

Near-domination in graphs

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Abstract

A vertex u of a graph “ t -dominates” a vertex v if there are at most t vertices different from u, v that are adjacent to v and not to u ; and a graph is “ t -dominating” if for every pair of distinct vertices, one of them t -dominates the other. Our main result says that if a graph is t -dominating, then it is close (in an appropriate sense) to being 0-dominating. We also show that an analogous statement for digraphs is false; and discuss some connections with the Erdős-Hajnal conjecture.

1 Introduction

In this paper, all graphs are finite and have no loops or parallel edges. We say a vertex u of a graph t -dominates a vertex v if there are at most t vertices different from u, v that are adjacent to v and not to u ; and a graph is t -dominating if for every pair of distinct vertices, one of them t -dominates the other.

Graphs that are 0-dominating are easily understood; they are called “threshold graphs” and have several different characterizations, which we discuss later. But for general fixed t , t -dominating graphs are not so transparent, and our main result states that every t -dominating graph has bounded “local difference” from a 0-dominating graph. Let us define this.

Let G, H be graphs on the same vertex set. We say that the *local difference* between G, H is d if d is the maximum, over all vertices v , of

$$|N_G(v) \setminus N_H(v)| + |N_H(v) \setminus N_G(v)|,$$

where $N_G(v), N_H(v)$ denote the set of neighbours of v in G, H respectively. Thus, if G, H have local difference d , then H can be obtained from G by changing the adjacency of some pairs of vertices, where the changed pairs form a graph with maximum degree d . Local difference is evidently a metric, and could be used, for instance, to describe “defective colouring”. A d -defective k -colouring of a graph is a partition of its vertex set into k subsets such that the subgraph induced on each subset has maximum degree at most d , and this is the same as saying the graph has local difference at most d from a k -colourable graph.

It is easy to see that if G is 0-dominating and G, H have local difference at most d then H is $2d$ -dominating, and we prove a kind of converse:

1.1 *For all integers $t \geq 0$, if G is t -dominating then there is a 0-dominating graph H with the same vertex set such that G, H have local difference at most $646t^4$.*

The proof is in three stages: we reduce the problem to “split graphs” (graphs with vertex set partitioned into a clique and a stable set); we reduce the split graph question to a problem about matrices; and then we solve the matrix problem. They are carried out in reverse order.

It is natural to ask whether there is an analogous result for set containment: given a collection \mathcal{F} of sets such that, for all $A, B \in \mathcal{F}$, either $A \subset B$ or $B \subset A$, we know immediately that \mathcal{F} is a chain. But what if we are only given that $\min(|A \setminus B|, |B \setminus A|) \leq t$? Is there some $f(t)$ so that we can add/delete at most $f(t)$ elements to each set to obtain a chain? Equivalently, is there a version of 1.1 that holds for outneighbourhoods in a digraph? In section 5, we show that the answer is no.

This research was motivated by the Erdős-Hajnal conjecture [4], that for every graph H , there exists $\epsilon > 0$ such that every graph G not containing H as an induced subgraph has a stable set or clique of cardinality at least $O(|V(G)|^\epsilon)$, and we discuss an application of our theorem to that conjecture in the final section.

2 0-domination

There are several different characterizations of 0-dominating graphs, as we explain now. A graph G is 0-dominating if and only if there are no four distinct vertices u, u', v, v' such that uu' and vv' are

edges and the pairs u, v' and v, u' are nonadjacent. Here u, v might or might not be adjacent, and also u', v' might or might not be adjacent; so this can restated as:

2.1 *A graph is 0-dominating if and only if no induced subgraph is isomorphic to $C_4, 2K_2$, or P_4 .*

(C_4 denotes the four-vertex cycle graph; $2K_2$ denotes its complement, the graph consisting of two disjoint edges; and P_4 denotes the four-vertex path.) A *split graph* is a graph G such that $V(G)$ can be partitioned into a clique and a stable set, and a graph is a split graph if and only if [6] it has no induced subgraph isomorphic to $C_4, 2K_2$ or C_5 . Thus every 0-dominating graph is a split graph.

A *half-graph* is a bipartite graph with bipartition X, Y say, such that X, Y can be ordered as $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ with the following property: for all i, i', j, j' with $1 \leq i \leq i' \leq m$ and $1 \leq j \leq j' \leq n$, if $x_i y_j$ is an edge then $x_{i'} y_{j'}$ is an edge. It is easy and well-known that a bipartite graph is a half-graph if and only if it has no induced subgraph isomorphic to $2K_2$. Let us say a *split half-graph* is a graph obtained from a half-graph with bipartition X, Y by adding edges to make X a clique.

Let G be 0-dominating, and let $V(G)$ be the disjoint union of a clique X and a stable set Y . Let B be the bipartite graph with bipartition X, Y formed by the edges of G between X and Y . Then B has no induced subgraph $2K_2$, since G has no P_4 , and it follows that B is a half-graph, and so G is a split half-graph. Since split half-graphs are 0-dominating, we have shown that:

2.2 *G is 0-dominating if and only if G is a split half-graph.*

Every nonnull split half-graph has either a vertex of degree zero or a vertex adjacent to all other vertices, and deleting this vertex gives another split half-graph. A graph G is a *threshold graph* if it can be built starting from the null graph by the two operations of adding an isolated vertex and adding a vertex adjacent to all current vertices; and since threshold graphs are 0-dominating, we have a third characterization:

2.3 *G is 0-dominating if and only if G is a threshold graph.*

The characterization 2.2 is the most useful for our purposes.

3 A matrix problem

Let us denote the set of all pairs (i, j) ($1 \leq i \leq m, 1 \leq j \leq n$) by $[m] \times [n]$. A subset F of $[m] \times [n]$ is *up-closed* if for all i, i', j, j' with $1 \leq i \leq i' \leq m$ and $1 \leq j \leq j' \leq n$, if $(i, j) \in F$ then $(i', j') \in F$; and F is *down-closed* if $([m] \times [n]) \setminus F$ is up-closed. The union of two up-closed subsets is up-closed, and so every subset of $[m] \times [n]$ has a unique maximal up-closed subset.

