

A multidimensional Ramsey Theorem

António Girão ^{*‡} Gal Kronenberg ^{*†} Alex Scott ^{*‡}

Abstract

In this paper, we prove a “multidimensional” generalisation of Ramsey’s Theorem to cartesian products of graphs, showing that a doubly exponential upper bound is enough in every dimension. More precisely, we prove that for every $r, n, d \in \mathbb{N}$, in any r -colouring of the edges of the Cartesian product $\square^d K_N$ of d copies of K_N there is a copy of $\square^d K_n$ such that the edges in each direction are monochromatic, provided $N \geq 2^{2^{C(d)rn^d}}$. As an application of our new approach we obtain improvements on the multidimensional Erdős-Szekeres Theorem proved by Fishburn and Graham 30 years ago thus confirming a conjecture posed by Bucić, Sudakov, Tran.

1 Introduction

The study of Ramsey theory is a longstanding and central part of combinatorics. As usual, for positive integers r, k , the Ramsey number $R_r(k)$ is the smallest n for which every r -colouring of K_n contains a monochromatic copy of K_k . Ramsey [16] showed in 1930 that these numbers exist. Since then, Ramsey numbers have been studied extensively, upper and lower bounds have been proved and many generalisations have been considered, see, e.g. [5, 7, 8, 13, 17, 21, 23]. Even for $r = 2$, the asymptotics are still not fully resolved. It is not hard to show that $R_2(k)$ grows at exponential rate, but the constant in the exponent is yet not known: the best current bounds are $(1 + o(1)) \frac{\sqrt{2(k+1)}}{e} 2^{k+1/2} \leq R_2(k+1) \leq e^{-c \log^2 k} \binom{2k}{k}$ the lower bound due to Spencer [21] and upper bound by Sah [17]. For larger r , the gap between the upper and lower bound is even larger, and even for $k = 3$, we do not understand the behaviour of $R_r(3)$ as $r \rightarrow \infty$. Very recently, Conlon and Ferber [6], Wigderson [24], and Sawin [18] found nice constructions which give the best known lower bounds for $R_r(k)$ when $r \geq 3$. In a recent breakthrough, Campos, Griffiths, Morris and Sahasrabudhe, reduced the long standing upper bound of 4^k to $(4 - \varepsilon)^k$. This is the first exponential improvement over the upper bound of Erdős and Szekeres, proved in 1935.

In this paper, we prove a “multidimensional” generalisation of Ramsey’s Theorem for cartesian products of graphs. Given two graphs H and G , we write $G \square H$ for the Cartesian product of H and G , namely the

^{*}Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, United Kingdom. E-mail: {girao,kronenberg,scott}@maths.ox.ac.uk.

[†]Supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska Curie grant agreement No. 101030925.

[‡]Research supported by EPSRC grant EP/V007327/1. AS would like to thank the Department of Mathematics at the University of Arizona where some of this work was completed.

graph with vertex set $V(G) \times V(H)$ in which (x, y) is joined to (x', y') if and only if $x = x'$ and $yy' \in E(H)$ or $xx' \in E(G)$ and $y = y'$. The Cartesian product is associative, so it makes sense to write $G_1 \square G_2 \square \dots \square G_d$ (without brackets) for the Cartesian product of d graphs; we write $\square^d G$ for the product $G \square \dots \square G$ of d copies of G . Note that in a Cartesian product of d graphs G_1, \dots, G_d , there is an edge between $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d)$ if and only if there is some i such that $v_i w_i$ is an edge of G_i and $v_j = w_j$ for $j \neq i$, and in this case we will say that the edge vw is in *direction* i .

Given a colouring c of the edges of $\square^d K_n$, we say that c is *monochromatic in every direction* if for each $i \in \{1, \dots, d\}$ there is some c_i such that all edges in direction i have colour c_i . For positive integers r, d , we define $R_r(d, n)$ to be the smallest N such that every r -colouring of the edges of $\square^d K_N$ contains a copy of $\square^d K_n$ that is monochromatic in every direction. Note that we cannot demand a copy of $\square^d K_n$ that has the *same* colour in every direction, as we are asking for a full-dimensional subgraph: for example, $\square^d K_N$ could be coloured with colour 1 for all edges in direction 1 and coloured with 2 for the edges in all other directions. It is easy to see that if we demand a monochromatic copy of $\square^\ell K_n$ then ℓ must be at most $\lceil d/r \rceil$. It will follow from Theorem 1.1 that this is also tight.

