) = \eta - 1$ . Consequently no child of  $p$  in  $(T_p, p)$  is  $t$ -bad for  $(S_p, p)$ , and so the number that are  $t$ -bad for one of the rooted trees  $(S_v, v)$  ( $v \in V(S')$ ) is at most  $\zeta' - t^{\eta+1}\zeta^\eta < \zeta' - t\zeta$ . This proves that we can choose  $q$  such that  $q \notin V(S_p)$ .

Let  $Q$  be the component containing  $q$  of the graph obtained from  $T'$  by deleting  $V(S)$ ; thus  $(Q, q)$  is  $(\zeta', \eta - h(p) - 1)$ -uniform, and so we may choose a  $(\zeta, \eta - h(p) - 1)$ -uniform rooted subtree  $(R_q, q)$  of  $(Q, q)$ . Note that  $q$  has no neighbours in  $V(Q)$  except its neighbours in  $T'$ , since  $(T', r)$  is path-induced. Since  $q$  is not  $t$ -bad for any of the rooted trees  $(S_v, v)$  ( $v \in V(S')$ ), it follows by 2.2 that for each  $v$  there is a  $(\zeta, \eta - h(v))$ -uniform rooted subtree  $(R_v, v)$  of  $(S_v, v)$  such that  $q$  has no  $G$ -neighbour in  $V(R_v)$  except possibly  $v$ , and  $q$  is  $G$ -adjacent to  $v$  if and only if they are  $T'$ -adjacent (that is,  $v = p$ ), since  $v \in V(S')$  and  $(S', r)$  is  $(\zeta', \eta)$ -decorated. Let  $S$  be the tree induced on  $V(S') \cup \{q\}$ , and let  $T$  be the union of  $T'$ , the trees  $R_v$  ( $v \in V(S') \cup \{q\}$ ) and the edge  $pq$ . Then  $S$  satisfies the theorem, because the tree  $T$  exists. This proves 3.1. ■

We deduce 1.5, which we restate in a strengthened form:

**3.2** Let  $\eta, t \geq 1$  and  $\zeta \geq 2$ . For every rooted tree  $(H, r)$  with height at most  $\eta$  and spread at most  $\zeta$ , let  $c = (\eta + 3)!|H|$ ; then  $\partial(G) \leq (|H|\zeta t)^c$  for every  $H$ -free graph  $G$  that does not contain  $K_{t,t}$  as a subgraph.



**Proof.** It suffices to prove the statement for all trees  $H$ , and it is helpful to assign a root  $s$  to  $H$ , so  $(H, s)$  is a rooted tree. Choose  $\eta \geq 1$  and  $\zeta \geq 2$  such that  $(H, s)$  has height at most  $\eta$  and spread at most  $\zeta$ . Let  $H$  have  $k$  vertices. Define  $\zeta_k = \zeta$ , and for  $i = k-1, k-2, \dots, 1$  let  $\zeta_i = i\zeta_{i+1}^\eta t^{\eta+1}$ .

Let  $G$  be an  $H$ -free graph that does not contain  $K_{t,t}$  as a subgraph. Suppose that  $G$  contains a one-vertex rooted tree that is  $(\zeta_1, \eta)$ -decorated in  $G$ . Choose a maximal rooted subtree  $(F, s)$  of  $(H, s)$  such that there is a rooted subtree  $(S, r)$  of  $G$ , isomorphic to  $(F, s)$ , such that  $(S, r)$  is  $(\zeta_i, \eta)$ -decorated in  $G$ , where  $i = |F|$ . By 3.1,  $i = k$ ; and so  $G$  contains an induced subgraph isomorphic to  $H$ , a contradiction.

Thus  $G$  contains no one-vertex rooted tree that is  $(\zeta_1, \eta)$ -decorated in  $G$ . Hence  $G$  contains no  $(\zeta_1, \eta)$ -uniform rooted tree as a path-induced rooted subgraph, and so by 2.4 (applied with  $(H, r)$  replaced by a  $(\zeta_1, \eta)$ -uniform rooted tree),  $\partial(G) \leq (\zeta_1 t)^d$  where  $d = (\eta + 1)!$ .