Now let $A = (a_{ij} : (i, j) \in [m] \times [n])$ be a 0/1 matrix. Its *support* is the set of all pairs $(i, j) \in [m] \times [n]$ with $a_{ij} = 1$. We say that A is *monotone* if its support is up-closed. For $i, i' \in \{1, \dots, m\}$ we say that i, i' are *row-inclusive* in A if one of the sets $\{j \in \{1, \dots, n\} : a_{ij} \neq 0\}$, $\{j \in \{1, \dots, n\} : a_{i'j} \neq 0\}$ is a subset of the other (and we define *column-inclusive* similarly). We say that the matrix A is *inclusive* if all $i, i' \in \{1, \dots, m\}$ are row-inclusive in A (and therefore all $j, j' \in \{1, \dots, n\}$ are column-inclusive in A). It follows that A is inclusive if and only if its rows and columns can be reordered to make a matrix that is monotone.

If $t \geq 0$ is an integer, we say that A is *t-restricted* if

- for all $i, i' \in \{1, \dots, m\}$ with $i < i'$, there are at most t values of $j \in \{1, \dots, n\}$ such that $a_{ij} = 1$ and $a_{i'j} = 0$; and
- for all $j, j' \in \{1, \dots, n\}$ with $j < j'$, there are at most t values of $i \in \{1, \dots, m\}$ such that $a_{ij} = 1$ and $a_{ij'} = 0$.

Thus a matrix A is 0-restricted if and only if it is monotone, that is, its support is up-closed. In order to prove 1.1 we need to prove something similar for t -restricted matrices; but before that we handle a special case.

Let A, B be two 0/1 matrices both indexed by $[m] \times [n]$. The *local difference* between A, B is the maximum, over all rows and columns, of the number of terms in that row or column in which A, B differ. One might hope that for all t , every t -restricted 0/1 matrix A has bounded local difference from some monotone matrix; but that is false, even for $t = 1$. For instance, with $m = 1$ and n large, a $1 \times n$ matrix A with entries $(1, \dots, 1, 0, \dots, 0)$ (with $n/2$ ones and $n/2$ zeroes) is 1-restricted, and yet has arbitrarily large local difference from every monotone matrix. This is perhaps unfair in that being 1-restricted has no content for a $1 \times n$ matrix, but we could pad it by adding more rows; say $n/2$ rows of all zeroes, then the given row, and then $n/2$ rows of all ones, making an $(n + 1) \times n$ matrix, which is also a counterexample. One might try assuming in addition that the rows are in increasing order of row-sum, and the same for columns (which will be the case when we apply these results to our graph problem), but a similar counterexample still can be made, as follows. Take the $(n + 1) \times n$ matrix just described, and change the $n/2$ entries $a_{i, \frac{n}{2}+i}$ ($1 \leq i \leq n/2$) to ones and the $n/2$ entries $a_{\frac{n}{2}+i+1, i}$ ($1 \leq i \leq n/2$) to zeroes.

Nevertheless, something like this is true; replace “monotone” by “inclusive”. We will prove:

3.1 *For all integers $t \geq 0$ and every t -restricted 0/1 matrix A , there is an inclusive matrix B , such that the local difference between A, B is at most $644t^4$.*

This means that we can change a bounded number of entries in every row and column and then reorder rows and columns to get a monotone matrix. But before we prove this, we handle a special case, matrices of bounded “breadth”, and next we define this. Let $A = (a_{ij} : (i, j) \in E)$ be a 0/1 matrix, with support E_1 , and let $E_0 = ([m] \times [n]) \setminus E_1$. Let X be the maximal down-closed subset of E_0 , and Y the maximal up-closed subset of E_1 . (X is unique, since the union of two down-closed sets is also down-closed, and similarly Y is unique.) Thus $X \cap Y = \emptyset$. A *diagonal* for A means a subset F of $[m] \times [n]$ such that for some integer c , F is the set of all pairs $(i, j) \in [m] \times [n]$ with $j = i + c$ and $(i, j) \notin X \cup Y$. Since X is down-closed and Y is up-closed, it follows that every diagonal F is an interval in the sense that if $(i, i + c), (i'', i'' + c) \in F$ where $i'' > i$ then $(i', i' + c) \in F$ for all i' with $i \leq i' \leq i''$. We call the maximum cardinality of all diagonals the *breadth* of A . We first show:

3.2 *For all integers $t, w \geq 0$ with $2w \geq t + 1$, and every t -restricted 0/1 matrix A with breadth at most w , there is an inclusive matrix B such that the local difference between A, B is at most $2(t + w)w^3$.*

Proof. Let A be a t -restricted 0/1 matrix indexed by $[m] \times [n]$, with breadth at most w , let its support be E_1 , let $E_0 = ([m] \times [n]) \setminus E_1$, and let X, Y be the maximal down-closed subset of E_0 and up-closed subset of E_1 respectively. Let $Z = ([m] \times [n]) \setminus (X \cup Y)$. We will give rules to change some of the entries a_{ij} for $(i, j) \in Z$. We need to satisfy two conditions:

- after the changes, all pairs i, i' will be row-inclusive; and
- for each row or column, at most $(2t + w)w^3$ entries in that row or column will be changed.

Our first task is to give the rules, but that needs a number of definitions. For $1 \leq j \leq n$, the j th *column* means the set $\{(i, j) : 1 \leq i \leq m\}$ for some $j \in \{1, \dots, n\}$, and a *row* is defined similarly. A *post* is the nonempty intersection of a column with Z , and a *beam* is the nonempty intersection of a row with Z . Every member of Z belongs to a unique post and a unique beam, and how we change the corresponding entry of A depends on the types of this post and beam. (Posts and beams again are intervals, in the natural sense.)

Let P, P' be posts, where P, P' are the intersection of the j th column with Z and the j' th column with Z , respectively. They are *parallel* if

- for all $i \in \{1, \dots, m\}$, $(i, j) \in Z$ if and only if $(i, j') \in Z$; and
- for all $i \in \{1, \dots, m\}$, if $(i, j) \in Z$ then $a_{ij} = a_{ij'}$.

Thus two parallel posts involve the same rows, and have the same entries in those rows. The *multiplicity* of a post P is the number of posts that are parallel to P (counting P itself). We define parallelness and multiplicity for beams similarly.

Let $(i, j) \in Z$. We associate four integers with (i, j) :

- $p^-(i, j)$, the number of $i' < i$ such that $(i', j) \in Z$;
- $q^-(i, j)$, the number of $j' < j$ such that $(i, j') \in Z$;
- $p^+(i, j)$, the number of $i' > i$ such that $(i', j) \in Z$;
- $q^+(i, j)$, the number of $j' > j$ such that $(i, j') \in Z$.

We observe that the post containing (i, j) has cardinality $p^-(i, j) + p^+(i, j) + 1$, and a similar statement holds for the beam.

(1) For each $(i, j) \in Z$, $\min(p^-(i, j), q^+(i, j)) < w$, and $\min(q^-(i, j), p^+(i, j)) < w$.

