

Induced subgraphs of graphs with large chromatic number.
XIII. New brooms

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Abstract

Gyárfás [3] and Sumner [10] independently conjectured that for every tree T , the class of graphs not containing T as an induced subgraph is χ -bounded, that is, the chromatic numbers of graphs in this class are bounded above by a function of their clique numbers. This remains open for general trees T , but has been proved for some particular trees. For $k \geq 1$, let us say a *broom* of *length* k is a tree obtained from a k -edge path with ends a, b by adding some number of leaves adjacent to b , and we call a its *handle*. A tree obtained from brooms of lengths k_1, \dots, k_n by identifying their handles is a (k_1, \dots, k_n) -*multibroom*. Kierstead and Penrice [5] proved that every $(1, \dots, 1)$ -multibroom T satisfies the Gyárfás-Sumner conjecture, and Kierstead and Zhu [7] proved the same for $(2, \dots, 2)$ -multibrooms.

In this paper give a common generalization; we prove that every $(1, \dots, 1, 2, \dots, 2)$ -multibroom satisfies the Gyárfás-Sumner conjecture .

1 Introduction

For a graph G , let $\chi(G)$ denote the chromatic number of G , and let $\omega(G)$ denote its clique number, that is, the number of vertices in its largest clique. We say a graph G *contains* H if some induced subgraph of G is isomorphic to H , and otherwise G is *H -free*.

The Gyárfás-Sumner conjecture [3, 10] asserts that:

1.1 Conjecture: *For every forest T and every integer κ , there exists c such that $\chi(G) \leq c$ for every T -free graph G with $\omega(G) \leq \kappa$.*

There has been surprisingly little progress on this conjecture. It is easy to see that if the conjecture holds for every component of a forest then it holds for the forest (the first component must be present; delete it and all vertices with a neighbour in it and repeat with the next component), and so it suffices to prove the conjecture when T is a tree. Gyárfás [3] proved the conjecture when T is a path, and Scott [9] proved it when T is a subdivision of a star; and recently, with Maria Chudnovsky, we [1] proved it for trees obtained from a subdivided star by adding one more vertex with one neighbour, and for trees obtained from a star and a subdivided star by adding a path between their centres. But the results that concern us most here are theorems of Gyárfás, Szemerédi and Tuza [4], Kierstead and Penrice [5], and Kierstead and Zhu [7], which are the only other results so far on the Gyárfás-Sumner conjecture, and which we explain next.

For $k \geq 1$, let us say a *broom of length k* is a tree obtained from a k -edge path with ends a, b by adding some number of leaves adjacent to b , and we call a its *handle*. A tree obtained from n brooms of lengths k_1, \dots, k_n respectively by identifying their handles is called a (k_1, \dots, k_n) -*multibroom*. Gyárfás, Szemerédi and Tuza (in the triangle-free case) and then Kierstead and Penrice (in the general case) proved that $(1, \dots, 1)$ -multibrooms satisfy the Gyárfás-Sumner conjecture, and Kierstead and Zhu proved that $(2, \dots, 2)$ -multibrooms satisfy it. In this paper we prove a common generalization of these results: every $(1, \dots, 1, 2, \dots, 2)$ -multibroom satisfies the Gyárfás-Sumner conjecture.

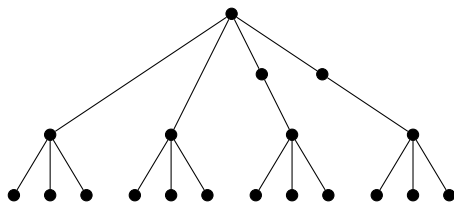


Figure 1: A $(1, 1, 2, 2)$ -multibroom

Let us state this more precisely. A (k, δ) -*broom* means a broom of length k with δ leaves different from its handle (thus, it is obtained by adding δ leaves adjacent to one end of a k -edge path). For $\delta \geq 1$, let $T(\delta)$ be the tree formed from the disjoint union of δ $(1, \delta)$ -brooms and δ $(2, \delta)$ -brooms by identifying their handles. We will prove that

1.2 *For all $\delta \geq 0$ and all $\kappa \geq 0$ there exists c such that every $T(\delta)$ -free graph with $\omega(G) \leq \kappa$ has chromatic number at most c .*

The proof method is by combining ideas of [5, 7] with some new twists.

2 Inductions

There are various inductions that will give us some assistance. We can use induction on κ , and on δ (in fact with a little work we can more-or-less assume that the result holds for every tree obtained from $T(\delta)$ by deleting a leaf), and there is a third induction, core maximization, that we explain later. Next we explain these inductions in more detail.

First and easiest, by induction on κ , we may assume that there exists τ such that $\chi(G) \leq \tau$ for every $T(\delta)$ -free graph with clique number less than κ . In particular, this tells us that if G is $T(\delta)$ -free with clique number at most δ , then for every vertex, the subgraph induced on its neighbours has chromatic number at most τ (since this subgraph has clique number less than τ). Consequently we can use 2.1 below, taking $T = T(\delta)$.

If v is a vertex of a graph G , and $k \geq 1$, $N^k(v)$ or $N_G^k(v)$ denotes the set of vertices of G with distance exactly k from v , and $N^k[v]$ denotes the set with distance at most k from v . If G is a nonnull graph and $k \geq 1$, we define $\chi^k(G)$ to be the maximum of $\chi(N^k[v])$ taken over all vertices v of G . (For the null graph G we define $\chi^k(G) = 0$.)

The following follows by repeated application of theorem 3.2 of [1] (a similar theorem for $(2, \dots, 2)$ -multibrooms is proved in [7]):

2.1 *Let T be a tree formed by identifying the handles of some set of brooms (of arbitrary lengths). For all $\kappa, \tau \geq 0$ there exists c with the following property. Let G be a T -free graph, with $\omega(G) \leq \kappa$, such that for every vertex, the subgraph induced on its neighbours has chromatic number at most τ . Then $\chi^2(G) \leq c$.*

Next, let us explore induction on the size of $T(\delta)$. That will allow us to exploit “matching-covered” sets. Let $X \subseteq V(G)$. We say that X is *matching-covered* in G if for each $x \in X$ there exists $y \in V(G) \setminus X$ adjacent to x and to no other vertex in X .

We would like to be able to assume that the result holds for all trees obtained from $T(\delta)$ by deleting a leaf; but only deleting one leaf, from one of its brooms, and so the smaller tree is not of the form $T(\delta')$ for $\delta' < \delta$, and so induction on δ is not fine enough. We could change the statement of the theorem, and prove it not only for $T(\delta)$, but for any tree that is a subtree of $T(\delta)$; but that would make things notationally more complicated later. There is another way to do it that is more convenient.

Let us say that G is (δ, κ) -good if G is $T(\delta)$ -free and $\omega(G) \leq \kappa$. An *ideal* of graphs is a class \mathcal{C} of graphs such that every induced subgraph of a member of \mathcal{C} also belongs to \mathcal{C} . If $X \subseteq V(G)$, we write $\chi(X)$ for $\chi(G[X])$ when there is no ambiguity.

2.2 *Let $\delta, \kappa \geq 1$. If \mathcal{C} is an ideal of (δ, κ) -good graphs with unbounded chromatic number, then there exist a subideal \mathcal{C}' of \mathcal{C} with unbounded chromatic number, and a number c such that every matching-covered set in every member of \mathcal{C}' has chromatic number at most c .*

Proof. Let R be a maximal subtree of $T(\delta)$ such that there exists c such that every R -free graph in \mathcal{C} with clique number at most κ has chromatic number at most c , and choose some such number c . By hypothesis there are members of \mathcal{C} with arbitrarily large chromatic number, so $R \neq T(\delta)$. Hence there is a subtree S of $T(\delta)$ with a leaf v such that $S \setminus v = R$. Let u be the neighbour of v in S . Let \mathcal{C}' be the subideal of all S -free graphs in \mathcal{C} . From the maximality of R , there are graphs in \mathcal{C} with arbitrarily large chromatic number.

Let $G \in \mathcal{C}$, and let X be matching-covered in G . Suppose that there is an induced subgraph of $G[X]$ isomorphic to R , and to simplify notation we assume it equals R . Choose $y \in V(G) \setminus X$ adjacent to u and to no other vertex in X ; then $G[V(R) \cup \{y\}]$ is isomorphic to S , a contradiction. Thus $G[X]$ does not contain R . Since $G[X] \in \mathcal{C}$, the choice of c implies that $\chi(X) \leq c$. Since all graphs in \mathcal{C} are (δ, κ) -good, this proves 2.2. \blacksquare

There is a third, very helpful, induction we can use, but it is more complicated. For integers $a, b \geq 1$, let us say an (a, b) -core in a graph G is a subset $Y \subseteq V(G)$ of cardinality ab , that admits a partition $\{A_1, \dots, A_b\}$ such that

- A_1, \dots, A_b each have cardinality a ;
- A_1, \dots, A_b are all stable sets of G ; and
- for $1 \leq i < j \leq b$, every vertex in A_i is adjacent to every vertex in A_j .

(An (a, b) -core is therefore a complete multipartite induced subgraph of specified size.) This partition is unique, since $a \geq 1$, and we speak of A_1, \dots, A_b as the *parts* of Y . Thus, if there is an (a, b) -core in G then $b \leq \omega(G)$. Let \mathbb{N} denote the set of nonnegative integers.

2.3 *Let $\delta, \kappa \geq 1$. If \mathcal{C} is an ideal of (δ, κ) -good graphs with unbounded chromatic number, then there exist a subideal \mathcal{C}' of \mathcal{C} with unbounded chromatic number, and integers $\alpha \geq 1$ and $\beta \geq 2$, and a non-decreasing function $\theta : \mathbb{N} \rightarrow \mathbb{N}$, with the following properties:*

- for all $a \geq 1$, every graph in \mathcal{C} with chromatic number more than $\theta(a)$ admits an (a, β) -core.
- no graph in \mathcal{C}' admits an $(\alpha, \beta + 1)$ -core.