Now  $\zeta_k = \zeta$ , and  $\zeta_{k-1} = (k-1)\zeta^\eta t^{\eta+1}$ . For all  $i$  with  $1 \leq i \leq k-2$ ,  $\zeta_{i+1} \geq i t^{\eta+1}$ , and so  $\zeta_i = i\zeta_{i+1}^\eta t^{\eta+1} \leq \zeta_{i+1}^{\eta+1}$ . Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq (k\zeta^\eta t^{\eta+1})^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So  $\partial(G) \leq (k\zeta t)^c$  where  $c = (k-2)(\eta+1)^2(\eta+1)! + (\eta+1)! \leq (\eta+3)!k$ . This proves 3.2. ■

## 4 Excluding $K_{s,t}$

In this section we prove 1.6, and before that we prove a weaker statement, with  $\partial(G)$  replaced by  $\chi(G)$ . For the latter we need the following lemma:

**4.1** *Let  $J$  be a digraph such that every vertex has outdegree at most  $k$ . Then the undirected graph underlying  $J$  has chromatic number at most  $2k + 1$ .*

**Proof.** Let  $G$  be the undirected graph underlying  $J$ . Since every subgraph of  $G$  has the property that its edges can be directed so that it has outdegree at most  $k$ , it follows that every such subgraph  $H$  has at most  $k|H|$  edges; and therefore (if it is non-null) has a vertex of degree at most  $2k$ . Consequently  $G$  is  $2k$ -degenerate, and so is  $(2k+1)$ -colourable. This proves 4.1. ■

We use 4.1 to prove the following (which we include here because the proof gives a relatively small constant  $c$ , although the fact that some  $c$  exists follows from 1.6):

**4.2** *Let  $H$  be a tree and  $s \geq 1$  an integer, and let  $c = (2s|H|)^{s+|H|}$ . Then for every  $H$ -free graph  $G$  and every integer  $t \geq 1$ , if  $G$  does not contain  $K_{s,t}$  as a subgraph then  $\chi(G) \leq ct$ .*

**Proof.** We will prove this by induction on  $|H|$  (for the same value of  $s$ ). Let  $H$  be a tree and  $s \geq 0$  an integer, and suppose the theorem holds for all smaller trees and the same value of  $s$ . We may assume that  $|H| \geq 3$ , since the theorem is true for trees with at most two vertices; let  $p \in V(H)$  have degree one, and let  $q$  be its  $H$ -neighbour. Let  $H'$  be obtained by deleting  $p$  from  $H$ . Let  $c' = (2s|H'|)^{s+|H'|}$ . We observe that

$$(1) \quad c \geq \max\left((|H|-2)^{s-1}, (s-1)(|H|-2), (2(s-2)(|H|-2)+1)c'+1\right).$$

Let  $t \geq 1$  be an integer, and let  $G$  be an  $H$ -free graph not containing  $K_{s,t}$  as a subgraph. We will show that  $\chi(G) \leq ct$ . Suppose that this is false, and choose a minimal induced subgraph  $G'$  of  $G$  with  $\chi(G') > ct$ . It follows that every vertex of  $G'$  has degree at least  $ct$  (since  $c$  is an integer).

Let  $v \in V(G')$ . We say a subset  $X \subseteq V(G') \setminus \{v\}$  is a  $v$ -bag if there is an isomorphism from  $H'$  to  $G[X \cup \{v\}]$  that maps  $q$  to  $v$ . (Thus each  $v$ -bag has cardinality  $|H| - 2$ .)

Let  $v \in V(G')$ , and suppose that there are  $s - 1$  pairwise disjoint  $v$ -bags, say  $X_1, \dots, X_{s-1}$ . Since  $G$  is  $H$ -free, every  $G$ -neighbour  $u$  of  $v$  either belongs to  $X_i$  or has a  $G$ -neighbour in  $X_i$ , for  $1 \leq i \leq s - 1$ . In particular, every  $G$ -neighbour  $u$  of  $v$  not in  $X_1 \cup \dots \cup X_{s-1}$  has a  $G$ -neighbour in each of  $X_1, \dots, X_{s-1}$ . But for each choice of  $x_i \in X_i$  ( $1 \leq i \leq s - 1$ ) there are at most  $t - 1$   $G$ -neighbours of  $v$   $G$ -adjacent to each of  $x_1, \dots, x_{s-1}$  (since they are also all adjacent to  $v$ , and  $G$  has no  $K_{s,t}$  subgraph). Consequently there are at most  $(t - 1)(|H| - 2)^{s-1}$   $G$ -neighbours of  $v$  not in  $X_1 \cup \dots \cup X_{s-1}$ ; and hence

$$(s - 1)(|H| - 2) + (t - 1)(|H| - 2)^{s-1} > ct.$$

Since  $ct = c + c(t - 1)$ , and  $(s - 1)(|H| - 2) < c$ , and  $(t - 1)(|H| - 2)^{s-1} \leq c(t - 1)$ , this contradicts (1); so there is no such choice of  $X_1, \dots, X_{s-1}$ .