It is not too hard to prove that $R_r(d, n)$ exists by an iterated application of Ramsey's Theorem. However, this would only give an upper bound of tower-type as a function of d . Our main goal is to show that a doubly exponential bound on n^d suffices.

Theorem 1.1. *Let d be a positive integer. There exists $C_d > 0$ such that for every n, r the following holds. For $N \geq r^{r^{C_d r n^d}}$, every r -colouring of $\square^d K_N$ contains a copy of $\square^d K_n$ which is monochromatic in every direction. That is, $R_r(d, n) \leq r^{r^{C_d r n^d}}$.*

As an immediate corollary, we see that any r -edge colouring of $\square^d K_N$ contains a *monochromatic* copy of $\square^\ell K_n$ for $\ell = \lceil d/r \rceil$.

Corollary 1.2. *Let n, d, r be positive integers and $\ell = \lceil d/r \rceil$. For $N \geq r^{r^{C_d r n^d}}$, every r -edge-colouring of $\square^d K_N$ contains a monochromatic copy of $\square^\ell K_n$. The value of ℓ is tight.*

Another foundational result in Ramsey theory appears in a paper Erdős and Szekeres [8] from 1935: any sequence of $n^2 + 1$ distinct real numbers contains either an increasing or decreasing subsequence of length $n + 1$. There are a number of different ways to generalise the Erdős-Szekeres Theorem to higher dimensions (see, for example, [2, 3, 11, 12, 14, 15, 19, 22]). Perhaps the most natural approach was developed thirty years ago by Fishburn and Graham [10].

A d -dimensional array is an injective function f from $A_1 \times \dots \times A_d$ to \mathbb{R} where A_1, \dots, A_d are non-empty subsets of \mathbb{Z} ; we say f has *size* $|A_1| \times \dots \times |A_d|$; if $|A_i| = n$ for each i , it will be convenient to say that f has *size* $[n]^d$. A multidimensional array is said to be *monotone* if for each direction all the 1-dimensional subarrays in that direction are increasing or decreasing. In other words, for every i , one of the following holds:

- For every choice of $a_j, j \neq i$, the function $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$ is increasing in x .
- For every choice of $a_j, j \neq i$, the function $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$ is decreasing in x .

Let $M_d(n)$ be the smallest N such that a d -dimensional array on $[N]^d$ contains a monotone d -dimensional subarray of size $[n]^d$. Fishburn and Graham [10] showed that $M_d(n)$ exists but their upper bounds were a tower of height d . Recently, Bucić, Sudakov, and Tran [1] proved considerably better upper bounds on $M_d(n)$, showing doubly exponential bounds in n^{d-1} for 2 and 3 dimensions, and triply exponential bounds in 4 or higher dimensions.

Theorem 1.3. (Bucić, Sudakov, and Tran)

- i) $M_2(n) \leq 2^{2^{(2+o(1))n}}$,
- ii) $M_3(n) \leq 2^{2^{(2+o(1))n^2}}$,
- iii) $M_d(n) \leq 2^{2^{2^{O_d(n^{d-1})}}}$, for $d \geq 4$.

Bucić, Sudakov, and Tran [1] asked whether a better bound could be proved in four and higher dimensions, speculating that the triply exponential bound could be reduced to a doubly exponential bound. Using the methods from our proof of Theorem 1.1, we resolve their question, showing that a doubly exponential upper bound holds in all dimensions.

Theorem 1.4. For every $d \geq 2$, there is $C_d > 0$, such that for every positive n , $M_d(n) \leq 2^{n^{C_d n^{d-1}}}$.

In our proof, instead of trying to find directly a monochromatic sub-product of cliques, we first find an intermediate structured object which we call d -consistent and then we show that a density-type argument on these structured objects allows us to get a monochromatic $\square^d K_n$. We then use this new approach to improve the upper bound in Theorem 1.3, leading to Theorem 1.4. This differs from the methods of [1], which makes significant use of the fact that the Erdős-Szekeres Theorem has a polynomial bound and so it can be iterated n^{d-1} times to obtain the bounds for $d \geq 4$ in Theorem 1.3. Ramsey's Theorem, however, has an exponential bound and so a different approach is required to avoid getting levels of exponentiation.

In our proof, we first find a new intermediate notion of a structured object which we call d -consistent and then we show that a density-type argument on these structured objects allows us to get a monochromatic $\square^d K_n$. We then use this new approach to improve the upper bound in Theorem 1.3, leading to Theorem 1.4.

