ANOTHER SIMPLE PROOF OF A THEOREM OF MILNER

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Abstract. In this note we give a short proof of a theorem of Milner concerning intersecting Sperner systems.

An *intersecting Sperner system* on $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$, no pair of which is either disjoint or nested. Milner [2] proved that an intersecting Sperner system on $[n]$ has at most $\binom{n}{\lceil (n+1)/2 \rceil}$ sets. Katona [1] gave a simple proof of Milner's theorem using the cycle method. We give a simpler proof that uses the cycle method in a different way.

We write $[n]^{(k)}$ for the set of subsets of size k of $[n]$. For $\mathcal{F} \subset [n]^{(k)}$ we write $\partial^+\mathcal{F}$ for the upper shadow $\{G \in [n]^{k+1} : G \supset F \text{ for some } F \in \mathcal{F}\}$ of $\mathcal{F}$ and $\partial^-\mathcal{F}$ for the lower shadow $\{G \in [n]^{k-1} : G \subset F \text{ for some } F \in \mathcal{F}\}$. By a simple counting argument, if $k < n/2$ then $|\partial^+\mathcal{F}| \geq |\mathcal{F}|$ and if $k > n/2$ then $|\partial^-\mathcal{F}| \geq |\mathcal{F}|$.

Theorem 1. *An intersecting Sperner system on* $[n]$ *has size at most*

$$\binom{n}{\lceil \frac{n+1}{2} \rceil} \tag{1}$$

Proof. Let $\mathcal{F} \subset \mathcal{P}(n)$ be an intersecting Sperner system of maximum size N. If n is odd, then $\mathcal{F}$ satisfies (1) by Sperner's lemma, so we may assume $n = 2k$ is even. Let $r = \min\{|A| : A \in \mathcal{F}\}$ and, for $0 \leq k \leq n$, $\mathcal{F}_k = \mathcal{F} \cap [n]^{(k)}$. If $r < n/2 = k$ then consider the system $\mathcal{F}' = (\mathcal{F} \setminus \mathcal{F}_r) \cup \partial^+\mathcal{F}_r$. This is an intersecting Sperner system which is at least as large as $\mathcal{F}$, since $|\partial^+\mathcal{F}_r| \geq |\mathcal{F}_r|$. Repeating the argument, we may assume that $|A| \geq n/2$ for $A \in \mathcal{F}$. Now let $r = \max\{|A| : A \in \mathcal{F}\}$. If $r > k+1$ then consider $\mathcal{F}' = (\mathcal{F} \setminus \mathcal{F}_r) \cup \partial^-\mathcal{F}_r$. Since all sets in $\mathcal{F}$ have size at least $n/2$, this is an intersecting Sperner system, and $|\mathcal{F}'| \geq |\mathcal{F}|$ because $|\partial^-\mathcal{F}_r| \geq |\mathcal{F}_r|$. Repeating, we may assume that $\mathcal{F} \subset [n]^{(k)} \cup [n]^{(k+1)}$.

Let $\mathcal{G} = \partial^+\mathcal{F}_k$. Since $\mathcal{G}$ and $\mathcal{F}_{k+1}$ are disjoint and $|\mathcal{F}_{k+1}| + |\mathcal{G}|$ is bounded by (1), the theorem follows if we show that $|\mathcal{G}| \geq |\mathcal{F}_k|$.

Date: February 8, 2010.

Consider a cyclic order $\mathbf{c}$ of $[n]$ and suppose $f(\mathbf{c})$ elements of $\mathcal{F}_k$ and $g(\mathbf{c})$ elements of $\mathcal{G}$ occur as intervals in $\mathbf{c}$. Since we do not have both an interval and its complement in $\mathcal{F}_k$, we have $f(\mathbf{c}) \leq n/2 = k$. However, every interval of length k can be extended to an interval of length $k+1$ in two ways, so $g(\mathbf{c}) \geq f(\mathbf{c}) + 1 \geq \frac{k+1}{k} f(\mathbf{c})$. Each element of $\mathcal{F}_k$ occurs in $k!^2$ cyclic orders and each element of $\mathcal{G}$ in $(k+1)!(k-1)!$ cyclic orders, so summing over all orders gives

$$(k+1)!(k-1)!|\mathcal{G}| = \sum_{\mathbf{c}} g(\mathbf{c}) \geq \frac{k+1}{k} \sum_{\mathbf{c}} f(\mathbf{c}) = \frac{k+1}{k} k!^2 |\mathcal{F}_k|,$$

and so $|\mathcal{F}_k| \leq |\mathcal{G}|$, as required. □

References

[1] G.O.H. Katona, A simple proof of a theorem of Milner, *J. Combin. Theory Ser. A* **83** (98), 138-140

[2] E.C. Milner, A combinatorial theorem on systems of sets, *J. London Math. Soc.* **43** (68), 204-206

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