Proof. For integers $a \geq 1$ and $b \geq 2$, let us say (a, b) is *unavoidable* if there exists c such that every graph in \mathcal{C} with chromatic number more than c admits an (a, b) -core. V. Rödl (see [6]) proved that for all integers $a \geq 1$, $(a, 2)$ is unavoidable. Choose β with $2 \leq \beta \leq \kappa + 1$ maximum such that for all $a \geq 1$, (a, β) is unavoidable. Since (a, β) is unavoidable for all $a \geq 1$, there is a function $\theta : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $a \geq 1$, every graph in \mathcal{C} with chromatic number more than $\theta(a)$ admits an (a, β) -core, and we can choose θ to be non-decreasing, so the first bullet holds.

By hypothesis there are graphs in \mathcal{C} with unbounded chromatic number, and they do not admit $(1, \kappa + 1)$ -cores (because they have clique number at most κ), so $\beta \leq \kappa$. From the maximality of β , there exists $\alpha \geq 1$ such that there are graphs in \mathcal{C} with arbitrarily large chromatic number that do not admit an $(\alpha, \beta + 1)$ -core. Let \mathcal{C}' be the ideal of graphs in \mathcal{C} that do not admit an $(\alpha, \beta + 1)$ -core; then the second bullet holds. This proves 2.3. \blacksquare

We combine these results in the following.

2.4 *Let $\delta \geq 1$. Suppose that for some value of $\kappa \geq 1$ there are (δ, κ) -good graphs with unbounded chromatic number. Then there exist $\tau \geq 0$, $\alpha \geq 1$, $\beta, \kappa \geq 2$, a non-decreasing function $\theta : \mathbb{N} \rightarrow \mathbb{N}$, and an ideal \mathcal{C} of graphs with unbounded chromatic number, such that for every $G \in \mathcal{C}$:*

- G is $T(\delta)$ -free;
- $\omega(G) \leq \kappa$;

- $\chi^2(G) \leq \tau$;
- every matching-covered set in G has chromatic number at most τ ;
- for all $a \geq 1$, if $\chi(G) > \theta(a)$ then G admits an (a, β) -core;
- G does not admit an $(\alpha, \beta + 1)$ -core.

Proof. Choose κ minimum such that there are (δ, κ) -good graphs with unbounded chromatic number. Thus $\kappa \geq 2$. Choose τ_1 such that every $(\delta, \kappa - 1)$ -good graph has chromatic number at most τ_1 . By 2.1 there exists τ_2 such that $\chi^2(G) \leq \tau_2$ for every (δ, κ) -good graph. By 2.2 there exist τ_3 and an ideal \mathcal{C}_1 of (δ, κ) -good graphs with unbounded chromatic number, such that every matching-covered set in G has chromatic number at most τ_3 . By 2.3, there exist a subideal \mathcal{C} of \mathcal{C}_1 with unbounded chromatic number, and α, β satisfying the last two bullets. Let $\tau = \max(\tau_1, \tau_2, \tau_3)$; then all six bullets are satisfied. This proves 2.4. ■

In view of 2.4, in order to prove 1.2 it suffices to show the following:

2.5 For all $\tau \geq 0$, and $\alpha, \delta \geq 1$, and $\beta \geq 2$, and for every non-decreasing function $\theta : \mathbb{N} \rightarrow \mathbb{N}$, there exists c such that if G satisfies

- (i) G is $T(\delta)$ -free;
- (ii) $\chi^2(G) \leq \tau$;
- (iii) every matching-covered set in G has chromatic number at most τ ;
- (iv) for all $a \geq 1$, if $\chi(G) > \theta(a)$ then G admits an (a, β) -core;
- (v) G does not admit an $(\alpha, \beta + 1)$ -core

then $\chi(G) \leq c$.

We could have added another constant κ and another condition that $\omega(G) \leq \kappa$, but it turns out not to be needed any more (a bound on $\omega(G)$ is implied by the second condition).

The five statements (i)–(v) of 2.5 are important for the rest of the paper, and we refer to them simply as (i)–(v). Henceforth, we fix $\tau \geq 0$, and $\alpha, \delta \geq 1$, and $\beta \geq 2$, and some non-decreasing function $\theta : \mathbb{N} \rightarrow \mathbb{N}$, for the remainder of the paper, and shall investigate the properties of a graph satisfying (i)–(v).

Let Y be a (ζ, β) -core in G . A vertex $v \in V(G) \setminus Y$ is *dense* to Y if v has at least α neighbours in each part of Y . We observe:

2.6 Let G satisfy (i)–(v), and let Y be a (ζ, β) -core in G . Then there are at most $\alpha\tau 2^{\beta\zeta}$ vertices in G that are dense to Y .

Proof. Let A_1, \dots, A_β be the parts of Y , and let $X_i \subseteq A_i$ with $|X_i| = \alpha$, for $1 \leq i \leq \beta$. The set N of vertices adjacent to all vertices in $X_1 \cup \dots \cup X_\beta$ has chromatic number at most τ by (ii), and includes no stable set of cardinality α , since G does not admit an $(\alpha, \beta + 1)$ -core by (v). Consequently $|N| \leq \alpha\tau$. Since there are only at most $2^{\beta\zeta}$ choices for X_1, \dots, X_β , and every vertex that is dense to Y belongs to the set N corresponding to some choice of X_1, \dots, X_β , it follows that there are at most $\alpha\tau 2^{\beta\zeta}$ vertices that are dense to Y . This proves 2.6. ■

3 Templates

We will use an extension of the template method of Kierstead-Penrice and Kierstead-Zhou, which was used in different (and not easily compatible) ways in those papers. Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers, and let G satisfy (i)–(v). If Y is a (ζ, β) -core in G , we say that a vertex $v \in V(G)$ is η -mixed on Y if

- v is not dense to Y ; and
- v has at least η neighbours in some part of Y .

Thus every vertex in Y is η -mixed on Y . A (ζ, η) -template in G is a pair (Y, H) , where Y is a (ζ, β) -core in G , and H is a set of vertices of G with $Y \subseteq H$ such that every vertex in H is η -mixed on Y . (Note that there may be vertices in $V(G) \setminus H$ that are η -mixed on Y .)

A (ζ, η) -template sequence in G is a sequence (Y_i, H_i) ($1 \leq i \leq n$) of (ζ, η) -templates, such that

- for $1 \leq i < j \leq n$, $H_i \cap H_j = \emptyset$;
- for $1 \leq i < j \leq n$, there is no edge between H_i and Y_j ; and
- for $1 \leq i < j \leq n$, no vertex in H_j is η -mixed on Y_i .

Later, we will denote $H_i \setminus Y_i$ by Z_i (see figure 2).

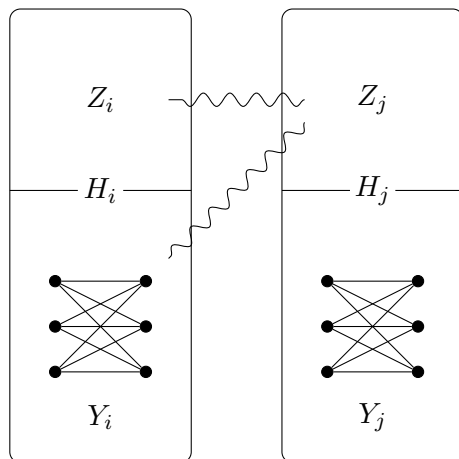


Figure 2: Two terms of a (ζ, η) -template sequence, with $j > i$. Wiggles indicate possible edges.

A (ζ, η) -template array \mathcal{T} in G consists of a (ζ, η) -template sequence (Y_i, H_i) ($1 \leq i \leq n$) together with a set $U(\mathcal{T}) \subseteq V(G)$, such that for every vertex $v \in U(\mathcal{T})$,

- $v \notin H_i$ for $1 \leq i \leq n$;
- v is not η -mixed on Y_i for $1 \leq i \leq n$; and
- v has a neighbour in $H_1 \cup \dots \cup H_n$.

We call (Y_i, H_i) ($1 \leq i \leq n$) the *sequence* of \mathcal{T} , and define $H(\mathcal{T}) = H_1 \cup \dots \cup H_n$ and $V(\mathcal{T}) = H(\mathcal{T}) \cup U(\mathcal{T})$.