We finish the introduction by pointing out that our results also give improvements for multidimensional lexicographic-monotone array. In their paper, Fishburn and Graham [10] introduced another natural generalisation for monotone sequences and the Erdős-Szekeres theorem, which they called a *lex-monotone array*. A d -dimensional array f is said to be *lex-monotone* if the following holds. There exists a permutation $\tau \in S_{[d]}$ and a sign vector $s \in \{-1, 1\}^d$, such that $f(\mathbf{x}) < f(\mathbf{y})$ if and only if the vector $(s_{\tau(i)}x_{\tau(i)})_{i \in [d]}$ is smaller in lexicographic ordering than vector $(s_{\tau(i)}y_{\tau(i)})_{i \in [d]}$, that is, if there is $i \in [d]$ such that $s_{\tau(j)}x_{\tau(j)} = s_{\tau(j)}y_{\tau(j)}$ for every $j < i$, and $s_{\tau(i)}x_{\tau(i)} < s_{\tau(i)}y_{\tau(i)}$. Let $L_d(n)$ be the smallest N such that a d -dimensional array on $[N]^d$ contains a lex-monotone d -dimensional subarray of size $[n]^d$. Fishburn and Graham gave a tower bound of order $d - 1$ on $L_d(n)$, which was improved by Bucić, Sudakov and Tran to a tower of order 5 for $d \geq 4$ (and triple exponential for $d = 3$). We note that using Theorem 1.4 together with Theorem 1.2 from [1] we obtain a triple exponential upper bound for all $d \geq 3$, that is, $L_d(n) \leq 2^{2^{2^{C_d n^{d-2}}}}$.

We prove Theorem 1.1 in Section 2 and Theorem 1.4 in Section 3. We conclude with some further discussion in Section 4.

2 Upper bound on $R_r(d, n)$

To simplify notation we will identify the vertex set of K_N with $[N] = \{1, \dots, N\}$ and the vertex set of $K_N \square \dots \square K_N$ with $[N]^d$. Suppose that, for each $i \in [d]$ we have a graph G_i and a subgraph $H_i \subseteq G_i$. For $a \in V(G_d)$, we write $H_1 \square \dots \square H_{d-1} \square a$ for the copy of $H_1 \square \dots \square H_{d-1} \subseteq G_1 \square \dots \square G_d$ with vertex set

$V(H_1) \times \cdots \times V(H_{d-1}) \times \{a\}$ (thus all vertices have a as their d th coordinate; we will usually omit the braces around $\{a\}$ for simplicity). We define subgraphs such as $a_1 \square \cdots \square a_{d-1} \square H_d$ analogously. We will sometimes refer to induced subgraphs by their vertex sets: thus, for an edge-coloured graph G we say that $S \subset V(G)$ contains a monochromatic K_k if there is some set $T \subset S$ such that the induced subgraph $G[T]$ is a monochromatic copy of K_k .

We begin with a short proof for the case $d = 2$, and then give a (more involved) argument for the general case $d \geq 3$.

Proof of Theorem 1.1 for $d = 2$. Note that $R_r(1, t)$ is just the usual Ramsey number for K_t , which is smaller than r^{rt} . Let $N := r^{r^{10rn^2}}$, and consider an r -colouring of the edges of $K_N \square K_N$. Fix a set $S \subset [N]$ of size $r^{r^{3rn^2+1}}$. By Ramsey's Theorem, for each $i \in [N]$ we can find a monochromatic copy of $K_{r^{3rn^2}}$ in $S \times i$. Note that there are at most $r \cdot \binom{|S|}{r^{3rn^2}} \leq r^{r^{7rn^2}}$ choices for the vertex set of each monochromatic copy of $K_{r^{3rn^2}}$ and its colour. Since $N/r^{r^{7rn^2}} \geq r^{2rn}$, a pigeonhole argument shows that there is a set A_1 of size r^{3rn^2} and a set $A_2 \subset [N]$ of size at least r^{2rn} such that, for each $a_2 \in A_2$ the set $A_1 \times a_2$ induces a monochromatic copy of $K_{r^{3rn^2}}$ (all with the same choice of colour). Applying Ramsey's Theorem once again, we can find in each set $b \times A_2$ a monochromatic copy of K_n (as $|A_2| = r^{2rn}$). As there are at most $r \cdot \binom{|A_2|}{n} \leq r^{2rn^2}$ choices for the vertex set and colour of this K_n , we can apply the pigeonhole principle again: there is a set $A'_1 \subset A_1$ of size at least $|A_1|/r^{2rn^2} \geq k$ and a set $A'_2 \subset A_2$ of size n such that, for every $a_1 \in A'_1$, the set $a_1 \times A'_2$ forms a monochromatic copy of K_n (and all with the same choice of colour). It is clear that $A'_1 \times A'_2$ forms a copy of $K_n \square K_n$ that is monochromatic in both directions. Hence $R_r(2, n) \leq N$, as we wanted to show. \square