Choose an integer  $r$  maximum such that there are  $r$  pairwise disjoint  $v$ -bags, say  $X_1, \dots, X_r$ . Consequently  $r \leq s - 2$ . Let  $Y_v = X_1 \cup \dots \cup X_r$ ; then from the maximality of  $r$ ,  $X \cap Y_v \neq \emptyset$  for every  $v$ -bag  $X$ . Moreover  $|Y_v| \leq (s - 2)(|H| - 2)$ .

Let  $J$  be the digraph with vertex set  $V(G')$  in which every vertex in  $Y_v$  is  $J$ -adjacent from  $v$ , for each  $v \in V(G')$ . Thus  $J$  has maximum outdegree at most  $(s - 2)(|H| - 2)$ , and so by 4.1, the undirected graph  $J'$  underlying  $J$  has chromatic number at most  $2(s - 2)(|H| - 2) + 1$ ; and so  $V(G') = V(J')$  can be partitioned into  $2(s - 2)(|H| - 2) + 1$  sets each of which is a stable set of  $J'$ . Let  $Z$  be one of these sets. Then  $G[Z]$  is  $H'$ -free (because otherwise there would be a vertex  $v \in Z$ , and a subset  $X \subseteq Z \setminus \{v\}$ , and an isomorphism from  $H'$  to  $G[X \cup \{v\}]$  mapping  $q$  to  $v$ , and hence with  $X \cap Y_v \neq \emptyset$ ; but no vertex of  $Y_v$  belongs to  $Z$ , since  $Z$  is stable in  $J'$ ). From the inductive hypothesis,  $\chi(Z) \leq c't$ , and hence

$$ct < \chi(G) = \chi(G') \leq (2(s - 2)(|H| - 2) + 1)c't$$

contrary to (1). This proves 4.2. ▀

To prove 1.6, we will need the following strengthening of 1.3, also proved in [7]:

**4.3** *For every forest  $H$ , and every integer  $s > 0$ , there is a tree  $S$  such that for every  $H$ -free graph  $G$ , if  $G$  contains  $S$  as a subgraph, then  $G$  contains  $K_{s,s}$  as a subgraph.*

Now we prove 1.6, which we restate:

**4.4** *For every forest  $H$  and every integer  $s > 0$ , there exists  $c > 0$  such that for every graph  $G$  and every integer  $t > 0$ , if  $G$  is  $H$ -free and does not contain  $K_{s,t}$  as a subgraph, then  $\partial(G) < ct$ .*

**Proof.** Let  $S$  be as in 4.3, and let  $c = |S|^s$ ; we will show that  $c$  satisfies the theorem. Let  $t > 0$  be an integer, and let  $G$  be an  $H$ -free graph that does not contain  $K_{s,t}$  as a subgraph. Suppose that  $\partial(G) \geq ct$ , and choose  $G$  minimal with these properties: then every vertex of  $G$  has degree at least  $ct$ .

(1) Let  $R$  be a tree. If every vertex of  $G$  has degree at least  $t|R|^s$ , then  $G$  contains a subgraph  $T$  isomorphic to  $R$ , and  $V(T)$  can be ordered as  $\{t_1, \dots, t_n\}$ , such that for  $1 \leq i \leq n$ ,  $t_i$  is  $G$ -adjacent to at most  $s - 1$  of  $t_1, \dots, t_{i-1}$ .