3.1 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers, and let G satisfy (i)–(v). Then there is a (ζ, η) -template array \mathcal{T} in G such that $V(G) \setminus V(\mathcal{T})$ has chromatic number at most $\theta(\zeta)$.*

Proof. Let (Y_i, H_i) ($1 \leq i \leq n$) be a (ζ, η) -template sequence with the property that for $1 \leq i \leq n$, H_i is the set of all vertices in G that are η -mixed on Y_i , and subject to this, with n maximum. Let $H = H_1 \cup \dots \cup H_n$, and let U be the set of vertices in $V(G) \setminus H$ with a neighbour in H . Let \mathcal{T} be the (ζ, η) -template array consisting of (Y_i, H_i) ($1 \leq i \leq n$) together with $U(\mathcal{T}) = U$. Let $W = V(G) \setminus V(\mathcal{T})$, and suppose that there is a (ζ, β) -core $Y_{n+1} \subseteq W$. Let H_{n+1} be the set of vertices in G that are η -mixed on Y_{n+1} . Then (Y_{n+1}, H_{n+1}) is a (ζ, η) -template, and no vertex in H_{n+1} belongs to H , since no vertex in H has a neighbour in Y_{n+1} , and every vertex in H_{n+1} has a neighbour in Y_{n+1} . Consequently (Y_i, H_i) ($1 \leq i \leq n+1$) is a (ζ, η) -template sequence, contrary to the maximality of n . Thus there is no (ζ, β) -core in W . From (iv), $\chi(W) \leq \theta(\zeta)$. This proves 3.1. ■

By setting $\phi(x) = x + \theta(\zeta)$ for $x \geq 0$, we deduce from 3.1 that:

3.2 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > c$.*

We will use the following elementary fact many times in the remainder of the paper, and we leave its proof to the reader:

3.3 *Let D be a digraph with maximum outdegree at most d . Then the graph underlying D has chromatic number at most $2d + 1$, and at most $d + 1$ if D is acyclic (that is, has no directed cycle).*

A (ζ, η) -template array \mathcal{T} with sequence (Y_i, H_i) ($1 \leq i \leq n$) is *partially 1-cleaned* if

- for all distinct $i, j \in \{1, \dots, n\}$, no vertex of H_j is dense to Y_i ; and
- for all distinct $i, j \in \{1, \dots, n\}$ and all $v \in U(\mathcal{T})$, if v is dense to Y_i then v has no neighbours in H_j ;

and *1-cleaned* if

- for $1 \leq i \leq n$, no vertex in $V(\mathcal{T})$ is dense to Y_i .

3.4 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a partially 1-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 3.2 (with ϕ replaced by ψ). Let $t = \alpha\tau 2^{\beta\zeta}$, and define $\phi(x) = \psi((2t + 1)x)$ for all $x \in \mathbb{N}$; we claim that ϕ satisfies 3.4. Let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.2, there is a (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > (2t + 1)c$. Let its sequence be (Y_i, H_i) ($1 \leq i \leq n$). Choose a partition U_i ($1 \leq i \leq n$) of $U(\mathcal{T})$, such that

- for $1 \leq i \leq n$, every vertex of U_i has a neighbour in H_i ; and
- for all distinct $i, j \in \{1, \dots, n\}$, if $v \in U_i$ is dense to Y_i and v has a neighbour in H_j then v is dense to Y_j .

(This can be arranged by assigning each vertex $v \in U(\mathcal{T})$ to some set U_i where v has a neighbour in H_i and is not dense to Y_i if possible, and otherwise assigning v to some set U_i where v has a neighbour in H_i .)

Let D be the digraph with vertex set $\{1, \dots, n\}$ in which for $1 \leq i, j \leq n$ with $i \neq j$, if some vertex in $H_j \cup U_j$ is dense to Y_i then j is adjacent from i . By 2.6, D has maximum outdegree at most t , and so by 3.3 the graph underlying D is $(2t + 1)$ -colourable. Consequently there is a partition I_1, \dots, I_{2t+1} of $\{1, \dots, n\}$ such that for $1 \leq s \leq 2t + 1$ and all distinct $i, j \in I_s$, no vertex of $H_j \cup U_j$ is dense to Y_i . Hence the subsequence of (Y_i, H_i) ($1 \leq i \leq n$) consisting of the terms with $i \in I_s$, together with the set $\bigcup_{i \in I_s} U_i$, is a partially 1-cleaned (ζ, η) -template array, \mathcal{T}_s say. But every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_s)$ for some s ; and so there exists $s \in \{1, \dots, 2t + 1\}$ such that

$$\chi(V(\mathcal{T}_s)) \geq \chi(V(\mathcal{T})) / (2t + 1) > c.$$

This proves 3.4. ▀

3.5 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a 1-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 3.4 (with ϕ replaced by ψ). Let $t = \alpha\tau 2^{\beta\zeta}$, and define $\phi(x) = \psi(x + t\tau)$ for all $x \in \mathbb{N}$; we claim that ϕ satisfies 3.5. Let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.4, there is a partially 1-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > c + t\tau$. Let its sequence be (Y_i, H_i) ($1 \leq i \leq n$); and for $1 \leq i \leq n$ let X_i be the set of all vertices in $V(\mathcal{T})$ that are dense to Y_i . Hence $X_i \subseteq U(\mathcal{T})$, and for all distinct $i, j \in \{1, \dots, n\}$, no vertex of X_i has a neighbour in H_j . Let $X = X_1 \cup \dots \cup X_n$. By 2.6 $|X_i| \leq t$ for each i . Hence we may partition X into t sets W_1, \dots, W_t such that $|X_i \cap W_j| \leq 1$ for all i, j with $1 \leq i \leq n$ and $1 \leq j \leq t$. For $v \in W_j$, let $v \in X_i$; then v has a neighbour in Y_i , and has no neighbours in $Y_{i'}$ for $i' \neq i$; and so each set W_j is matching-covered in G . By (iii), $\chi(W_j) \leq \tau$ for each j , and so $\chi(W_1 \cup \dots \cup W_t) \leq t\tau$. This proves that $\chi(X) \leq t\tau$. Let \mathcal{T}' be the (ζ, η) -template array in G with the same sequence as \mathcal{T} and with $U(\mathcal{T}') = U(\mathcal{T} \setminus X)$; then \mathcal{T}' is 1-cleaned, and

$$\chi(V(\mathcal{T}')) \geq \chi(V(\mathcal{T})) - t\tau > c.$$

This proves 3.5. ▀

4 Edges between templates

4.1 *Let $\eta \geq 1$ and $\zeta \geq \max(\eta + \delta, \alpha)$ be integers, and let G satisfy (i)–(v). Let \mathcal{T} be a 1-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). For each $v \in V(\mathcal{T})$, there are at most 2δ values of $i \in \{1, \dots, n\}$ such that v has a neighbour in Y_i .*

Proof. Suppose there exists $I \subseteq \{1, \dots, n\}$ with $|I| = 2\delta + 1$ such that v has a neighbour in Y_i for each $i \in I$, and let i_0 be the maximum element of I . Let $I' = I \setminus \{i_0\}$. It follows that $v \notin H_i$ for all $i \in I'$. Consequently for $i \in I'$, v is not η -mixed on Y_i , and not dense to Y_i (since the template array is 1-cleaned and $v \in V(\mathcal{T})$); and it follows that v has at most $\eta - 1$ neighbours in each part of Y_i . Let $i \in I'$, and let the parts of Y_i be A_1, \dots, A_β . Since v has a neighbour in Y_i , we may assume that v has a neighbour in A_1 . Since v has at most $\eta - 1$ neighbours in A_2 , and $|A_2| = \zeta$, it follows that v has at least δ non-neighbours in A_2 , and so there is a $(1, \delta)$ -broom with handle v in $G[Y_i \cup \{v\}]$. But also, since v has at least δ non-neighbours in A_1 , there is a $(2, \delta)$ -broom with handle v in $G[Y_i \cup \{v\}]$. By selecting the $(1, \delta)$ -broom with handle v in $G[Y_i \cup \{v\}]$ for δ values of $i \in I'$, and selecting the $(2, \delta)$ -broom for the remaining δ values of $i \in I'$, and taking the union of all these brooms, we find that G contains $T(\delta)$, contrary to (i). This proves 4.1. \blacksquare

For the remainder of the paper, let us define $\gamma = (2\delta\tau + 1)(2\delta + 1)$.

4.2 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Let G satisfy (i)–(v), and let \mathcal{T} be a 1-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). For each $v \in V(\mathcal{T})$, there are fewer than γ values of $i \in \{1, \dots, n\}$ such that v has a neighbour in H_i .*

Proof. Suppose then that G , \mathcal{T} , and (Y_i, H_i) ($1 \leq i \leq n$) are as in the theorem, and there are at least γ values of $i \in \{1, \dots, n\}$ such that v has a neighbour in H_i . By 4.1, there are at most $2\delta + 1$ values of i such that v has a neighbour in Y_i . Consequently there exists $I_1 \subseteq \{1, \dots, n\}$ with $|I_1| = 2\delta(2\delta + 1)\tau$ such that for each $i \in I_1$, v has a neighbour in H_i and v has no neighbour in Y_i . For $i \in I_1$, let $u_i \in H_i \setminus Y_i$ be adjacent to v .

Let D be the digraph with vertex set I_1 in which for distinct $i, j \in I_1$, i is adjacent from j in D if u_j has a neighbour in Y_i (and consequently $i < j$). From 4.1, D has maximum outdegree at most 2δ and is acyclic, and so by 3.3 the graph underlying D has chromatic number at most $2\delta + 1$. Hence there exists $I_2 \subseteq I_1$ with $|I_2| = |I_1|/(2\delta + 1) = 2\delta\tau$ such that for all distinct $i, j \in I_2$, u_j has no neighbour in Y_i . Since $\chi(\{u_i : i \in I_2\}) \leq \tau$ by (ii), there exists $I_3 \subseteq I_2$ with $|I_3| = 2\delta$ such that for all $i < j$ with $i, j \in I_3$, u_i and u_j are nonadjacent. For each $i \in I_3$, since u_i is η -mixed on Y_i , and $\eta \geq \delta$, and v has no neighbour in Y_i , it follows that there is a $(1, \delta)$ -broom in $G[\{v, u_i\} \cup Y_i]$ with handle v . Since u_i is η -mixed on Y_i , and has fewer than α neighbours in some part of Y_i , and $\zeta \geq \delta + \alpha$, it follows that there are two distinct parts A_1, A_2 of Y_i such that u_i has a neighbour in A_1 and has at least δ non-neighbours in A_2 . Consequently there is a $(2, \delta)$ -broom in $G[\{v, u_i\} \cup Y_i]$ with handle v . But then, choosing the $(1, \delta)$ -broom for δ values of $i \in I_3$ and choosing the $(2, \delta)$ -broom for the other δ values of $i \in I_3$, and taking their union, we find that G contains $T(\delta)$, contrary to (i). This proves 4.2. \blacksquare