We now turn to the general argument for $d \geq 3$. Consider an r -edge-coloured product of complete graphs. We say a 1-dimensional K_k is 1-consistent if it is monochromatic. For $d \geq 2$, we say $\square^d K_k$ is d -consistent if the following two conditions hold:

- for every $a \in [k]$, the subgraph $(\square^{d-1} K_k) \square a$ has the same colouring: that is, for every edge \mathbf{xy} in $\square^{d-1} K_k$, the colour of the edge between $\mathbf{x} \times a$ and $\mathbf{y} \times a$ is the same for every $a \in [k]$; and
- for some (and thus for every) a , the subgraph $\square^{d-1} K_k \square a$ is $(d-1)$ -consistent.

In other words, for each $i \in [d]$, the ' i -dimensional subspaces' $(\square^i K_N) \square a_{i+1} \square \cdots \square a_d$ have the same colouring for every choice of a_{i+1}, \dots, a_d .

The proof splits into two lemmas. Given suitable $N \gg k \gg n$ and an r -edge colouring of $\square^d K_N$, we first find a d -consistent subgraph $H = \square^d K_k$. We then show that H contains a monochromatic copy of $\square^d K_n$.

Lemma 2.1. *For every $d \geq 1$ there is a constant $g(d)$ such that the following holds. Let r, k be positive integers and $0 < \epsilon < 1/2$. Suppose that $N \geq \epsilon^{-g(d)k^{d-1}} \cdot r^{g(d)rk^d}$ and $\square^d K_N$ has a d -consistent r -edge-colouring. Then every set S of at least ϵN^d vertices contains a copy of $\square^d K_k$ which is monochromatic in every direction.*

Proof. We argue by induction on d that $g(d) = (12 + 2r)^{d-1}$ will do. It is clear that when $d = 1$ it is sufficient to have $N \geq r^{rk}/\epsilon$, which we do as $g(1) = 1$. So we assume that $d \geq 2$ and we have handled smaller cases. Let T be the set of elements $\mathbf{v} \in [N]^{d-1}$ such that $\mathbf{v} \times [N]$ contains at least $\epsilon N/2$ elements of S . A counting argument shows that $|T| \geq \epsilon N^{d-1}/2$. Let $A \subset [N]$ be a random subset of size $(10/\epsilon)r^{rk}$. For each $\mathbf{v} \in T$, with probability at least $2/3$ the set $\mathbf{v} \times A$ contains at least r^{rk} elements of S . So we can choose A so that the set

$$T' := \{\mathbf{v} \in T : |(\mathbf{v} \times A) \cap S| \geq r^{rk}\}$$

has size at least $(2/3)|T| \geq (\epsilon/3)N^{d-1}$. By Ramsey's Theorem, for each $\mathbf{v} \in T'$, there is a monochromatic copy of K_k contained in $(\mathbf{v} \times A) \cap S$, say on vertices $B_{\mathbf{v}} \subset A$. There are at most $r \binom{(10/\epsilon)r^{rk}}{k} \leq \epsilon^{-9k} r^{rk^2+1}$ choices for $B_{\mathbf{v}}$ and the colour of the corresponding copy of K_k , so there are $B \subset A$ and $U \subset T'$ such that

$$|U| \geq \epsilon^{9k} r^{-rk^2-1} |T'| \geq (\epsilon/3) \epsilon^{9k} r^{-rk^2-1} N^{d-1}$$