We prove this by induction on  $|R|$ . We may assume that  $|R| > 1$ ; let  $p \in V(R)$  have degree one in  $R$ , and let  $q$  be its  $R$ -neighbour. Let  $R'$  be obtained from  $R$  by deleting  $p$ . From the inductive hypothesis,  $G$  contains a subgraph  $T'$  isomorphic to  $R'$ , and its vertex set can be ordered as  $\{t_1, \dots, t_{n-1}\}$ , such that for  $1 \leq i \leq n - 1$ ,  $t_i$  is  $G$ -adjacent to at most  $s - 1$  of  $t_1, \dots, t_{i-1}$ . Choose  $v \in V(T')$  such that some isomorphism from  $R'$  to  $T'$  maps  $q$  to  $v$ . If some  $G$ -neighbour  $u$  of  $v$  does not belong to  $V(T')$  and has at most  $s - 1$   $G$ -neighbours in  $V(T')$ , then we may set  $t_n = u$  as required; so we may assume that every  $G$ -neighbour  $u$  of  $v$  in  $G$  either belongs to  $V(T')$  or has at least  $s$   $G$ -neighbours in  $V(T')$ . Let  $X \subseteq V(T')$  with  $|X| = s$ . If there are at least  $t$  vertices in  $V(G)$  that are  $G$ -adjacent to every vertex in  $X$ , then  $G$  contains  $K_{s,t}$  as a subgraph, a contradiction. So for each such  $X$ , there are at most  $t - 1$  vertices in  $V(G)$  that are  $G$ -adjacent to every vertex in  $X$ . Since there are most  $|R'|^s$  choices of  $X$ , there are at most  $(t - 1)|R'|^s$  vertices in  $V(G) \setminus V(T')$  that have at least  $s$   $G$ -neighbours in  $V(T')$ . Consequently  $v$  has at most  $(t - 1)|R'|^s$   $G$ -neighbours not in  $V(T')$ . But it has at most  $|R'|$   $G$ -neighbours in  $V(T')$  and so the degree of  $v$  in  $G$  is at most  $(t - 1)|R'|^s + |R'| < t|R|^s$ . This proves (1).

Each vertex of  $G$  has degree at least  $ct = t|S|^s$ ; let us apply (1) taking  $R = S$ . We deduce that  $G$  contains a subgraph  $T$  isomorphic to  $S$ , and its vertex set can be ordered as  $\{t_1, \dots, t_n\}$ , such that for  $1 \leq i \leq n$ ,  $t_i$  is  $G$ -adjacent in  $G$  to at most  $s - 1$  of  $t_1, \dots, t_{i-1}$ . By 4.3,  $G[V(T)]$  contains  $K_{s,s}$  as a subgraph. Choose  $i$  maximum such that  $t_i$  belongs to this subgraph; then  $t_i$  is  $G$ -adjacent to at least  $s$  vertices that are earlier in the ordering, a contradiction. This proves 4.4. ■

## 5 Long holes

There is another result in the paper by Bonamy et al. [1]:

**5.1** Let  $\ell \geq 2$  be an integer; then there exists  $c > 0$  that  $\partial(G) \leq \tau(G)^c$  for every graph  $G$  with no induced cycle of length at least  $\ell$ .

In this section we give a simpler proof of this result.

Let  $\eta, t \geq 1$  be integers. We say a rooted tree  $(H, r)$  is  $(t, \eta)$ -tapering if  $(H, r)$  has height  $\eta$ , and every vertex  $v \in V(H)$  of height  $i < \eta$  has exactly  $t^{\eta-i}$  children. For each  $v \in V(H)$ , let  $h(v)$  be its height in  $(H, r)$ .

Let  $G$  be a graph. A map  $\phi$  from  $V(H)$  into  $V(G)$  is a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$  if

- for all distinct  $u, v \in V(H)$ , if  $u, v \in V(H)$  are  $H$ -adjacent then  $\phi(u), \phi(v)$  are distinct and  $G$ -adjacent;
- for each  $u \in V(H)$ , if  $v, w$  are distinct children of  $u$  in  $(H, r)$ , then  $\phi(v) \neq \phi(w)$ ;
- for every path  $P$  of  $H$  with one end  $r$ , the vertices  $\phi(v)$  ( $v \in V(P)$ ) are all distinct; and

- for every path  $P$  of  $H$  with one end  $r$ , and for all distinct  $u, v \in V(P)$ ,  $\phi(u), \phi(v)$  are  $G$ -adjacent if and only if  $u, v$  are  $H$ -adjacent.

Let  $\phi$  be a  $(t, \eta)$ -infusion into  $G$ . We define  $V(\phi) = \{\phi(v) : v \in V(H)\}$ , and we define the *root* of  $\phi$  to be  $\phi(r)$ . We say  $u \in V(G)$  is *t-bad* for  $\phi$  if there exists  $v \in V(H)$  with  $h(v) < \eta$ , such that  $u$  is distinct from and  $G$ -adjacent to  $\phi(w)$  for more than  $(t-1)t^{\eta-h(v)-1}$  children  $w$  of  $v$  in  $(H, r)$ . Then we have:

**5.2** *Let  $t, \eta \geq 1$  be integers, let  $(H, r)$  be a  $(t, \eta)$ -tapering rotted tree, let  $G$  be a graph not containing  $K_{t,t}$  as a subgraph, and let  $\phi$  be a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$ . There are at most  $t^{\eta^2}$  vertices in  $G$  that are t-bad for  $\phi$ .*

The proof is like that for 2.3, using that  $H$  has at most  $t^{\eta-1}$  vertices that have children, and we omit it.