4.3 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Then there exists s with the following property. Let G satisfy (i)–(v), and let \mathcal{T} be a 1-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). For $1 \leq j \leq n$, there are at most s values of $i \in \{1, \dots, n\}$ such that some vertex in H_j has at least δ neighbours in H_i .*

Proof. Let $s_3 = 2\delta\tau$, let $s_2 = (2(\delta + 1)\gamma + 1)s_3$, let $s_1 = s_2 + \gamma$, and let $s = \zeta\beta s_1$. Now let $\eta, \zeta, G, \mathcal{T}$ and (Y_i, H_i) ($1 \leq i \leq n$) be as in the theorem, and suppose that for some $j \in \{1, \dots, n\}$ and some subset $I \subseteq \{1, \dots, n\}$ with $|I| > s$, and every $i \in I$, there exists $u_i \in H_j$ with at least δ neighbours

in H_i . Since each u_i has a neighbour in Y_j , and $|Y_j| = \zeta\beta$, there exists $I_1 \subseteq I$ with $|I_1| = s_1$ and a vertex $y \in Y_j$ adjacent to every u_i ($i \in I_1$). Since by 4.2, y has neighbours in H_i for at most γ values of i , there exists $I_2 \subseteq I_1$ with $|I_2| = |I_1| - \gamma = s_2$ such that y has no neighbours in H_i for $i \in I_2$ (and in particular $j \notin I_2$). For each $i \in I_2$, choose a set $W_i \subseteq H_i$ with $|W_i| = \delta$ such that every vertex in W_i is adjacent to u_i .

Let D be the digraph with vertex set I_2 in which for distinct $i, i' \in I_1$, i' is adjacent from i if some vertex in $u_i \cup W_i$ has a neighbour in $H_{i'}$. (Thus if $u_i = u_{i'}i$ then we have $i \rightarrow i'$ and $i' \rightarrow i$.) By 4.2, D has maximum outdegree at most $(\delta + 1)\gamma$, and so by 3.3 the graph underlying D has chromatic number at most $2(\delta + 1)\gamma + 1$. Hence there exists $I_3 \subseteq I_2$ with $|I_3| = |I_2| / (2(\delta + 1)\gamma + 1) = s_3$ such that for all distinct $i, i' \in I_3$, no vertex in $u_i \cup W_i$ has a neighbour in $H_{i'}$ (and in particular the vertices u_i ($i \in I_3$) are all distinct). By (ii) the set $\{u_i : i \in I_3\}$ has chromatic number at most τ , so there exists $I_4 \subseteq I_3$ with $|I_4| = 2\delta$ such that the vertices u_i ($i \in I_4$) are pairwise nonadjacent.

For each $i \in I_4$, there is a $(1, \delta)$ -broom in $G[\{y, u_i\} \cup W_i]$ with handle y . Moreover, choose $w_i \in W_i$; then since w_i is η -mixed on Y_i , there is a $(2, \delta)$ -broom in $G[\{y, u_i, w_i\} \cup Y_i]$ with handle y . By choosing the $(1, \delta)$ -broom for δ values of i , and the $(2, \delta)$ -broom for the other δ values of i , and taking their union, we find that G contains $T(\delta)$, contrary to (i). This proves 4.3. \blacksquare

For $d \geq 0$, let us say a 1-cleaned (ζ, η) -template array \mathcal{T} with sequence (Y_i, H_i) ($1 \leq i \leq n$) is *partially* $(2, d)$ -cleaned if

- for all $i \in \{1, \dots, n\}$, every vertex of H_i has at most d neighbours in $H(\mathcal{T}) \setminus H_i$

and *2-cleaned* if

- for all distinct $i, j \in \{1, \dots, n\}$, no vertex of H_i has a neighbour in H_j , and
- for $1 \leq i \leq n$, $H_i \setminus Y_i$ is stable.

4.4 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Then there exist $d \geq 0$ and a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$, with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a partially $(2, d)$ -cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > c$.*

Proof. Let s be as in 4.3, and let $d = \gamma(\delta - 1)$. Let ψ satisfy 3.5 (with ϕ replaced by ψ), and define $\phi(x) = \psi((2s + 1)x)$ for $x \geq 0$. Now let $c \geq 0$, and let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. By 3.5 there is a 1-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > (2s + 1)c$. Let its sequence be (Y_i, H_i) ($1 \leq i \leq n$). Let D be the digraph with vertex set $\{1, \dots, n\}$ in which for distinct i, j with $1 \leq i, j \leq n$, i is adjacent from j if some vertex of H_j has at least δ neighbours in H_i . By 4.3, D has maximum outdegree at most s , and so by 3.3 the graph underlying D has chromatic number at most $2s + 1$. Consequently there is a partition I_1, \dots, I_{2s+1} of $\{1, \dots, n\}$ such that for $1 \leq r \leq 2s + 1$, if $i, j \in I_r$ are distinct then each vertex of H_j has at most $\delta - 1$ neighbours in H_i . By 4.2 it follows that for each $j \in I_r$, each vertex of H_j has at most $\gamma(\delta - 1) = d$ neighbours in $\bigcup_{i \in I_r} H_i \setminus H_j$. For each $r \in \{1, \dots, 2s + 1\}$, let \mathcal{T}_r be the (ζ, η) -template array with sequence the subsequence of (Y_i, H_i) ($1 \leq i \leq n$) consisting of the terms with $i \in I_r$, and with $U(\mathcal{T}_r)$ the set of vertices in $U(\mathcal{T})$ with a neighbour in $\bigcup_{i \in I_r} H_i$. Thus each \mathcal{T}_r is partially $(2, d)$ -cleaned; and since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_r)$ for some r , there exists $r \in \{1, \dots, 2s + 1\}$ such that $\chi(V(\mathcal{T}_r)) \geq \chi(V(\mathcal{T})) / (2s + 1) > c$. This proves 4.4. \blacksquare

4.5 Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a 2-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(U(\mathcal{T})) > c$.

Proof. Let d, ψ satisfy 4.4 (with ϕ replaced by ψ). Let $t = (d + 1)\beta\zeta\tau$, and define $\phi(x) = \psi((x + \beta + 1)t)$ for $x \geq 0$; we claim this satisfies the theorem. For let $c \geq 0$, and let G satisfy (i)–(v) with $\chi(G) > \phi(c)$. By 4.4 there is a partially $(2, d)$ -cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(V(\mathcal{T})) > (c + \beta + 1)t$. Let its sequence be (Y_i, H_i) ($1 \leq i \leq n$). For $1 \leq i \leq n$, every vertex of H_i has a neighbour in Y_i , and since $|Y_i| = \beta\zeta$, and by (ii) the set of vertices in G adjacent to any given vertex of Y_i has chromatic number at most τ , it follows that $\chi(H_i) \leq \beta\zeta\tau$. Let J_1 be the subgraph of G with vertex set $H(\mathcal{T})$ and edge set all edges of G with an end in H_i and an end in H_j for distinct i, j ; and let J_2 be the subgraph of G with vertex set $H(\mathcal{T})$ and edge set all edges of G with both ends in H_i for some i . We have just seen that J_2 has chromatic number at most $\beta\zeta\tau$; and since \mathcal{T} is partially $(2, d)$ -cleaned, J_1 has maximum degree at most d and so is $(d + 1)$ -colourable. Hence $G_1 \cup G_2$ has chromatic number at most $(d + 1)\beta\zeta\tau = t$. Consequently there is a partition W_1, \dots, W_t of $H_1 \cup \dots \cup H_n$ into t stable sets. For $1 \leq j \leq t$, let \mathcal{T}_j be the (ζ, η) -template with sequence $(Y_i, (H_i \cap W_j) \cup Y_i)$ ($1 \leq i \leq n$), where $U(\mathcal{T}_j)$ is the set of vertices in $U(\mathcal{T})$ with a neighbour in W_j . Then each \mathcal{T}_j is 2-cleaned, and since every vertex of $V(\mathcal{T})$ belongs to $V(\mathcal{T}_j)$ for some j , there exists $j \in \{1, \dots, t\}$ such that

$$\chi(V(\mathcal{T}_j)) \geq \chi(V(\mathcal{T}))/t > c + \beta + 1.$$

But $\chi(H(\mathcal{T}_j)) \leq \beta + 1$, since \mathcal{T}_j is 2-cleaned; and so $\chi(U(\mathcal{T}_j)) > c$. This proves 4.5. ■

5 Shadowing, and growing daisies

Let \mathcal{T} be a (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). A *shadowing* of \mathcal{T} is a sequence B_1, \dots, B_n of pairwise disjoint subsets of $U(\mathcal{T})$, with union $U(\mathcal{T})$, such that for $1 \leq i \leq n$, every vertex in B_i has a neighbour in H_i . Every template array has a shadowing, and in general it has many. Let us say a shadowing B_1, \dots, B_n has *degree at most s* if for every vertex $v \in V(\mathcal{T})$, there are at most s values of $i \in \{1, \dots, n\}$ such that v has a neighbour in B_i . If $X \subseteq U(\mathcal{T})$, we say the shadowing has *degree at most s relative to X* if for every vertex $v \in V(\mathcal{T})$, there are at most s values of $i \in \{1, \dots, n\}$ such that v has a neighbour in $B_i \cap X$.

Let B_1, \dots, B_n be a shadowing of \mathcal{T} . A *daisy* (with respect to \mathcal{T} and the given shadowing) is an induced subgraph D of G isomorphic to the $\delta + 2$ -vertex star $K_{1, \delta+1}$, such that

- exactly one vertex u of D belongs to $H(\mathcal{T})$; let $u \in H_i$ say;
- u has degree one in D , and the neighbour v of u in D belongs to $U(\mathcal{T})$;
- there exists $j \neq i$ with $1 \leq j \leq n$ such that $V(D) \setminus \{u, v\} \subseteq B_j$.