and $B_{\mathbf{v}} = B$ for every $\mathbf{v} \in U$. Now let $\tilde{\epsilon} = (\epsilon/3) \epsilon^{9k} r^{-rk^2-1} \geq \epsilon^{12k} r^{-rk^2-1}$, so $|U| \geq \tilde{\epsilon} N^{d-1}$. We now apply the inductive hypothesis to $[N]^{d-1}$, with the colouring inherited from $[N]^{d-1} \times a$ (which by consistency is the same for any $b \in B$) with the subset U playing the role of S . Since $N \geq \epsilon^{-g(d)k^{d-1}} r^{g(d)rk^d} \geq \tilde{\epsilon}^{-g(d-1)k^{d-2}} \cdot r^{g(d-1)rk^{d-1}}$, we obtain a $(d-1)$ -dimensional product P of copies of K_k which is monochromatic in every direction. Then $P \times B$ gives a d -dimensional product of copies of K_k which is monochromatic in every direction (as $P \times b$ is coloured in the same way for every $b \in B$, and all sets $\mathbf{v} \times B$ ($\mathbf{v} \in B$) give monochromatic copies of K_k with the same colour). \square

Lemma 2.2. *For every positive integer d there is $f(d)$ such that for every r, k the following holds. Let $N := r^{f(d)rk^d}$. Then in every r -colouring of $\square^d K_N$ there is a d -consistent $\square^d K_k$.*

Proof. We argue by induction on d that $f(d) = 2^{d-1}d!$ will do. For $d = 1$, this is true by Ramsey's Theorem. Now, let $M := r^{f(d-1)rk^{d-1}}$ and consider the subgraph $(\square^{d-1} K_M) \square K_N$. By induction any r -colouring of $\square^{d-1} K_M$ contains a $(d-1)$ -consistent $\square^{d-1} K_k$. Therefore, for every $a \in [N]$ the subgraph $\square^{d-1} K_M \times a$ contains a $(d-1)$ -consistent $\square^{d-1} K_k \times a$. As there are most $r^{(d-1)k^{2+(d-2)}}$ possible r -colourings of $\square^{d-1} K_k$ and at most $\binom{M}{k}^{d-1}$ possible vertex sets, there are at most $r^{(d-1)k^{2+(d-2)}} \binom{M}{k}^{d-1} \leq r^{(d-1)k^d} M^{k(d-1)}/k < r^{f(d)rk^d}/k = N/k$ possible combinations. Thus, by the pigeonhole principle, there is a $(d-1)$ -consistent colour pattern c of $\square^{d-1} K_k$ and a set $A \subset [N]$ of size k , such that for every $a \in A$, the subgraph $\square^{d-1} K_k \times a$ has colour pattern c , and all these boxes lie on the same vertex set in the first $d-1$ dimensions, as we wanted to show. \square

Proof of Theorem 1.1. We will show that the statement holds with $C_d = 3dg(d-1) + f(d) + 1$, where f and g are any functions satisfying the previous two lemmas. Let $N := r^{r^{C_d} r^{rn^d}}$. Let $t := r^{3g(d-1)rn^d}$ and $u := r^{rn}$. Applying Lemma 2.2 to $\square^d K_N$, we obtain a d -consistent copy B of $\square^d K_t$. Relabelling, and restricting the final coordinate to u choices, we may assume that we have a d -consistent colouring of $\square^{d-1} K_t \square K_u$.

For every $a_1 \times \dots \times a_{d-1}$, where each $a_i \in [t]$, we can apply Ramsey's Theorem to $a_1 \square \dots \square a_{d-1} \square K_u$ to get a monochromatic copy of K_n . There are $\binom{u}{n} \cdot r$ choices of colour and coordinates C for this copy; setting $\epsilon = 1/(ru^n)$ we see that there is some set S of at least ϵt^{d-1} vertices in $\square^{d-1} K_t$ for which both colour and coordinates agree.

Finally, we apply Lemma 2.1 to $S \subset \square^{d-1} K_t$ and obtain a copy K of $\square^{d-1} K_n$ contained in S that is monochromatic in every direction. This lemma holds for $t \geq \epsilon^{-g(d-1)n^{d-2}} r^{g(d-1)rn^{d-1}}$, which holds by our choice of constants, as

$$\epsilon^{-g(d-1)n^{d-2}} = (ru^n)^{g(d-1)n^{d-2}} = r^{(rn^2+1)g(d-1)n^{d-2}}.$$

Then $K \square C$ is a copy of $\square^d K_n$ that is monochromatic in every direction. \square

3 Upper bound on $M_d(n)$

We note first that an immediate application of Theorem 1.1 gives an upper bound of $M_d(n) \leq 2^{2^{C_a n^d}}$ by giving an edge xy colour red if $f(x) < f(y)$ and blue otherwise. However, we wish to get a dependence of $d - 1$ in the exponent in Theorem 1.4, and so prove it separately.