The next result strengthens 1.7:

**5.3** *Let  $\eta \geq 2$  be an integer, and let  $G$  be a graph with no induced cycle of length more than  $\eta$ . For every integer  $t \geq 1$ , if  $G$  does not contain  $K_{t,t}$  as a subgraph then  $\partial(G) \leq t^{\eta^2}$ .*

**Proof.** We may assume that  $t \geq 2$ . Let  $t \geq 1$  be an integer, and let  $G$  be a graph with no induced cycle of length more than  $\eta$  that does not contain  $K_{t,t}$ . Let  $(H, r)$  be a  $(t, \eta)$ -tapering rooted tree (not necessarily contained in  $G$ ).

(1) *If  $u \in V(G)$  and  $v_i$  is a  $G$ -neighbour of  $u$  for  $1 \leq i \leq t^\eta$ , all distinct, and for each  $i$  there is a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$  with root  $v_i$ , such that  $u \notin V(\phi_i)$ , and  $u$  is not t-bad for  $\phi_i$ , then there is a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$ , with root  $u$ .*

Let  $(H', r)$  be a  $(t, \eta - 1)$ -tapering rooted subtree of  $(H, r)$ . It follows (analogously to 2.2) that for  $1 \leq i \leq t^\eta$ , there is a  $(t, \eta - 1)$ -infusion  $\phi'_i$  of  $(H', r)$  into  $G$  such that  $u$  has no  $G$ -neighbour in  $V(\phi'_i)$  except  $v_i$ . Let us number the components of  $H \setminus \{r\}$  as  $H_1, \dots, H_{t^\eta}$ . Let  $\psi(r) = v$ , and for  $1 \leq i \leq t^\eta$  and each  $v \in V(H_i)$ , define  $\psi(v) = \phi'_i(w)$  where  $w$  is the parent of  $v$  in  $(H, r)$ . Then  $\psi$  is a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$ , with root  $v$ . This proves (1).

In these circumstances we say that  $\psi$ , constructed as in the proof of (1), is *derived from* the sequence  $(\phi_i : 1 \leq i \leq t^\eta)$ .

If  $P$  is a path of  $H$  with length  $\eta$  and one end  $r$ , and  $\phi$  is a  $(t, \eta)$ -infusion of  $(H, r)$  into  $G$ , then  $\phi$  maps  $P$  to an induced path  $\phi(P)$  of  $G$  with length  $\eta$  and with one end the root of  $\phi$ . We call  $\phi(P)$  a *column* of  $\phi$ . We observe that if  $\psi$  is derived from  $(\phi_i : 1 \leq i \leq t^\eta)$  as above, then for every column  $Q$  of  $\psi$ , there is a column  $Q'$  of one of  $\phi_i (1 \leq i \leq t^\eta)$ , say of  $\phi'_i$ , such that  $Q \setminus \psi(r)$  is a subpath of  $Q'$ . Let us call  $(\phi', Q')$  a *shift* of  $(\phi, Q)$ .

Let  $\mathcal{A}_1$  be the set of all  $(t, \eta)$ -infusions of  $(H, r)$  into  $G$ . Inductively for  $i > 1$ , let  $\mathcal{A}_i$  be the set of all  $(t, \eta)$ -infusions  $\phi$  such that for some choice of  $\phi_i (1 \leq i \leq t^\eta) \in \mathcal{A}_{i-1}$ ,  $\phi$  is derived from  $(\phi_i : 1 \leq i \leq t^\eta)$ . Thus  $\mathcal{A}_i \subseteq \mathcal{A}_{i-1}$  for each  $i$ . There are two cases: either  $\mathcal{A}_i$  is empty for some  $i$ , or it remains nonempty for all values of  $i$ . Suppose first that  $\mathcal{A}_i$  is nonempty for all  $i$ , and let  $\mathcal{A}$  be the intersection of all the sets  $\mathcal{A}_i (i \geq 1)$ . Choose  $\phi_1 \in \mathcal{A}$ , and let  $Q_1$  be a column of  $\phi_1$ . Since  $\phi_1$  is derived from some members of  $\mathcal{A}$ , there exists  $\phi_2 \in \mathcal{A}$  with root  $u_2$ , and a column  $Q_2$  of  $\phi_2$ , such that