Suppose that $p^-(i, j), q^+(i, j) \geq w$. It follows that $(i - w, j), (i, j + w) \in Z$. Since X is down-closed and $(i - w, j) \notin X$, it follows that for $0 \leq h \leq w$, $(i - w + h, j + h) \notin X$; and similarly, since $(i, j + w) \notin Y$ it follows that for $0 \leq h \leq w$, $(i - w + h, j + h) \notin Y$. Consequently the set $\{(i - w + h, j + h) : (0 \leq h \leq w)\}$ is a subset of a diagonal. But it has cardinality $w + 1$, a contradiction. Consequently $\min(p^-(i, j), q^+(i, j)) < w$, and similarly $\min(p^+(i, j), q^-(i, j)) < w$. This proves (1).

(2) For every post P , if $(i, j) \in P$ then the multiplicity of P is at most $q^-(i, j) + q^+(i, j) + 1$.

The beam containing (i, j) has cardinality $q^-(i, j) + q^+(i, j) + 1$; but it intersects all posts parallel to P , and so its cardinality is at least the multiplicity of P . This proves (2).

Now we can give the rules. For each $(i, j) \in [m] \times [n]$ define b_{ij} as follows. If $(i, j) \notin Z$ then $b_{ij} = a_{ij}$, so we may assume that $(i, j) \in Z$; let (i, j) belong to a post P and a beam Q .

- If both $p^-(i, j), q^-(i, j) \geq w$ then $b_{ij} = 1$;
- if both $p^+(i, j), q^+(i, j) \geq w$ then $b_{ij} = 0$;
- if both $p^-(i, j), p^+(i, j) < w$ and P has multiplicity at least $2w$ then $b_{ij} = a_{ij}$;
- if both $q^-(i, j), q^+(i, j) < w$ and Q has multiplicity at least $2w$ then $b_{ij} = a_{ij}$;
- if both $p^-(i, j), p^+(i, j) < w$ and P has multiplicity less than $2w$ then $b_{ij} = 0$;
- if both $q^-(i, j), q^+(i, j) < w$ and Q has multiplicity less than $2w$ then $b_{ij} = 0$.

We claim the rules are consistent; for let $(i, j) \in Z$. By (1), only one of the first two rules applies to (i, j) , and if one of the first two applies then none of the other rules apply. If the third rule applies to (i, j) , then the fifth does not; and since P has multiplicity at least $2w$, it follows by (2) that $q^-(i, j) + q^+(i, j) + 1 \geq 2w$, so at least one of $q^-(i, j), q^+(i, j) \geq w$ and consequently the fourth and sixth rules do not apply. Similarly if the fourth rule applies then the fifth and sixth do not. Finally, both the fifth and sixth may apply simultaneously, but they assign the same value to b_{ij} . Thus the rules are consistent. Furthermore, we observe that every $(i, j) \in Z$ falls under one of the rules, by (1), and so the matrix $B = (b_{ij})$ is well-defined.

Now we must check the two bullets given at the start of this proof. We say B is *increasing in row i* if $b_{ij} \leq b_{ij'}$ for all j, j' with $1 \leq j < j' \leq n$, and we define *increasing in column j* similarly.

(3) *Let P be a post, a subset of the j th column. If $|P| < 2w$ then B is increasing in column j .*

Suppose that there exist i, i' with $1 \leq i < i' \leq m$, such that $b_{ij} = 1$ and $b_{i'j} = 0$. It follows that $(i, j) \notin X$, and so $(i', j) \notin X$, since X is down-closed; and similarly $(i, j), (i', j) \notin Y$, and consequently $(i, j), (i', j) \in Z$. Let Q, Q' be the beams containing $(i, j), (i', j)$ respectively. Since $|P| < 2w$, both Q, Q' have multiplicity less than $2w$. Since $b_{ij} = 1$, not both $q^-(i, j), q^+(i, j) < w$ from the sixth rule. If $p^+(i, j) \geq w$, then by (1) $q^-(i, j) < w$, so $q^+(i, j) \geq w$ and $b_{ij} = 0$, a contradiction. Thus $p^+(i, j) < w$.

Suppose that also $p^-(i, j) < w$. Since $b_{ij} = 1$ it follows that P has multiplicity at least $2w$, and $a_{ij} = b_{ij} = 1$. If $a_{i'j} = 0$, then since P has multiplicity at least $2w \geq t + 1$, there are $t + 1$ values of $j' \in \{1, \dots, n\}$ such that $a_{ij'} = 1$ and $a_{i'j'} = 0$, contradicting that A is t -restricted. Thus $a_{i'j} = 1$, and in particular $a_{i'j} \neq b_{i'j}$. Since P has multiplicity at least $2w$ it follows that one of $p^-(i', j), p^+(i', j) \geq w$, and hence $p^-(i', j) \geq w$, since $p^+(i', j) < p^+(i, j) < w$. By (1), $q^+(i', j) < w$. Since P has multiplicity at least $2w$, (2) implies that $q^-(i', j) + q^+(i', j) + 1 \geq 2w$, and so $q^-(i', j) \geq w$. This contradicts the first rule, since $b_{i'j} = 0$. This proves that $p^-(i, j) \geq w$.

We have seen that not both $q^-(i, j), q^+(i, j) < w$, and so from (1), $q^-(i, j) \geq w$. But $q^-(i', j) \geq q^-(i, j)$ since X is down-closed, and so $q^-(i', j) \geq w$; and also $p^-(i', j) > p^-(i, j) \geq w$, and yet $b_{i'j} = 0$, contrary to the rules. This proves (3).

(4) *Let $1 \leq i < i' \leq m$; then i, i' are row-inclusive in B .*

Suppose not, then in particular, there exists $j \in \{1, \dots, n\}$ such that $b_{ij} = 1$ and $b_{i'j} = 0$. Now $(i, j) \notin X$ since $b_{ij} = 1$, and since X is down-closed it follows that $(i', j) \notin X$. Also $(i', j) \notin Y$ since $b_{i'j} = 0$, and so $(i, j) \notin Y$ since Y is up-closed. Consequently $(i, j), (i', j) \in Z$. Let P be the post

containing them both, and let Q, Q' be the beams containing $(i, j), (i', j)$ respectively. Since B is not increasing in the j th column, $|P| \geq 2w$ by (3). If B is increasing in the i th row, and B is increasing in the i' th row, then (since B is 0/1-valued) i, i' are row-inclusive in B , a contradiction. So B is not increasing in one of the i th row or the i' th row, and so one of $|Q|, |Q'| \geq 2w$ by (3).