The proof follows along the same route as the proof of Theorem 1.1. However, we need an modified version of Lemma 2.1 where the exponent is slightly better. In what follows, we will assume all the arrays hereafter are injective, which we can ensure by a small perturbation to the values. We say a 1-dimensional array is *consistent* if it is monotone. For $d \geq 2$, we say an array $f : [N]^d \rightarrow \mathbb{R}$ is d -consistent if the following two conditions hold:

- for every $a \in [N]$, f restricted to $[N]^{d-1} \times \{a\}$ has the same order pattern i.e. for every $\mathbf{x}, \mathbf{y} \in [N]^{d-1}$, if $f(\mathbf{x} \times a) < f(\mathbf{y} \times a)$, for some a then the same holds for all $a \in [N]$; and
- for some (and thus for every) a , the $(d-1)$ -dimensional subarray $f : [N]^{d-1} \times a \rightarrow \mathbb{R}$ is $(d-1)$ -consistent.

As in Section 2, we will show that being d -consistent with the appropriate parameters is enough for our purpose.

Lemma 3.1. *For every $d \geq 1$, there is a constant $g(d)$ such that the following holds. Let r, k be positive integers and $0 < \epsilon < 1/2$. Suppose that $N \geq \epsilon^{-g(d)k^{d-1}} \cdot k^{g(d)k^{d-1}}$ and the array $f : [N]^d \rightarrow \mathbb{R}$ is d -consistent. Then every $S \subseteq [N]^d$ of size at least ϵN^d contains a monotone subarray of size $[k]^d$.*

Proof. We follow the proof of Lemma 2.1, with k^2 playing the role of r^{rk} , $g(d) = 2 \cdot 15^{d-1}$, and with the remark that in the 1-dimensional case, the Erdős-Szekeres Theorem implies that it is enough to have $N \geq \epsilon^{-1}(2k)^2$. We include the full details here for completeness.

We argue by induction on d that $g(d) = 2 \cdot 15^{d-1}$ will do. It is clear that when $d = 1$ it is sufficient to have $N \geq k^2/\epsilon$, which we do as $g(1) = 2$. So we assume that $d \geq 2$ and we have handled smaller cases. Let T be the set of elements $\mathbf{v} \in [N]^{d-1}$ such that $\mathbf{v} \times [N]$ contains at least $\epsilon N/2$ elements of S . A counting argument shows that $|T| \geq \epsilon N^{d-1}/2$. Let $A \subset [N]$ be a random subset of size $(10/\epsilon)k^2$. For each $\mathbf{v} \in T$, with probability at least $2/3$ the set $\mathbf{v} \times A$ contains at least k^2 elements of S . So we can choose A so that the set

$$T' := \{\mathbf{v} \in T : |(\mathbf{v} \times A) \cap S| \geq k^2\}$$

has size at least $(2/3)|T| \geq (\epsilon/3)N^{d-1}$. By the Erdős-Szekeres Theorem, for each $\mathbf{v} \in T'$, there is a monotone subarray of size k contained in $(\mathbf{v} \times A) \cap S$, say on vertices $B_{\mathbf{v}} \subset A$. There are at most $2^{\binom{(10/\epsilon)k^2}{k}} \leq \epsilon^{-9k} k^{2k}$ choices for $B_{\mathbf{v}}$ and the direction of monotonicity (i.e. increasing or decreasing), so there are $B \subset A$ and $U \subset T'$ such that

$$|U| \geq \epsilon^{9k} k^{-2k} |T'| \geq (\epsilon/3) \epsilon^{9k} k^{-2k} N^{d-1}$$

and $B_{\mathbf{v}} = B$ for every $\mathbf{v} \in U$. Now let $\tilde{\epsilon} = (\epsilon/3) \epsilon^{9k} k^{-2k} \geq \epsilon^{12k} k^{-2k}$, then $|U| \geq \tilde{\epsilon} N^{d-1}$. We now apply the inductive hypothesis to the subset $U \subseteq [N]^{d-1}$, with the values inherited from $[N]^{d-1} \times b$ (which by consistency are the same for any $b \in B$). Since

$$N \geq \epsilon^{-g(d)k^{d-1}} k^{g(d)k^{d-1}} \geq \tilde{\epsilon}^{-g(d-1)k^{d-2}} \cdot k^{g(d-1)k^{d-2}},$$

we obtain a $(d-1)$ -dimensional monotone subarray of size $[k]^{d-1}$. Then $P \times B$ gives a monotone subarray of size $[k]^d$. \square

It is thus enough to find a d -consistent array. The following lemma is analogue of Lemma 2.2 for arrays.