We call u the *root*, v the *eye*, and the vertices in $V(D) \setminus \{u, v\}$ the *petals* of the daisy. We need the following, proved in [1], but we repeat the proof because it is short:

5.1 Let $d \geq 0$ be an integer, let G be a graph with chromatic number more than d , and let $X \subseteq V(G)$ be stable, such that $\chi(G \setminus X) < \chi(G)$. Then some vertex in X has at least d neighbours in $V(G) \setminus X$.

Proof. Let $\chi(G) = k + 1$, and so $k \geq d$. Let $\phi : V(G) \setminus X \rightarrow \{1, \dots, k\}$ be a k -colouring of $G \setminus X$. For each $x \in X$, if x has at most $d - 1$ neighbours in $V(G) \setminus X$ then we may choose $\phi(x) \in \{1, \dots, k\}$, different from $\phi(v)$ for each neighbour $v \in V(G) \setminus X$ of x ; and this extends ϕ to a k -colouring of G , which is impossible. Thus for some $x \in X$, x has at least d neighbours in $V(G) \setminus X$. This proves 5.1. ■

We deduce:

5.2 *Let \mathcal{T} be a (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). Let $X \subseteq U(\mathcal{T})$ with $\chi(X) > s\beta\delta\zeta\tau^2$. Let B_1, \dots, B_n be a shadowing of degree at most s relative to X . Then there is a daisy in $G[H(\mathcal{T}) \cup X]$.*

Proof. For $1 \leq i \leq n$, let W_i be the set of vertices in X that have a neighbour in H_i and have none in $H_1 \cup \dots \cup H_{i-1}$. Since every vertex in W_i has distance at most two from some vertex in Y_i , and $|Y_i| = \beta\zeta$, it follows that $\chi(W_i) \leq \beta\zeta\tau$ for each i . Consequently there is a partition $X_1, \dots, X_{k\beta\zeta\tau}$ of X such that $W_i \cap X_j$ is stable for $1 \leq i \leq n$ and $1 \leq j \leq k\beta\zeta\tau$. We may assume that $\chi(X_1) \geq \chi(X)/(\beta\zeta\tau) > s\delta\tau$. Choose i minimum such that $\chi(X_1 \setminus (W_1 \cup \dots \cup W_i)) < \chi(X_1)$, and let $H = G[X_1 \setminus (W_1 \cup \dots \cup W_{i-1})]$. Then since $\chi(H \setminus (X_1 \cap W_i)) < \chi(H)$, and $X_1 \cap W_i$ is stable, and $\chi(H) > s\delta\tau$, 5.1 implies that some vertex $v \in W_i$ has a set P of at least $s\delta\tau$ neighbours in $W_{i+1} \cup \dots \cup W_n$. Choose $u \in H_i$ adjacent to v . Since the vertices in P have no neighbours in H_i , they are nonadjacent to u . By hypothesis there are at most s values of $j \in \{1, \dots, n\}$ such that v has a neighbour in $B_j \cap X$, and so there exists $j \in \{1, \dots, n\}$ such that $|P \cap B_j| \geq \delta\tau$. Since $\chi(P) \leq \tau$ (because the vertices in P are all adjacent to v), there is a stable subset $P' \subseteq P \cap B_j$ with $|P'| \geq \delta$. Now $j \neq i$ since no vertices in P have a neighbour in H_i ; and so $G[\{u, v\} \cup P']$ is a daisy. This proves 5.2. ■

Let \mathcal{T} be a (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$); and let B_1, \dots, B_n be a shadowing. A *bunch* of daisies is a set $\{D_j : j \in J\}$ of daisies where $J \subseteq \{1, \dots, n\}$ and for each $j \in J$, D_j has root u_j , eye v_j and set of petals P_j , with the following properties:

- $P_j \subseteq B_j$ for each $j \in J$;
- there exists $i \in \{1, \dots, n\} \setminus J$ such that $u_j \in H_i$ for each $j \in J$;
- for all distinct $j, j' \in J$, $P_j \cup \{v_j\}$ is disjoint from $P_{j'} \cup \{v_{j'}\}$, and there is no edge joining these two sets; and
- for all distinct $j, j' \in J$, u_j has no neighbour in $P_{j'}$.

(Thus, the roots may not all be distinct, and the root of one daisy may be adjacent to the eye of another.)

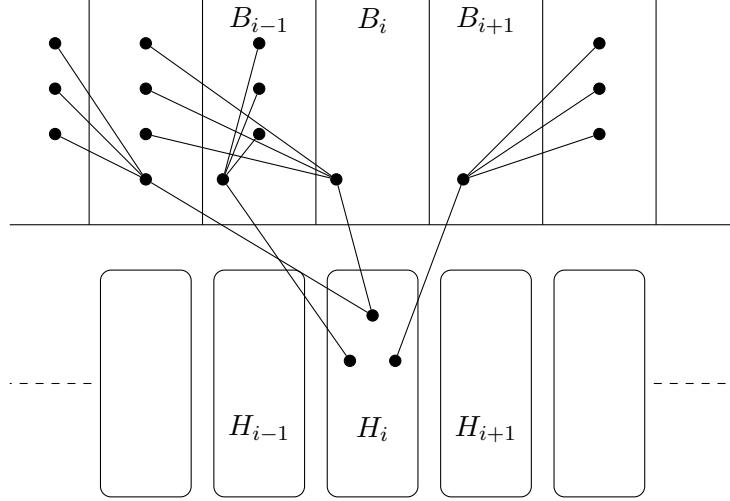


Figure 3: A bunch of daisies.

We deduce:

5.3 Let \mathcal{T} be a (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). Let $t \geq 0$, and let $X \subseteq U(\mathcal{T})$ with

$$\chi(X) > 2st\zeta^2\delta(2(\delta+2)s+3)\beta^2\tau^3.$$

Let B_1, \dots, B_n be a shadowing of degree at most s relative to X . Then there is a bunch $\{D_1, \dots, D_t\}$ of daisies, such that $V(D_j) \subseteq H(\mathcal{T}) \cup X$ for $1 \leq j \leq t$.

Proof. Let $m_1 = t\beta\zeta\tau$, and $m = t\beta\zeta\tau(2(\delta+2)s+3)$. Let D be the digraph with vertex set $\{1, \dots, n\}$ in which for distinct $i, j \in \{1, \dots, n\}$, j is adjacent from i if there is a daisy with root in H_i , eye in X and set of petals in $X \cap B_j$.

(1) There exists $i \in \{1, \dots, n\}$ with outdegree in D at least m .

Suppose not; then by 3.3 the graph underlying D is $2m$ -colourable. Consequently there exists $I \subseteq \{1, \dots, n\}$ with

$$\chi(X') \geq \chi\left(\bigcup_{1 \leq i \leq n} B_i \cap X\right)/(2m) > s\delta\beta\zeta\tau^2$$

where $X' = \bigcup_{i \in I} B_i \cap X$, such that for all distinct $i, j \in I$, there is no daisy with root in H_i , eye in X and set of petals in $X \cap B_j$. In particular, applying 5.2 to the (ζ, η) -template array \mathcal{T}' with sequence $(Y_i, H_i)(i \in I)$ and $U(\mathcal{T}') = X'$, it follows that $\chi(X') \leq s\beta\delta\zeta\tau^2$, a contradiction. This proves (1).

From (1), there exist $i \in \{1, \dots, n\}$ and $J \subseteq \{1, \dots, n\} \setminus \{i\}$, with $|J| = m$, such that for each $j \in J$ there is a daisy D_j with root $u_j \in H_i$, eye $v_j \in X$ and set of petals $P_j \subseteq B_j \cap X$. Now let D' be the digraph with vertex set J in which for all distinct $j, j' \in J$, j is adjacent from j' if some vertex in D_j belongs to or has a neighbour in $P_{j'}$. Then D' has maximum outdegree at most $(\delta+2)s+1$, and so by 3.3 the graph underlying D' is $2(\delta+2)s+3$ -colourable. Hence there exists $J_1 \subseteq J$ with $|J_1| = m/(2(\delta+2)s+3) = m_1$, such that for all distinct j, j' , no vertex in D_j has a neighbour in

P_j . In particular, the vertices $v_j(j \in J_1)$ are all distinct. Since $\{v_j : j \in J_1\}$ has chromatic number at most $\beta\zeta\tau$, there exists $J_2 \subseteq J_1$ with $|J_2| = m_1/(\beta\zeta\tau) = t$ such that the vertices $v_j(j \in J_2)$ are pairwise nonadjacent. But then $\{D_j : j \in J_2\}$ is a bunch of daisies of cardinality t . This proves 5.3. \blacksquare

6 Privatization

Let A, B be disjoint subsets of $V(G)$; we say A covers B if every vertex in B has a neighbour in A . We claim:

6.1 *Let $A, B \subseteq V(G)$ be disjoint, and let A cover B . Let $d \geq 0$ be an integer. Then there exist $A' \subseteq A$ and $B' \subseteq B$ such that*

- A' covers $B \setminus B'$;
- B' is the union of d matching-covered sets;
- every vertex in B' has at most one neighbour in A' ; and
- every vertex in A' has exactly d neighbours in B' .

Proof. We proceed by induction on d . The result is trivial for $d = 0$, because we can set $A' = A$ and $B' = \emptyset$; so we assume that $d > 0$ and the result holds for $d - 1$. Hence there exist $A' \subseteq A$ and $B'' \subseteq B$ such that

- A' covers $B \setminus B''$;
- B'' is the union of $d - 1$ matching-covered sets;
- every vertex in B'' has at most one neighbour in A' ; and
- every vertex in A' has exactly $d - 1$ neighbours in B'' .