Suppose that $|Q| \geq 2w$. Then one of $q^-(i, j), q^+(i, j) \geq w$, and one of $p^-(i, j), p^+(i, j) \geq w$, and since $b_{ij} = 1$, it follows from (1) and the rules that $p^-(i, j), q^-(i, j) \geq w$. But $q^-(i', j) \geq q^-(i, j)$ since X is down-closed, and so $q^-(i', j) \geq w$; and also $p^-(i', j) > p^-(i, j) \geq w$, and yet $b_{i'j} = 0$, contrary to the first rule. Thus $|Q| < 2w$, and so $|Q'| \geq 2w$.

Hence one of $q^-(i', j), q^+(i', j) \geq w$, and one of $p^-(i', j), p^+(i', j) \geq w$, and since $b_{i'j} = 0$, it follows from (1) and the rules that $p^+(i', j), q^+(i', j) \geq w$. But $q^+(i, j) \geq q^+(i', j)$, and so $q^+(i, j) \geq w$; and also $p^+(i, j) > p^+(i', j) \geq w$, and yet $b_{ij} = 1$, contrary to the rules. This proves (4).

(5) *The local difference between A, B is at most $2(t+w)w^3$.*

Let $1 \leq j \leq n$; by the symmetry between axes, it suffices to show that there are at most $(2t+w)w^3$ values of i such that $a_{ij} \neq b_{ij}$. For every such value of i , it follows that $(i, j) \in Z$; so we may assume that there is a post P included in the j th column with $|P| > (2t+w)w^3$, and choose i_1, i_2 with $1 \leq i_1 \leq i_2 \leq m$ such that

- for $1 \leq i < i_1$, $(i, j) \in X$;
- for $i_1 \leq i \leq i_2$, $(i, j) \in P$; and
- for $i_2 < i \leq m$, $(i, j) \in Y$.

Let $i_1 \leq i \leq i_2$, and suppose that $a_{ij} = 0$ and $b_{ij} = 1$. From the rules, it follows that $p^-(i, j), q^-(i, j) \geq w$; and so from (1), $p^+(i, j) < w$. Consequently $i_2 - w + 1 \leq i \leq i_2$, and so there are at most w such values of i . Now let $i_1 \leq i \leq i_2$, and suppose that $a_{ij} = 1$ and $b_{ij} = 0$. From the rules, it follows that either

- $p^+(i, j), q^+(i, j) \geq w$, or
- both $p^-(i, j), p^+(i, j) < w$ and P has multiplicity less than $2w$, or
- both $q^-(i, j), q^+(i, j) < w$ and Q_i has multiplicity less than $2w$, where Q_i is the beam containing (i, j) .

At most w values of i satisfy the first bullet, as before. If i satisfies the second bullet then $|P| \leq 2w - 1$, a contradiction since $|P| > (2t+w)w^3$; so we may assume that i satisfies the third bullet. There are therefore only w^2 possibilities for the pair $q^-(i, j), q^+(i, j)$. Let $h, k \geq 0$ with $h, k < w$, and let $I(h, k)$ be the set of all i that satisfy $i_1 \leq i \leq i_2$, and $a_{ij} = 1$, and $b_{ij} = 0$, and $q^-(i, j) = h$ and $q^+(i, j) = k$, and Q_i has multiplicity less than $2w$. We need to bound $|I(h, k)|$. We can evidently get an exponential bound, since there are only 2^{2w+1} possibilities for the entries of Q_i , and only $2w - 1$ values of $i \in I(h, k)$ in which all the entries are the same; but we can do better.

If $i \in I(h, k)$, A may or may not be increasing in the i th row. Let I_1 be the set of $i \in I(h, k)$ such that A is increasing in the i th row, and $I_2 = I(h, k) \setminus I_1$. For each $i \in I_1$, there are only $h + 1$ possibilities for the entries of Q_i , and at most $2w - 1$ distinct $i \in I_1$ with the same set of entries, and so $|I_1| \leq (h + 1)(2w - 1)$. If $i \in I_2$ then there exists j' with $j - h \leq j' \leq j + k - 1$ such that $a_{ij'} = 1$

and $a_{i,j'+1} = 0$. There are only $h + k$ choices for j' , and for each j' there are at most t values of $i \in I_2$ such that $a_{ij'} = 1$ and $a_{i,j'+1} = 0$, since A is t -restricted; and so $|I_2| \leq t(h + k)$. Consequently $|I(h, k)| \leq (h + 1)(2w - 1) + t(h + k)$. Since $h, k < w$, it follows that $|I(h, k)| \leq w(2w - 1) + 2(w - 1)t$.

In summary, at most w values of i satisfy the first bullet above; none satisfy the second bullet; and (since there are at most w^2 choices for h, k), at most $(w(2w - 1) + 2(w - 1)t)w^2$ satisfy the third. Consequently there are at most $(w(2w - 1) + 2(w - 1)t)w^2 + w$ values of i such that $a_{ij} = 1$ and $b_{ij} = 0$. As we saw, there are at most w values of i such that $a_{ij} = 0$ and $b_{ij} = 1$; so there are at most $(w(2w - 1) + 2(w - 1)t)w^2 + 2w$ values of i such that $a_{ij} \neq b_{ij}$. This proves (5).

From (4) and (5), this proves 3.2. ■

Now we eliminate the hypothesis about bounded breadth, by means of the following.

3.3 *For all integers $t \geq 0$ and every t -restricted 0/1 matrix A , there is a t -restricted 0/1 matrix B with breadth at most $4t$, such that the local difference between A, B is at most $4t$.*

Proof. Let A be a 0/1 matrix indexed by $[m] \times [n]$, let its support be E_1 , let $E_0 = ([m] \times [n]) \setminus E_1$, and let X, Y be the maximal down-closed subset of E_0 and up-closed subset of E_1 respectively. Let $Z = ([m] \times [n]) \setminus (X \cup Y)$. Let P_1 be the set of all $(i, j) \in [m] \times [n]$ such that there exist at least $2t$ values of $i' < i$ with $(i', j) \in E_1$, and let Q_1 be the set of all (i, j) such that there are at least $2t$ values of $j' < j$ with $(i, j') \in E_1$. Let P_0 be the set of all (i, j) such that there exist at least $2t$ values of $i' > i$ with $(i', j) \in E_0$, and let Q_0 be the set of all (i, j) such that there are at least $2t$ values of $j' > j$ with $(i, j') \in E_0$.