Lemma 3.2. *For every positive integer d there is $f(d)$ such that for every k the following holds. Let $N := k^{f(d)k^{d-1}}$. Then in every d -dimensional array of size $[N]^d$ there is a d -consistent subarray of size $[k]^d$.*

Proof. We argue by induction on d that $f(d) = 2^{d-1}d! + 1$ will do. For $d = 1$, this is true by the Erdős-Szekeres Theorem. Now let $M := k^{f(d-1)k^{d-2}}$. By induction any array on $[M]^{d-1}$ contains a $(d-1)$ -consistent array of size $[k]^{d-1}$. Therefore, for every $a \in [N]$ the $(d-1)$ -dimensional array on $[M]^{d-1} \times \{a\}$ contains a $(d-1)$ -consistent array on $A_1 \times \cdots \times A_{d-1} \times \{a\}$, where $|A_i| = n$ for all $i \in [d-1]$. As there are most $k^{(d-1)k^{1+(d-2)}}$ possible orderings of an array of size k^{d-1} , and at most $\binom{M}{k}^{d-1}$ choices for the sets A_i , there are at most $k^{(d-1)k^{1+(d-2)}} \binom{M}{k}^{d-1} < k^{(d-1)k^{d-1}} M^{k(d-1)}/k \leq k^{f(d)k^{d-1}}/k = N/k$ possible combinations. Thus, by the pigeonhole principle there are an ordering \mathcal{O} of $[k]^{d-1}$ and a set $A \subset [N]$ of size k , where $[k]^{d-1} \times \{a\}$ have the same ordering \mathcal{O} for every $a \in A$ and all these $(d-1)$ -dimensional arrays lie on the same vertex set in the first $d-1$ dimensions, as we wanted to show. \square

Proof of Theorem 1.4. We will show that the statement holds with $C_d = 4dg(d-1) + f(d)$, where f and g are any functions satisfying Lemma 3.1 and Lemma 3.2. Let $N := n^{nC_{d-1}}$. Let $t := n^{3g(d-1)n^{d-1}}$ and $u := n^2$. Applying Lemma 3.2 to $[N]^d$, with t playing the role of k , we obtain a d -consistent array on $B_1 \times \cdots \times B_d := B$, where $|B_i| = t$ for all $i \in [d]$. Relabelling, and restricting the final coordinate to u choices, we may assume that we have a d -consistent array on $[t]^{d-1} \times [u]$. For every $a_1 \times \cdots \times a_{d-1} \in [t]^{d-1}$, we can apply the Erdős-Szekeres Theorem to the 1-dimensional array on $a_1 \times \cdots \times a_{d-1} \times [u]$ to get a monotone subarray of size n . By a simple counting argument there are at most $\binom{u}{n} \cdot 2$ choices for the coordinates of the subarray and whether it is increasing or decreasing, and hence there is some choice of these which occurs on a fraction of at least $(2 \cdot \binom{u}{n})^{-1} := \hat{\epsilon}$ of the vertices of $[t]^{d-1}$, say on a set S . Let $A \subseteq [u]$ be the common choice of coordinates for the monotonic subarrays corresponding to S .

Finally, we apply Lemma 3.1 to $S \subseteq [t]^{d-1}$. By the choices of $t, u, \hat{\epsilon}$ we have that $t \geq \hat{\epsilon}^{-g(d-1)n^{d-2}}$. $n^{g(d-1)n^{d-2}}$, and so we obtain a monotone $(d-1)$ -dimensional array T of size $[u]^{d-1}$. Since B is d -consistent, $T \times \{a\}$ is monotone for every $a \in A$ (with the same choice of direction of monotonicity). By construction, this gives a monotone d -dimensional array on $T \times A$. \square

4 Concluding remarks

In Theorem 1.1, we have given a doubly exponential upper bound on the d -dimensional Ramsey numbers. From below, we have only a singly exponential bound (which follows easily by considering random colourings). It would be very interesting to close the gap. In particular, it would be good to know whether there is a simple exponential upper bound, or whether the numbers grow more quickly.