Choose A' minimal with this property. Consequently for each $u \in A'$ there exists $v_u \in B \setminus B''$ such that u is the unique neighbour of v_u in A' . Let $X = \{v_u : u \in A'\}$ and let $B' = B'' \cup X$. Then X is matching-covered, and every vertex in A' has a unique neighbour in X and exactly $d - 1$ in B'' , and so exactly d in B' . Consequently B' satisfies the theorem. This proves 6.1. \blacksquare

Let \mathcal{T} be a 2-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). We recall that $H(\mathcal{T})$ denotes $\bigcup_{1 \leq i \leq n} H_i$; and let $Y(\mathcal{T})$ denote $\bigcup_{1 \leq i \leq n} Y_i$ and $Z(\mathcal{T}) = H(\mathcal{T}) \setminus Y(\mathcal{T})$. A *privatization* for \mathcal{T} is a subset $\Pi \subseteq U(\mathcal{T})$ such that

- Π is the union of $\delta\tau$ matching-covered sets;
- every vertex in Π has exactly one neighbour in $Z(\mathcal{T})$ and none in $Y(\mathcal{T})$; and
- every vertex in $Z(\mathcal{T})$ has exactly $\delta\tau$ neighbours in Π .

We deduce:

6.2 Let G satisfy (i)–(v), and let \mathcal{T} be a 2-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$). Then there is a 2-cleaned (ζ, η) -template array \mathcal{T}' , with sequence (Y_i, H'_i) ($1 \leq i \leq n$) and a privatization Π for \mathcal{T}' , such that

- $H'_i \subseteq H_i$ for $1 \leq i \leq n$;
- $U(\mathcal{T}') \subseteq U(\mathcal{T})$; and
- $\chi(U(\mathcal{T}') \setminus \Pi) \geq \chi(U(\mathcal{T})) - \delta\tau^2$.

Proof. We will obtain the desired \mathcal{T}' by removing some elements of each H_i and also removing some elements of $U(\mathcal{T})$. We cannot remove from H_i any element of Y_i , but the only role of the elements of $H_i \setminus Y_i$ is to provide neighbours for the vertices in $U(\mathcal{T})$; so we can happily remove some of them if we also remove from $U(\mathcal{T})$ the vertices which no longer have neighbours in any of the (shrunk) sets H_i .

Let B be the set of vertices in $U(\mathcal{T})$ with no neighbour in $Y(\mathcal{T})$. By 6.1, since $Z(\mathcal{T})$ covers B , there exist $A' \subseteq Z(\mathcal{T})$ and $B' \subseteq B$ such that

- A' covers $B \setminus B'$;
- B' is the union of $\delta\tau$ matching-covered sets;
- every vertex in B' has at most one neighbour in A' ; and
- every vertex in A' has exactly $\delta\tau$ neighbours in B' .

Let Π be the set of vertices in B' that have a neighbour in A' ; for $1 \leq i \leq n$ let $H'_i = (H_i \cap A') \cup Y_i$; and let \mathcal{T}' be the template array with sequence (Y_i, H'_i) ($1 \leq i \leq n$) and $U(\mathcal{T}') = (U(\mathcal{T}) \setminus B') \cup \Pi$. Since $\chi(B') \leq \delta\tau^2$ by (iii), this proves 6.2. ■

The advantage of privatization is the following lemma, used when we have a shadowing of bounded degree.

6.3 Let $\eta \geq 1$ and $\zeta \geq \max(\eta, \alpha)$, and let $q, s \geq 0$. Let G satisfy (i)–(v), and let \mathcal{T} be a 2-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$), that admits a privatization Π . Let B_1, \dots, B_n be a shadowing of degree at most s relative to $U(\mathcal{T}) \setminus \Pi$. Let $1 \leq i \leq n$, and let $\{D_j : j \in J\}$ be a bunch of daisies, each with root in H_i and with $V(D_j) \cap \Pi = \emptyset$, with

$$|J| = 2q\zeta\beta(1 + (q + s)(\delta^2 + 1) + 2\delta + \delta\tau)\tau.$$

Then there exist $u \in H_i \cup B_i$ and $J' \subseteq J$ with $|J'| = q$, such that for each $j \in J'$, u is adjacent to the eye of D_j and nonadjacent to the petals of D_j .

Proof. For each $j \in J$, let u_j, v_j, P_j be the root, eye, and set of petals of D_j respectively. Let D be the digraph with vertex set J_1 in which for all distinct $j, j' \in J_1$, j' is adjacent from j if u_j is adjacent to $v_{j'}$. If some vertex of D has outdegree at least q we are done, so we assume not. Hence by 3.3 the graph underlying D has chromatic number at most $2q$, and so there exists $J_1 \subseteq J$ with $|J_1| = |J|/(2q)$ such that u_j is nonadjacent to $v_{j'}$ for all distinct $j, j' \in J_1$. In particular, the vertices u_j ($j \in J_1$) are all distinct.

Since $|Y_i| = \zeta\eta$, it follows that $u_j \notin Y_i$ for at least $|J_1| - \zeta\eta$ values of $j \in J_1$. Now each such u_j has a neighbour in Y_i , and so there exist $J_2 \subseteq J_1$ with

$$|J_2| = (|J_1| - \zeta\eta)/|Y_i| = ((q+s)(\delta^2+1) + 2\delta + \delta\tau)\tau$$

and a vertex $y \in Y_i$, such that $u_j \notin Y_i$ and y is adjacent to u_j for each $j \in J_2$. Since $\{u_j : j \in J_1\}$ is τ -colourable, there exists $J_3 \subseteq J_2$ with

$$|J_3| = |J_2|/\tau = (q+s)(\delta^2+1) + 2\delta + \delta\tau$$

such that the vertices $u_j (j \in J_3)$ are pairwise nonadjacent. Consequently for all distinct $j, j' \in J_3$ the daisies $D_j, D_{j'}$ are vertex-disjoint and no edge joins them.

The set of vertices in Π with distance two from y has chromatic number at most τ ; fix some partition of this set into τ stable sets. For each $j \in J_3$, $u_j \in Z(\mathcal{T})$ and so u_j has $\delta\tau$ neighbours in Π . We claim that all these neighbours belong to B_i . For $x \in \Pi$ be adjacent to u_j , and let $x \in B_k$ say. Then x has a neighbour in H_k , from the definition of B_k ; but x has a unique neighbour in $H(\mathcal{T})$, since $x \in \Pi$, and this neighbour in u_j , and so $u_j \in H_k$. Since $u_j \in H_i$ it follows that $k = i$. Thus u_j has $\delta\tau$ neighbours in Π , and they all belong to B_i .

The set of vertices in Π with distance two from y has chromatic number at most τ ; fix some partition of this set into τ stable sets. Consequently, for each $j \in J_3$, there are δ neighbours of u_j that belong to the same stable set of the partition. Since $|J_3| \geq \delta\tau$, there exists $J_4 \subseteq J_3$ with $|J_4| = \delta$ and a stable subset Π' of $\Pi \cap B_i$, such that for each $j \in J_4$, u_j has at least δ neighbours in Π' . For each $j \in J_4$, choose $\Pi_j \subseteq \Pi' \cap B_i$ with $|\Pi_j| = \delta$, such that every vertex in Π_j is adjacent to u_j . Thus $G[\{y, u_j\} \cup \Pi_j]$ is a $(1, \delta)$ -broom with handle y , for each $j \in J_4$; and there are no edges between these brooms not incident with y .

Let $\{y\} \cup \bigcup_{j \in J_4} \Pi_j = Q$ say. If some vertex in Q is adjacent to q of the vertices $v_j (j \in J_3)$ we are done, so we assume not. Also by hypothesis each vertex in Q has a neighbour in P_j for at most s values of $j \in \{1, \dots, n\}$. Since $|Q| = \delta^2 + 1$, there are at most $(q+s)(\delta^2+1)$ values of $j \in J_3 \setminus J_4$ such that some vertex in Q has a neighbour in $P_j \cup \{v_j\}$. Since $|J_3| - |J_4| - (q+s)(\delta^2+1) \geq \delta$, there exists $J_5 \subseteq J_3 \setminus J_4$ with $|J_5| = \delta$ such that for each $j \in J_5$, no vertex in Q is adjacent to v_j or has a neighbour in $B_j \setminus \Pi$. It follows that $G[\{y, u_j, v_j\} \cup P_j]$ is a $(2, \delta)$ -broom with handle y for each $j \in J_5$. By taking the union of the $(1, \delta)$ -brooms $G[\{y, u_j\} \cup \Pi_j]$ for each $j \in J_4$ and the $(2, \delta)$ -brooms $G[\{y, u_j, v_j\} \cup P_j]$ for each $j \in J_5$, we find that G contains $T(\delta)$, a contradiction. This proves 6.3 \blacksquare

7 Edges between $H(\mathcal{T})$ and $U(\mathcal{T})$.

Our next goal is to bound the number of neighbours each vertex of $U(\mathcal{T})$ has in $H(\mathcal{T})$.