(1) *There do not exist $(i_1, j_1) \in P_1$ and $(i_0, j_0) \in Q_0$ such that $i_0 \geq i_1$ and $j_0 \geq j_1$. Also, there do not exist $(i_1, j_1) \in Q_1$ and $(i_0, j_0) \in P_0$ such that $i_0 \geq i_1$ and $j_0 \geq j_1$.*

Suppose that such $(i_1, j_1), (i_0, j_0)$ exist. From the symmetry between axes, we may assume that $(i_1, j_1) \in P_1$, and hence $(i_0, j_0) \in Q_0$. Consequently $(i_0, j_1) \in P_1 \cap Q_0$. Choose $I_1 \subseteq \{1, \dots, i_0 - 1\}$ with cardinality $2t$ such that $(i, j_1) \in E_1$ for each $i \in I_1$, and choose $J_0 \subseteq \{j_1 + 1, \dots, n\}$ with cardinality $2t$ such that $(i_0, j) \in E_0$ for each $j \in J_0$. From the symmetry between zeroes and ones, we may assume without loss of generality that $(i_0, j_1) \in E_0$. If E_1 contains at least half of the pairs (i, j) with $i \in I_1$ and $j \in J_0$, then there exists $i \in I_1$ such that $(i, j) \in E_1$ for at least t values of $j \in J_0$, and hence for at least $t + 1$ values of $j \in J_0 \cup \{j_0\}$; and since $(i, j_0) \in E_0$ for each such j , this contradicts that A is t -restricted. On the other hand, if E_0 contains more than half of the pairs (i, j) with $i \in I_1$ and $j \in J_0$, then there exists $j \in J_0$ such that $(i, j) \in E_0$ for at least $t + 1$ values of $i \in I_1$; and since $(i_1, j) \in E_1$ for each such i , this also contradicts that A is t -restricted. This proves (1).

(2) *There do not exist $(i_1, j_1) \in P_1 \cup Q_1$ and $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i_1 + 4t$ and $j_0 \geq j_1 + 4t$.*

Suppose that such $(i_1, j_1), (i_0, j_0)$ exist. From the symmetry between axes, we may assume that $(i_1, j_1) \in P_1$. By (1) it follows that $(i_0, j_0) \notin Q_0$, and so $(i_0, j_0) \in P_0$. Choose $I_1 \subseteq \{1, \dots, i_1 - 1\}$ with cardinality $2t$ such that $(i, j_1) \in E_1$ for each $i \in I_1$, and choose $I_0 \subseteq \{i_0 + 1, \dots, m\}$ with cardinality $2t$ such that $(i, j_0) \in E_0$ for each $i \in I_0$. If E_0 contains more than half of the pairs (i, j) with $i \in I_1$ and $j_1 < j < j_0$, then there exists j with $j_1 < j < j_0$ such that $(i, j) \in E_0$ for at least

$t + 1$ values of $i \in I_1$, contradicting that A is t -restricted. If E_1 contains at least half of the pairs (i, j) with $i \in I_1$ and $j_1 < j < j_0$, then there exists $i \in I_1$ such that $(i, j) \in E_1$ for at least half of the values of j with $j_1 < j < j_0$, and since $(i, j_1) \in E_1$, it follows that $(i, j_0) \in Q_1$, contrary to (1) since $(i_0, j_0) \in P_0$. This proves (2).

Construct a matrix $B = (b_{ij})$ as follows. Let $(i, j) \in [m] \times [n]$.

- If $(i, j) \in Z$ and there is no $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i$ and $j_0 \geq j$ then $b_{ij} = 1$;
- if $(i, j) \in Z$ and there exists $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i$ and $j_0 \geq j$, and there is no $(i_1, j_1) \in P_1 \cup Q_1$ such that $i_1 \leq i$ and $j_1 \leq j$, then $b_{ij} = 0$;
- if neither of these applies then $b_{ij} = a_{ij}$.

We need to show that B is t -restricted, B has breadth at most $4t$, and the local difference between A, B is at most $4t$. Let the support of B be F_1 , let $F_0 = ([m] \times [n]) \setminus F_1$, and let X', Y' be the maximal down-closed subset of F_0 and up-closed subset of F_1 respectively.

(3) For all $(i, j) \in [m] \times [n]$, if there is no $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i$ and $j_0 \geq j$ then $(i, j) \in Y'$. Also, if there is no $(i_1, j_1) \in P_1 \cup Q_1$ such that $i_1 \leq i$ and $j_1 \leq j$, then $(i, j) \in X'$.

Suppose there is no $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i$ and $j_0 \geq j$. Let F be the set of all $(i', j') \in [m] \times [n]$ such that $i \leq i'$ and $j \leq j'$; then for every such pair, $(i', j') \in Z \cup Y$, and there is no $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i'$ and $j_0 \geq j'$, and so $(i', j') \in F_1$. Hence F is an up-closed subset of F_1 , and since Y' is the unique maximal up-closed subset of F_1 , it follows that $F \subseteq Y'$, and in particular $(i, j) \in Y'$ as required. The second statement follows similarly. This proves (3).

(4) B is t -restricted.

Suppose that there exist i, i' with $1 \leq i < i' \leq m$ such that $b_{ij} = 1$ and $b_{i'j} = 0$ for $t + 1$ values of $j \in \{1, \dots, n\}$. For each such j , $(i, j) \notin Y'$ since $(i', j) \in F_0$; $(i, j) \notin X'$ since $(i, j) \notin F_0$; and similarly $(i', j) \notin X', Y'$. By (3), $b_{ij} = a_{ij}$ and $b_{i'j} = a_{i'j}$, contradicting that A is t -restricted. From the symmetry between axes, this proves (4).

(5) B has breadth at most $4t$.

Let $Z' = ([m] \times [n]) \setminus (X' \cup Y')$. Since $X \subseteq X'$ and $Y \subseteq Y'$, it follows that $Z' \subseteq Z$. Suppose that for some i, j all of the pairs $(i + h, j + h)$ ($0 \leq h \leq 4t$) belong to Z' . Thus $(i, j) \in Z$, and since $(i, j) \notin X'$, it follows from (3) that there exists $(i_1, j_1) \in P_1 \cup Q_1$ such that $i_1 \leq i$ and $j_1 \leq j$. Similarly, since $(i + 4t, j + 4t) \notin Y'$, there exists $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \geq i + 4t$ and $j_0 \geq j + 4t$. But then $i_0 \geq i_1 + 4t$ and $j_0 \geq j_1 + 4t$, contrary to (2). This proves (5).

(6) The local difference between A, B is at most $4t$.