Problem 4.1. *Fix $r, d \geq 2$. Is $R_2(2, n)$ superexponential in n ?*

The same gap between lower and upper bounds is seen in the multidimensional Erdős-Szekeres Theorem. Bucić, Sudakov and Tran [1] gave a doubly exponential upper bound for $d = 2, 3$; and Theorem 1.4 gives a doubly exponential upper bound for $d \geq 4$ (improving on the previous triply exponential upper bound [1]). But a frustrating gap between singly and doubly exponential bounds remains.

It is easy to see that $M_2(n) \leq R_2(2, n)$ for every n . It would be interesting to see if $R_2(2, n)$ can be bounded above by a function of $M_2(n)$. For example, is the following true.

Problem 4.2. Does there exist $C > 0$ such that $\log R_2(2, n) \leq (\log(M_2(n)))^C$?

5 Acknowledgements

The authors would like to thank Zachary Hunter and Matija Bucic for their helpful comments. We also thank the anonymous referee for their suggestions.

References

- [1] M. Bucic, B. Sudakov and T. Tran, Erdős-Szekeres theorem for multidimensional arrays, *to appear in J. European Math. Soc.*, [arXiv:1910.13318](#) [2](#), [3](#), [7](#)
- [2] H. Burkill and L. Mirsky, Monotonicity, *J. Math Anal. Appl.* **41** (1973), 391–410. [2](#)
- [3] H. Burkill and L. Mirsky, Combinatorial problems on the existence of large submatrices I, *Discrete Math.* **6** (1973), 15–28. [2](#)
- [4] M. Campos, S. Griffiths, R. Morris, and Julian Sahasrabudhe. *An exponential improvement for diagonal Ramsey*. arXiv preprint [arXiv:2303.09521](#) (2023).
- [5] D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **2** 170 (2009), 941–960. [1](#)
- [6] D. Conlon, and A. Ferber, Lower bounds for multicolor Ramsey numbers, *Advances in Mathematics* **378** (2021), 107528. [1](#)
- [7] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294. [1](#)
- [8] P. Erdős, and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470. [1](#), [2](#)
- [9] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.*, **53(4)** (1947), 292–294.
- [10] P. C. Fishburn and R. L. Graham, Lexicographic Ramsey Theory, *J. Comb. Theory, Ser. A* **62(2)** (1993), 280–298. [2](#), [3](#)
- [11] K. Kalmanson, On a theorem of Erdős and Szekeres, *J. Comb. Theory Ser. A* **15** (1973), 343–346. [2](#)
- [12] J. B. Kruskal, Monotonic subsequences, *Proc. Amer. Math. Soc.* **4** (1953), 264–274. [2](#)
- [13] H. Lefmann, A note on Ramsey numbers, *Studia Sci. Math. Hungar.* **22** (1987), 445–446. [1](#)
- [14] N. Linial and M. Simkin, Monotone subsequences in high-dimensional permutations, *Comb. Prob. Comput.* **27** (2018), 69–83. [2](#)
- [15] A. P. Morse, Subfunction structure, *Proc. Amer. Math. Soc.* **21** (1969), 321–323. [2](#)
- [16] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc., Ser. 2* **30** (1930), 264–286. [1](#)
- [17] A. Sah, Diagonal Ramsey via effective quasirandomness, *Preprint available at arXiv: 2005.09251*. [1](#)

- [18] W. Sawin, An improved lower bound for multicolor Ramsey numbers and the half-multiplicity Ramsey number problem. arXiv preprint arXiv:2105.08850 (2021). [1](#)
- [19] R. Siders, Monotone subsequences in any dimension, *J. Comb. Theory Ser. A* **85** (1999), 243–253. [2](#)
- [20] J. M. Steele, Variations on the monotone subsequence theme of Erdős and Szekeres. In *Discrete probability and algorithms* 1995 (pp. 111–131). Springer, New York, NY.
- [21] J. Spencer, Ramsey’s theorem—a new lower bound, *J. Combin. Theory Ser. A* **18** (1975), 108–115. [1](#)
- [22] T. Szabó and G. Tardos, A multidimensional generalization of the Erdős-Szekeres lemma on monotone subsequences, *Comb. Prob. Comput.* **10** (2001), 557–565. [2](#)
- [23] A. Thomason, An upper bound for some Ramsey numbers, *J. Graph Theory* **12** (1988), 509–517. [1](#)
- [24] Y. Wigderson, An improved lower bound on multicolor Ramsey numbers, *Proc. Amer. Math. Soc.* **149(6)** (2021), 2371–2374. [1](#)