7.1 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha) + \delta$ be integers; then there exists $\ell \geq 0$ with the following property. Let G satisfy (i)–(v), and let \mathcal{T} be a 2-cleaned (ζ, η) -template array in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$), that admits a privatization. Let X be the set of vertices in $U(\mathcal{T})$ that have at least $(\beta+1)\gamma\delta$ neighbours in $H(\mathcal{T})$. Then $\chi(X) \leq \ell$.*

Proof. Let

$$\begin{aligned}
s &= 2(2\gamma + 1)\delta\tau + \gamma, \\
q &= 2\delta(2\gamma(\delta + 1) + 1) + \gamma, \\
m &= 2q\zeta\beta(1 + (q + s)(\delta^2 + 1) + 2\delta + \delta\tau)\tau, \text{ and} \\
\ell &= 2sm\zeta^2\delta(2(\delta + 2)s + 3)\beta^2\tau^3.
\end{aligned}$$

We claim that ℓ satisfies the theorem. For let \mathcal{T} , (Y_i, H_i) ($1 \leq i \leq n$) and X be as in the theorem. For each $v \in X$, since v has at least $(\beta + 1)\gamma\delta$ neighbours in $H(\mathcal{T})$, and there are only at most γ values of i such that v has a neighbour in H_i , it follows that there exists $i \in \{1, \dots, n\}$ such that v has at least $(\beta + 1)\delta$ neighbours in H_i . Choose a shadowing B_1, \dots, B_n such that for $1 \leq i \leq n$, every vertex in $B_i \cap X$ has at least $(\beta + 1)\delta$ neighbours in H_i .

(1) *The shadowing B_1, \dots, B_n has degree less than s .*

Suppose not, and choose $y \in V(\mathcal{T})$ and $J \subseteq \{1, \dots, n\}$ with $|J| = s$ such that for each $j \in J$ there exists $u_j \in X \cap B_j$ adjacent to y . Since y has a neighbour in H_j for at most γ values of j , there exists $J_1 \subseteq J$ with $|J_1| = |J| - \gamma$ such that y has no neighbour in H_j for each $j \in J_1$. Since the subgraph induced on $\{u_j : j \in J_1\}$ is τ -colourable, there exists $J_2 \subseteq J_1$ with $|J_2| = |J_1|/\tau = 2(2\gamma + 1)\delta$ such that the vertices $u_j (j \in J_2)$ are pairwise nonadjacent. Let D be the digraph with vertex set J_2 in which for all distinct $j, j' \in J_2$, j' is adjacent from j if u_j has a neighbour in $H_{j'}$. Since D has maximum outdegree at most γ , by 3.3 the graph underlying D is $(2\gamma + 1)$ -colourable, and so there exists $J_3 \subseteq J_2$ with $|J_3| = |J_2|/(2\gamma + 1) = 2\delta$ such that for all distinct $j, j' \in J_3$, u_j has no neighbour in $H_{j'}$. For each $j \in J_3$, since $u_j \in X$, u_j has at least $\delta(\beta + 1)$ neighbours in H_j ; and since H_j is $(\beta + 1)$ -colourable, there is a stable set P_j of δ such neighbours. Thus $G[\{y, u_j\} \cup P_j]$ is a $(1, \delta)$ -broom with handle y . Let v_j be a neighbour of u_j in H_j , choosing $v_j \in Y_j$ if possible. If $v_j \in Y_j$, let A be a part of Y_j not containing v_j ; then since $u_j \notin H_j$, u_j has at most $\eta - 1$ neighbours in A , and since $\zeta \geq \eta - 1 + \delta$, there is a set $Q_j \subseteq A$ of δ vertices all nonadjacent to u_j . If $v_j \notin Y_j$, then u_j has no neighbour in Y_j , and since v_j is η -mixed on Y_j and $h \geq \delta$, it follows that v_j has a set Q_j of δ neighbours in some part of Y_j . In either case $G[\{y, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$ -broom with handle y . By choosing the $(1, \delta)$ -broom for δ values of $j \in J_3$, and the $(2, \delta)$ -broom for the other δ values of $j \in J_3$, and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (1).

(2) *There is no bunch of daisies $\{D_j : j \in J\}$ with $|J| = m$ such that $V(D_j) \subseteq X \cup H(\mathcal{T})$ for each $j \in J$.*

Suppose such a bunch $\{D_j : j \in J\}$ exists. Let $i \in \{1, \dots, n\}$ such that the root of D_j belongs to H_i for each $j \in J$. Let Π be a privatization. (Thus $\Pi \cap X = \emptyset$, since $(\beta + 1)\gamma\delta \geq 2$.) By 6.3 applied to the (ζ, η) -template array \mathcal{T} with sequence $(Y_j, H_j) (1 \leq j \leq n)$ and $U(\mathcal{T}) = X \cup \Pi$, there exist $y \in H_i \cup B_i$ and $J_1 \subseteq J$ with $|J_1| = q$, such that for each $j \in J_1$, y is adjacent to the eye (u_j say) of D_j and has no neighbour in the set of petals (P_j say) of D_j . Thus $G[\{y, u_j\} \cup P_j]$ is a $(1, \delta)$ -broom with handle y , for each $j \in J_1$. Since there are at most γ values of $j \in J_1$ such that y has a neighbour in H_j , there exists $J_2 \subseteq J_1$ with $|J_2| = |J_1| - \gamma$ such that y has no neighbour in H_j for $j \in J_2$.

Let D be the digraph with vertex set J_2 in which for all distinct $j, j' \in J_2$, j' is adjacent from

j if some vertex in $\{u_j\} \cup P_j$ has a neighbour in $H_{j'}$. Since D has maximum outdegree at most $\gamma(\delta + 1)$, by 3.3 the graph underlying D is $(2\gamma(\delta + 1) + 1)$ -colourable, and so there exists $J_3 \subseteq J_2$ with $|J_3| = |J_2| / (2\gamma(\delta + 1) + 1) = 2\delta$ such that for all distinct $j, j' \in J_3$, no vertex in $\{u_j\} \cup P_j$ has a neighbour in $H_{j'}$.

For each $j \in J_3$, choose a neighbour v_j of u_j , such that

- if u_j has a neighbour in Y_j then $v_j \in Y_j$;
- if u_j has no neighbour in Y_j and has a neighbour in H_j then $v_j \in H_j$;
- if u_j has no neighbour in H_j then $v_j \in P_j$.

We claim that in each case, y is nonadjacent to v_j , and there is a stable set $Q_j \subseteq H_j$ of neighbours of v_j , all nonadjacent to u_j , with $|Q_j| = \delta$. To see this, if $v_j \in H_j$ the proof is as in the proof of (1), so we assume that $v_j \in P_j$, and therefore u_j has no neighbours in H_j . Since $P_j \subseteq X$, v_j has at least $\delta(\beta + 1)$ neighbours in H_j , and since H_j is $(\beta + 1)$ -colourable, the claim follows. In particular, $G[\{y, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$ -broom for each $j \in J_3$. By choosing the $(1, \delta)$ -broom for δ values of $j \in J_3$, and the $(2, \delta)$ -broom for the other δ values of $j \in J_3$, and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (2).

From (1), (2) and 5.3, it follows that $\chi(X) \leq \ell$. This proves 7.1. ■

The bound $(\beta + 1)\gamma\delta$ will be very useful in the remainder of the proof, and for convenience let us define $\varepsilon = (\beta + 1)\gamma\delta$, for the remainder of the paper. Let us say a 2-cleaned (ζ, η) -template array \mathcal{T} in G is 3-cleaned if every vertex in $U(\mathcal{T})$ has fewer than ε neighbours in $H(\mathcal{T})$. We deduce:

7.2 *Let $\eta \geq \delta$ and $\zeta \geq \max(\eta, \alpha + \delta)$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ with the following property. For all $c \geq 0$, if G satisfies (i)–(v) and $\chi(G) > \phi(c)$ then there is a 3-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(U(\mathcal{T})) > c$.*

Proof. Let ψ satisfy 4.5 (with ϕ replaced by ψ), and let ℓ be as in 7.1. For all $c \geq 0$ define $\phi(c) = \psi(c + \delta\tau^2 + \ell)$; we claim this satisfies the theorem. For let $c \geq 0$, and let G satisfy (i)–(v) with $\chi(G) > \phi(c)$. By 4.5, there is a 2-cleaned (ζ, η) -template array \mathcal{T} in G such that $\chi(U(\mathcal{T})) > c + \delta\tau^2 + \ell$. By 6.2, there is a 2-cleaned (ζ, η) -template array \mathcal{T}_1 in G that admits a privatization, such that $\chi(U(\mathcal{T}_1)) > c + \ell$. Let X be the set of vertices in $U(\mathcal{T}_1)$ that have at least ε neighbours in $H(\mathcal{T}_1)$. Let ℓ be as in 7.1; then by 7.1, $\chi(X) \leq \ell$. Let \mathcal{T}' be the (ζ, η) -template array with the same sequence as \mathcal{T}_1 and with $U(\mathcal{T}') = U(\mathcal{T}_1) \setminus X$. It follows that \mathcal{T}' is 3-cleaned, and $\chi(U(\mathcal{T}')) \geq \chi(U(\mathcal{T}_1)) - \ell > c$. This proves 7.2. ■

8 Edges within a shadowing

Now we have come to the final stage of the proof: we investigate the edges between different sets of a shadowing. First we prove that there is some template array such that every shadowing has bounded degree; and privatize it; and then we will show that for a privatized template array, if every shadowing has bounded degree then the graph has bounded chromatic number.

So far, our technique in this paper has been to start with a template array, and make nicer and nicer ones at the cost of reducing the chromatic number of $U(\mathcal{T})$. This has more-or-less reached its limit, with 7.2, so now we need to do something different. To prove the next result, we will start with a 3-cleaned template array \mathcal{T} , and apply 7.2 to $G[U(\mathcal{T})]$ to get a second one, with vertex set a subset of $U(\mathcal{T})$; and repeat, generating a nested sequence of template arrays.

8.1 *Let $\eta \geq \alpha + 2(\delta + 1)^3(\varepsilon + 1)^2$ and $\zeta \geq \eta + \delta$ be integers. Then there is a non-decreasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$, and an integer s , with the following property. Let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. Then there is a 3-cleaned (ζ, η) -template array \mathcal{T} in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$), such that $\chi(U(\mathcal{T})) > c$, and such that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i .*

Proof.