Let $1 \leq i \leq m$, and let J be the set of all j with $1 \leq j \leq n$ such that $b_{ij} \neq a_{ij}$. We need to bound $|J|$. Let J_1 be the set of all $j \in J$ such that $b_{ij} = 1$ and $a_{ij} = 0$, and let J_0 be the set of all

$j \in J$ such that $b_{ij} = 0$ and $a_{ij} = 1$. If $j \in J_1$, then there is no $(i_0, j_0) \in P_0 \cup Q_0$ such that $i_0 \leq i$ and $j_0 \geq j$, and in particular $(i, j) \notin Q_0$; and so $(i, j') \in E_0$ for at most $2t - 1$ values of $j' < j$. Since $J_1 \subseteq E_0$, it follows (by choosing $j \in J_1$ maximal, if possible) that $|J_1| \leq 2t$.

If $j \in J_0$, then there is no $(i_1, j_1) \in P_1 \cup Q_1$ such that $i_1 \leq i$ and $j_1 \leq j$, and in particular $(i, j) \notin Q_1$, and so $(i, j') \in E_1$ for at most $2t - 1$ values of $j' > j$; and so $|J_0| \leq 2t$. Summing, it follows that $|J| \leq 4t$. From the symmetry between axes, this proves (6).

From (4), (5), (6), this completes the proof of 3.3. ■

Combining the previous two theorems, we deduce 3.1, which we restate::

3.4 *For all integers $t \geq 0$ and every t -restricted 0/1 matrix A , there is an inclusive matrix B , such that the local difference between A, B is at most $644t^4$.*

Proof. By 3.3 there is a t -restricted 0/1 matrix A' with breadth at most $4t$, such that the local difference between A, A' is at most $4t$. By 3.2 applied to A' , with $w = 4t$, there is an inclusive matrix B such that the local difference between A', B is at most $640t^4$. But then the local difference between A, B is at most $640t^4 + 4t \leq 644t^4$. This proves 3.4. ■

4 Near-dominating graphs

Now we use the results of the previous section to study t -dominating graphs. It is tempting to try to apply 3.1 directly to the adjacency matrix A of a t -dominating graph, but that does not work. 3.1 would give us an inclusive matrix B that has bounded local difference from A , and so we could reorder its rows and columns to make it monotone; but we need the corresponding permutations of the rows and columns to be the same, and we need the monotone matrix to be symmetric, and the direct application of 3.1 gives neither of these things. We will apply 3.1 in another way. First we show:

4.1 *Let G be a t -dominating split graph. Then there is a split half-graph H with $V(H) = V(G)$ such that the local difference between G, H is at most $644t^4$.*

Proof. Let G be a t -dominating split graph, and let $V(G)$ be the disjoint union of a clique M and a stable set N . Let $M = \{u_1, \dots, u_m\}$, where for $1 \leq i < j \leq m$ the degree of u_i is at most that of u_j , and let $N = \{v_1, \dots, v_n\}$, where for $1 \leq i < j \leq n$ the degree of v_i is at most that of v_j . Let $A = (a_{ij})$ be the 0/1 matrix where $a_{ij} = 1$ if u_i, v_j are adjacent and $a_{ij} = 0$ otherwise. We claim that A is t -restricted. For let $1 \leq i < i' \leq m$, and suppose that there are t values of $j \in \{1, \dots, n\}$ such that $a_{ij} = 1$ and $a_{i'j} = 0$. Thus for each such j , v_j is adjacent to u_i and not to $u_{i'}$. Since the degree of $u_{i'}$ is at least that of u_i , it follows that there are at least t vertices adjacent to $u_{i'}$ and not to u_i , contradicting that G is t -dominating. A similar argument shows that there do not exist $j, j' \in \{1, \dots, n\}$ with $j < j'$ such that for t values of i , $a_{ij} = 1$ and $a_{ij'} = 0$. Hence A is t -restricted. By 3.4, there is an inclusive matrix B such that the local difference between A, B is at most $644t^4$. Let H be the graph with vertex set $M \cup N$ in which u_i, v_j are adjacent if $b_{ij} = 1$, and M is a clique and N is a stable set. It follows that H is a split half-graph, and its local difference from G is at most $644t^4$. This proves 4.1. ■

4.2 Let G be a t -dominating graph. Then there is a t -dominating split graph H with $V(H) = V(G)$ and with local difference at most $2t$ from G .

Proof. Let $V(G) = \{v_1, \dots, v_n\}$, where for $1 \leq i < j \leq n$, the degree of v_j is at least that of v_i . Choose i minimum such that some vertex in $\{v_1, \dots, v_i\}$ has at least $2t+1$ neighbours in $\{v_1, \dots, v_i\}$. (If this is not possible then every vertex has degree at most $2t$, and so G has local difference at most $2t$ from the graph H obtained by deleting all the edges, which is a t -dominating split graph.) Let $N = \{v_1, \dots, v_{i-1}\}$ and $M = \{v_i, \dots, v_n\}$. Thus every vertex in N has at most $2t$ neighbours in N . Choose $x \in N \cup \{v_i\}$ and $X \subseteq N \setminus \{x\}$ with $|X| \geq 2t$ (and $X \neq \emptyset$ if $t = 0$), such that X is the set of neighbours of x in N . For all $j \geq i$, there are at most t vertices adjacent to x and not to v_j , since the degree of v_j is at least that of x and G is t -dominating. In particular, v_j is adjacent to at least half the vertices in X . Let $j \geq i$, and let Y be the set of vertices in M that are different from and nonadjacent to v_j . Since every vertex in Y is adjacent to at least half the vertices in X , and $X \neq \emptyset$, some vertex $v_h \in X$ is adjacent to at least half the vertices in Y . But there are at most t vertices adjacent to v_h and not to v_j , since the degree of v_h is at most that of v_j and G is t -dominating; and consequently $|Y|/2 \leq t$. Thus every vertex $v \in M$ is nonadjacent to at most $2t$ vertices in $M \setminus \{v\}$. Let H be the split graph obtained from G by deleting all edges with both ends in N and making adjacent all pairs of vertices in M ; then the local difference between G, H is at most $2t$.

We claim that H is t -dominating. For let $u, v \in V(H) = V(G)$. If $u, v \in M$ then since one of u, v t -dominates the other in G , the same is true in H (since in H , u, v have the same neighbours in $M \setminus \{u, v\}$). Similarly if $u, v \in N$ then one of u, v t -dominates the other. If $u \in M$ and $v \in N$ then v 0 -dominates and hence t -dominates u . This proves that H is t -dominating, and so proves 4.2. ■

Combining 4.1 and 4.2, we obtain our main result, which we restate:

4.3 Let G be a t -dominating graph. Then there is a split half-graph H with $V(H) = V(G)$ and with local difference at most $646t^4$ from G .