$(\phi_2, Q_2)$  is a shift of  $(\phi_1, Q_1)$ . Similarly we can choose an infinite sequence  $(\phi_i, Q_i)$  ( $i = 1, 2, 3, \dots$ ) such that each  $\phi_i \in \mathcal{A}$  and each  $(\phi_i, Q_i)$  is a shift of its predecessor. Let  $v_i$  be the root of  $\phi_i$  for each  $i$ . Then  $v_i, v_{i+1}, \dots, v_{i+\eta}$  are the vertices in order of  $Q_i$  for each  $i$ ; and so form an induced path of  $G$ . Since  $G$  is finite, there exists  $j > 0$  such that  $v_j$  is adjacent to one of  $v_1, \dots, v_{j-2}$ ; choose a minimum such value of  $j$ , and choose  $i \leq j - 2$  maximum such that  $v_i, v_j$  are adjacent. Then  $\{v_i, \dots, v_j\}$  induces a cycle of  $G$  of length more than  $\eta$ , a contradiction.

So the second case holds, that is,  $\mathcal{A}_i$  is empty for some  $i$ . Choose  $k + 1$  minimum such that  $\mathcal{A}_{k+1} = \emptyset$ . For  $1 \leq i \leq k$  let  $X_i$  be the set of all vertices  $v$  such that  $v$  is the root of a member of  $\mathcal{A}_i$  and not the root of any member of  $\mathcal{A}_{i+1}$ . Thus the sets  $X_1, \dots, X_k$  are pairwise disjoint. Let  $X_0$  be the set of vertices that are not the root of any member of  $\mathcal{A}_1$ ; so the sets  $X_0, \dots, X_k$  form a partition of  $V(G)$ . For each edge  $e$  of  $G$  with an end in one of  $X_1, \dots, X_k$ , choose  $i$  maximum such that  $e$  has an end in  $X_i$ , let  $v$  be an end of  $e$  in  $X_i$ , and call  $v$  the *head* of  $e$ . For each  $v \in X_i$ , choose  $\phi_v \in \mathcal{A}_i$  with root  $v$ . (Thus  $\phi_v \notin \mathcal{A}_{i+1}$  from the definition of  $X_i$ .) Define

- $A$  is the set of all edges of  $G$  with both ends in  $X_0$ ;
- $B$  is the set of all edges  $uv$  with head  $v$  such that  $u \notin V(\phi_v)$  and  $u$  is not bad for  $\phi_v$ ;
- $C$  is the set of all edges  $uv$  with head  $v$  such that  $u \notin V(\phi_v)$  and  $u$  is bad for  $\phi_v$ ;
- $D$  is the set of all edges  $uv$  with head  $v$  such that  $u \in V(\phi_v)$ .

Since there is no  $(t, \eta)$ -infusion of  $(H, r)$  into  $G[X_0]$ , it follows that  $G[X_0]$  does not contain a  $(\zeta, \eta)$ -uniform tree as a path-induced rooted subgraph, where  $\zeta = t^\eta$ , and so  $\partial(G[X_0]) \leq (\zeta t)^{(\eta+1)!}$  from 2.4. Hence

$$|A| \leq (\zeta t)^{(\eta+1)!} |G|.$$

For each  $u \in V(G)$ , with  $u \in X_i$  say, there do not exist  $t^\eta$  neighbours  $v$  of  $u$  such that  $uv$  has head  $v$  and belongs to  $B$ , since there is no  $(t, \eta)$ -infusion of  $(H, r)$  with root  $u$  that is derived from members of  $\mathcal{A}_i$ . Hence

$$|B| \leq t^\eta |G|.$$

For each  $v \in V(G)$ , there are at most  $t^{\eta^2}$  neighbours  $u$  of  $v$  such that the edge  $uv$  has head  $v$  and belongs to  $C$ , by 5.2; so

$$|C| \leq t^{\eta^2} |G|.$$

Finally, for each  $v \in V(G)$ , there are at most  $t^\eta$  neighbours  $u$  of  $v$  such that the edge  $uv$  has head  $v$  and belongs to  $D$ ; so

$$|D| \leq t^\eta |G|.$$

Summing, we obtain

$$|E(G)| \leq \left( (t^{\eta+1})^{(\eta+1)!} + t^\eta + t^{\eta^2} + t^\eta \right) |G| \leq \left( t^{(\eta+2)!} + t^{\eta^2} \right) |G| \leq t^{7\eta^2} / 2.$$

Consequently  $\partial(G) \leq t^{7\eta^2}$ . This proves 5.3. ■

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