Proof. Let G be t -dominating. By 4.2 there is a t -dominating split graph G' with $V(G') = V(G)$ and with local difference at most $2t$ from G . By 4.1, there is a split half-graph H with $V(H) = V(G)$ such that the local difference between G', H is at most $644t^4$. Thus the local difference between G, H is at most $644t^4 + 2t \leq 646t^4$. This proves 4.3. ■

5 A counterexample

Here is a concept similar to t -domination for graphs and t -restriction for matrices. Let G be a bipartite graph with bipartition (A, B) ; we say G is t -nested on (A, B) if for all distinct $u, v \in B$, one of them has at most t neighbours nonadjacent to the other. Thus, 0 -nested graphs are half-graphs. (Being t -nested is “half” of being t -restricted; t -restriction also requires the analogous statement with A, B exchanged. We could also view t -nestedness as a version of t -domination for digraphs, if we regard the matrix as the adjacency matrix of a digraph.)

Let G, H both be bipartite graphs with bipartition (A, B) ; we say the *bipartite local difference* between G, H is the maximum over $v \in B$ of $|N_G(v) \setminus N_H(v)| + |N_H(v) \setminus N_G(v)|$. One might hope that for all t there exists $f(t)$ such that if G is t -nested on (A, B) then there is a half-graph H with

the same bipartition (A, B) , with bipartite local difference at most $f(t)$ from G . (Indeed, this looked like an easier question than our main result, and we tried it first as a warm-up.) But this is false, even if $t = 1$. Here is a counterexample.

Let \mathcal{S} be the set of all 0/1 sequences with at most k terms, and let T be the binary tree defined by \mathcal{S} ; thus $V(T) = \mathcal{S}$, and s, s' are adjacent if their lengths differ by one and one of them is an initial subsequence of the other. Let I be the set of all $s \in \mathcal{S}$ of length less than k , and B those of length exactly k . (Thus B is the set of leaves of T , and I the set of internal vertices.) Each $s \in B$ is a sequence of k terms (s_1, \dots, s_k) say, and we define $n(s) = \sum_{1 \leq i \leq k} s_i 2^{k-i}$. Thus all the numbers $n(s)$ ($s \in B$) are different and range from 0 to $2^k - 1$. For $s, s' \in B$, we write $s < s'$ if $n(s) < n(s')$.

For each $s \in B$ take a set W_s of $2^{k+1} + 2k$ new vertices, so that all the sets $V(T)$ and W_s ($s \in B$) are pairwise disjoint. Let $W = \bigcup_{s \in B} W_s$, and let $A = I \cup W$. We construct a bipartite graph G with bipartition (A, B) as follows. Let $s \in B$. For $s' \in B$, s has no neighbour in $W_{s'}$ if $s' > s$, and s is adjacent to every vertex in $W_{s'}$ if $s' \leq s$. For $s' \in I$, we decide the adjacency of s, s' in G by the following rule. Choose i minimum such that either i is greater than the number of terms of s' , or the i th terms of s and of s' are different, and let x be the i th term of s ; we make s, s' adjacent if and only if $x = 0$. (In other words, if we consider the paths P_s and $P_{s'}$ from the root to s and s' in T , we have that s, s' are adjacent in G if the first vertex of $P_{s'}$ that does not lie on P_s gets label 0.)

This graph is 1-nested; and indeed, for $s, s' \in B$ with $s < s'$, there is at most one $v \in I$ adjacent to s and not to s' , and no such $v \in W$. Suppose that there is a 0-nested graph H with bipartition (A, B) that has bipartite local difference less than k from G . For each $s \in S$, its degree in G is at least $(2^{k+1} + 2k)n(s)$ and at most $(2^{k+1} + 2k)n(s) + 2^{k+1} - 1$; and so its degree in H is at least $(2^{k+1} + 2k)n(s) - k$ and at most

$$(2^{k+1} + 2k)n(s) + 2^{k+1} - 1 + k < (2^{k+1} + 2k)(n(s) + 1) - k.$$

Consequently, for distinct $s, s' \in B$, if $s < s'$ then the degree of s in H is less than that of s' . Since H is 0-nested, it follows that every vertex of A adjacent to s in H is also adjacent to s' .

Let t_0 be the null sequence, and let L_0, R_0 be the sets of members of B with first term 0 and 1 respectively. It follows that t_0 is adjacent in G to every vertex in L_0 , and nonadjacent to every vertex in R_0 ; and yet $s < s'$ for every $s \in L_0$ and $s' \in R_0$. Consequently, in H either t_0 is nonadjacent to every member of L_0 , or adjacent to every member of R_0 . If the first, let t_1 be the one-term sequence (0), and otherwise $t_1 = (1)$. Let L_1 be the set of members of B such that t_1 is an initial segment, and the second term is 0, and let R_1 be those such that t_1 is an initial segment with second term 1. Again, t_1 is adjacent in G to every member of L_1 , and nonadjacent to every member of R_1 , and in H either t_1 is nonadjacent to all vertices in L_1 or adjacent to all in R_1 ; let t_2 be the corresponding two-term sequence. By continuing this process we obtain a sequence $t_0, t_1, t_2, \dots, t_{k-1} \in I$, and a vertex $t_k \in B$, where for $0 \leq i \leq k$, t_i has i terms and t_{i-1} is an initial segment of t_i , and for $0 \leq i \leq k-1$, t_k is adjacent to t_i in exactly one of G, H . Since $t_k \in B$, this contradicts that the bipartite local difference between G, H is at most $k-1$.

This could be viewed another way: for each vertex in B , take its set of neighbours in A . Then we obtain a collection of subsets \mathcal{C} of A such that for every two of them, say X, Y , one of $|X \setminus Y|, |Y \setminus X| \leq 1$. But if we want to change this last 1 to a 0 by adding and subtract elements of A from the sets of \mathcal{C} , then some set has to have an arbitrarily large number of elements added or subtracted.

6 The Erdős-Hajnal conjecture

Let us say an *ideal* of graphs is a class \mathcal{C} of graphs, such that if $G \in \mathcal{C}$ and H is isomorphic to an induced subgraph of G then $H \in \mathcal{C}$; and an ideal is *proper* if some graph is not in it. The Erdős-Hajnal conjecture [4] asserts:

6.1 Conjecture: *For every proper ideal \mathcal{C} , there exist $c, \epsilon > 0$ such that for every graph $G \in \mathcal{C}$, G has a clique or stable set of cardinality at least $c|V(G)|^\epsilon$.*

We are interested in the way the (optimal) coefficient ϵ depends on \mathcal{C} . In particular, when does taking $\epsilon = 1$ work? If \mathcal{C} is the set of all graphs not containing one particular graph H as an induced subgraph, then there are almost no choices of H for which $\epsilon = 1$ works — only those graphs H with at most two vertices, as is easily seen. For instance, let S_t be the star with centre of degree t , and \mathcal{C} the class of all graphs that do not contain S_t as an induced subgraph; then $\epsilon = O(1/t)$, since the Ramsey number $R(k, t)$ is at least $(k/\log k)^{(t+1)/2}$ for fixed t and large k [2] (and ϵ is known to exist because of a result of Alon, Pach and Solymosi [1]).

But there are ideals defined by excluding more than one graph. If we take \mathcal{C} to be the class of graphs that contain neither S_t nor its complement as an induced subgraph, then [3] for every graph $G \in \mathcal{C}$, either G or its complement has maximum degree bounded by a function of t , and so there exist c, ϵ with $\epsilon = 1$.

Another ideal of interest is the class of all t -dominating graphs, for fixed t . Every split graph has a clique or stable set containing at least half its vertices, and so by 4.2, in every t -dominating graph G , there is a subset $X \subseteq V(G)$ with $|X| \geq |V(G)|/2$ such that either the subgraph induced on X has maximum degree at most $2t$ or its complement graph does. Consequently G has a clique or stable set with cardinality at least $|V(G)|/(4t + 2)$; and so we may take $\epsilon = 1$ in 6.1 for this class.

Take a “substar” (a graph obtained from a star by deleting some edges) and the complement of a substar, and let \mathcal{C} be the class containing neither of these graphs; in this case $\epsilon = 1$ does not work in general (consider a disjoint union of $n^{1/2}$ cliques each with $n^{1/2}$ vertices), but $\epsilon = 1/2$ works. This is a special case of the following. (If G is a graph, $\omega(G)$ and $\alpha(G)$ denote the cardinalities of its largest clique and largest stable set respectively, and we denote $\max(\omega(G), \alpha(G))$ by $\rho(G)$.) We recall that threshold graphs were defined in section 2, and are the same as 0-dominating graphs, because of 2.3.

6.2 *Let H_1, H_2 be threshold graphs, with $|V(H_1)| + |V(H_2)| = m$ and $\omega(H_1) + \alpha(H_2) = k$. For every graph G , if G has no induced subgraph isomorphic to H_1 or to H_2 , then $|V(G)| \leq (2^m - 1)\rho(G)^{k-2}$.*

Proof. We proceed by induction on m . If one of H_1, H_2 has at most one vertex the claim is trivial, so we assume they both have at least two vertices. A vertex is *isolated* if it has degree zero, and *universal* if it is adjacent to all other vertices. For $i = 1, 2$, let H'_i be obtained from H_i by deleting an isolated vertex if there is one, and if not let $H'_i = H_i$; and let H''_i be obtained from H_i by deleting a universal vertex if there is one, and if not let $H''_i = H_i$. Let G contain neither of H_1, H_2 . For $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X . Let $v \in V(G)$, and let N be the set of neighbours of v , and $M = V(G) \setminus (N \cup \{v\})$.

Suppose first that one of H_1, H_2 has an isolated vertex, and one has a universal vertex. Since $G[M]$ contains neither of H'_1, H'_2 , it follows that $|M| \leq (2^{m-1} - 1)\rho(G)^{k-2}$; and similarly $|N| \leq (2^{m-1} - 1)\rho(G)^{k-2}$. Consequently

$$|V(G)| = |N| + |M| + 1 \leq 2(2^{m-1} - 1)\rho(G)^{k-2} + 1 \leq (2^m - 1)\rho(G)^{k-2}$$

as required.

Thus either H_1, H_2 both have no isolated vertex, or they both have no universal vertex. We can reduce the second case to the first by replacing H_1, H_2 and G by their complements, and exchanging H_1, H_2 (to preserve the property that $\omega(H_1) + \alpha(H_2) = k$); so we may assume the first case holds. It follows that H_1, H_2 both have universal vertices, since they are threshold graphs; and in particular, $\omega(H_1'') + \alpha(H_2'') = k - 1$.

Choose $v \in V(G)$ with maximum degree, and let N be its set of neighbours. Since $G[N]$ contains neither of H_1'', H_2'' , it follows from the inductive hypothesis that

$$|N| \leq (2^{m-2} - 1)\rho(G)^{k-3} \leq 2^{m-2}\rho(G)^{k-3} - 1.$$

Since v has maximum degree, it follows that G is $2^{m-2}\rho(G)^{k-3}$ -colourable, and so some stable set has cardinality at least $|V(G)|/(2^{m-2}\rho(G)^{k-3})$. In particular $\rho(G)$ is at least this quantity, and so $2^{m-2}\rho(G)^{k-2} \geq |V(G)|$, and again the result follows. This proves 6.2. \blacksquare

One can also ask, what is the size of the largest clique or stable set in almost all graphs in an ideal? This may be much larger than we can guarantee for *every* graph in the ideal. Thus while for every $\epsilon > 0$ there exists t such that there are K_t -free n -vertex graphs with no stable set of size at least n^ϵ , it is known [5] that for every t there exists $C_t > 0$ such that almost every K_t -free n -vertex graph has a stable set of size at least $C_t n$.

It remains an open problem to determine whether for some $\epsilon > 0$ and every graph H , there exists $C_H > 0$ such that almost every graph G that does not contain H as an induced subgraph has a stable set or clique of size at least $C_H |V(G)|^\epsilon$. This would be true (see Lemma 3 of [7] and the discussion around it) if for some $\epsilon > 0$ and every integer $t \geq 0$, there exists $C_t > 0$ such that, for every graph G , either

- there are two vertices u, v and a stable set S with $|S \cap (N(u) - N(v))|, |S \cap (N(v) - N(u))|, |S \cap N(u) \cap N(v)|$ all of cardinality t , or
- there are two vertices u, v and a clique T with $|T \cap (N(u) - N(v))|, |T \cap (N(v) - N(u))|, |T \setminus (N(u) \cup N(v))|$ all of cardinality t , or
- G contains a clique or stable set of size at least $C_t |V(G)|^\epsilon$.

Our results are a first step in proving this statement.

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