Let

$$\begin{aligned}
s_3 &= (\delta(\delta + 1) + 1)\varepsilon + \delta \\
s_2 &= ((2\delta\varepsilon + 1)\delta\tau + \varepsilon)s_3 \\
s_1 &= (2\varepsilon + 1)\tau s_2 \\
s &= s_1 + \varepsilon \\
t_4 &= 2\delta \\
t_3 &= 2\delta t_4 \\
t_2 &= (\alpha\tau 2^{\beta\zeta} + 1)t_3 \\
t_1 &= t_2^{s_2}, \text{ and} \\
t &= 1 + 2^{s_2}\delta\tau + t_1.
\end{aligned}$$

Let ψ satisfy 7.2 (with ϕ replaced by ψ). Define $\phi^0(x) = x$ for $x \geq 0$, and inductively for $i \geq 1$, let $\phi^i(x) = \psi(\phi^{i-1}(x))$ for $x \geq 0$. Let $\phi = \phi^t$; we claim that ϕ, s satisfy the theorem.

Let G satisfy (i)–(v), with $\chi(G) > \phi(c)$. Define $U^0 = V(G)$; thus $\chi(U_0) > \phi^t(c)$. For $1 \leq j \leq t$ define \mathcal{T}^j and U^j inductively as follows. Let j satisfy $1 \leq j \leq t$, and suppose we have defined U^{j-1} , and $\chi(U^{j-1}) > \phi^{t-j+1}(c)$. By 7.2 applied to $G[U^{j-1}]$, there is a 3-cleaned (ζ, η) -template array \mathcal{T}^j in $G[U^{j-1}]$ such that $\chi(U(\mathcal{T}^j)) > \phi^{t-i}(c)$. Let $U^j = U(\mathcal{T}^j)$. This completes the inductive definition. Let \mathcal{T} be the (ζ, η) -template array with the same sequence as \mathcal{T}^1 , and with $U(\mathcal{T}) = U^t$; we will show that \mathcal{T} satisfies the theorem. Certainly it is 3-cleaned, and $\chi(U(\mathcal{T})) > c$.

We remark that for all $j < j' \leq t$, every vertex in $H(\mathcal{T}^{j'})$ has fewer than ε neighbours in $H(\mathcal{T}^j)$ (since $H(\mathcal{T}^{j'}) \subseteq U(\mathcal{T}^j)$ and \mathcal{T}^j is 3-cleaned), and for all j every vertex in U^t has fewer than ε neighbours in $H(\mathcal{T}^j)$ (for the same reason); but when $j' > j$ we know nothing about the number of neighbours a vertex in $H(\mathcal{T}^j)$ has in $H(\mathcal{T}^{j'})$.

For $1 \leq j \leq t$, let the sequence of \mathcal{T}^j be (Y_i^j, H_i^j) ($1 \leq i \leq n_j$). We assume for a contradiction that there exist $y \in U^t$ and a subset $I \subseteq \{1, \dots, n_1\}$ with $|I| = s$, such that for each $i \in I$, there exists $u_i \in U^t$ adjacent to y and with a neighbour in H_i^1 . Since there are at most ε values of i such that y has a neighbour in H_i^1 , there exists $I_1 \subseteq I$ with $|I_1| = |I| - \varepsilon = s_1$ such that for each $i \in I_1$, y has no neighbour in H_i^1 .

(1) *There exist $I_2 \subseteq I_1$ with $|I_2| = s_2$, and for each $i \in I_2$, a $(2, \delta)$ -broom B_i of $G[\{y, u_i\} \cup H_i^1]$*

with handle y , such that every edge joining two of these brooms is incident with y .

Let D be the digraph with vertex set I_1 in which all for all distinct $i, i' \in I_1$, i' is adjacent from i if u_i has a neighbour in $H_{i'}^1$. Since D has maximum outdegree at most ε , by 3.3 the graph underlying D has chromatic number at most $2\varepsilon + 1$. Hence there exists $I'_1 \subseteq I_1$ with $|I'_1| = s_1/(2\varepsilon + 1)$, such that for all all distinct $i, i' \in I'_1$, u_i has no neighbour in $H_{i'}^1$. Since the set $\{u_i : i \in I'_1\}$ has chromatic number at most τ (because each u_i is adjacent to y), there exists $I_2 \subseteq I'_1$ with $|I_2| = |I'_1|/\tau$ such that the vertices $u_i (i \in I_2)$ are pairwise nonadjacent. For each $i \in I_2$, choose $w_i \in H_i^1$ adjacent to u_i , with $w_i \in Y_i^1$ if possible. If $w_i \in Y_i^1$, let A be a part of Y_i^1 not containing w_i ; then since u_i has fewer than η neighbours in A , and w_i is adjacent to every vertex in A , there exists $R_i \subseteq A$ of cardinality δ all nonadjacent to u_i . If $w_i \notin Y_i^1$, then u_i has no neighbour in Y_i ; since $w_i \in H_i^1$, w_i has at least $\eta \geq \delta$ neighbours in some part of Y_i^1 ; and so there exists a stable set $R_i \subseteq Y_i^1$ of cardinality δ , all adjacent to w_i and not to u_i . In either case, $G[\{y, u_i, w_i\} \cup R_i]$ is a $(2, \delta)$ -broom with handle y . This proves (1).

Let J_1 be the set of all $j \in \{2, \dots, t\}$ such that some vertex in $H(\mathcal{T}^j)$ is adjacent to at least s_3 of the vertices $u_i (i \in I_2)$.

(2) If $j \in \{2, \dots, t\} \setminus J_1$, there is a subset $X^j \subseteq H(\mathcal{T}^j)$ and a subset $I^j \subseteq I_2$, such that

- $|X^j| = |I^j| = (2\delta\varepsilon + 1)\delta\tau$;
- y has no neighbour in X^j ;
- every vertex in X^j has a unique neighbour in $\{u_i : i \in I^j\}$; and
- every vertex in $\{u_i : i \in I^j\}$ has a unique neighbour in X^j .

Since for all $i \in I_2$, u_i has a neighbour in $H(\mathcal{T}^j)$, there exists $X \subseteq H(\mathcal{T}^j)$, minimal such that each $u_i (i \in I_2)$ has a neighbour in X . Since every vertex in X is adjacent to fewer than s_3 vertices in $\{u_i : i \in I_2\}$, it follows that $|X| \geq s_2/s_3 = (2\delta\varepsilon + 1)\delta\tau + \varepsilon$. From the minimality of X , for each $x \in X$ there exists $i(x) \in I_2$ such that x is the unique neighbour of $u_{i(x)}$ in X . Since y has at most ε neighbours in $H(\mathcal{T}^j)$, there exists $X^j \subseteq X$ with $|X^j| = (2\delta\varepsilon + 1)\delta\tau$ such that y has no neighbours in X^j . Let $I^j = \{i(x) : x \in X^j\}$; then this proves (2).

(3) $|\{2, \dots, t\} \setminus J_1| \leq 2^{s_2}\delta\tau$.

Suppose not. For each $j \in \{2, \dots, t\} \setminus J_1$, there are at most 2^{s_2} possibilities for the set I^j , and so there exists $J' \subseteq \{2, \dots, t\} \setminus J_1$ with $|J'| = \delta\tau$, and a subset $I_3 \subseteq I_2$ (necessarily with $|I_3| = (2\delta\varepsilon + 1)\delta\tau$), such that $I^j = I_3$ for each $j \in J'$. Let $X = \bigcup_{j \in J'} X^j$; then $\chi(X) \leq \tau$. Take a partition Z_1, \dots, Z_τ of X into stable sets. For each $i \in I_3$, since u_i has $\delta\tau$ neighbours in X , there exists $r \in \{1, \dots, \tau\}$ such that u_i has at least δ neighbours in Z_r . Since there are only τ possibilities for r , there exist $I_4 \subseteq I_3$ with $|I_4| = |I_3|/\tau = (2\delta\varepsilon + 1)\delta$, and $r \in \{1, \dots, \tau\}$, such that for each $i \in I_4$, u_i has at least δ neighbours in Z_r ; let N_i be a set of δ such neighbours. Since for each $j \in J'$, every vertex in X_j has a unique neighbour in $\{u_i : i \in I_2\}$, the sets $N_i (i \in I_4)$ are pairwise disjoint and their union is stable. Let D be the digraph with vertex set I_4 , in which for all distinct $i, i' \in I_4$, i' is adjacent from i if some vertex in N_i has a neighbour in $V(B_{i'})$ (as defined in (1)). Then D has maximum outdegree

$\delta\varepsilon$, since $|N_i| = \delta$ and each vertex in N_i has at most ε neighbours in H^1 (and none in $V(B_{i'}) \setminus H^1$). By 3.3 the graph underlying D has chromatic number at most $2\delta\varepsilon + 1$, and so there exists $I_5 \subseteq I_4$ with

$$|I_5| = |I_4|/(2\delta\varepsilon + 1) = \delta,$$

such that for all distinct $i, i' \in I_5$, no vertex in N_i has a neighbour in $V(B_{i'})$. It follows that G contains $T(\delta)$, a contradiction. This proves (3).

From (3), it follows that $|J_1| \geq t - 1 - 2^{s_2}\delta\tau = t_1$. For each $j \in J_1$, choose $v^j \in H(\mathcal{T}^j)$ adjacent to at least s_3 of the vertices $u_i (i \in I_2)$. Consequently there exists $J_2 \subseteq J_1$ with $|J_2| = t_2 \leq |J_1|2^{-s_2}$ and a subset $I_3 \subseteq I_2$ with $|I_3| = s_3$, such that every vertex $v^j (j \in J_2)$ is adjacent to every vertex $u_i (i \in I_3)$. For each $j \in J_2$, there exists $i \in \{1, \dots, n_j\}$ such that $v^j \in H_i^j$; let us write $H^j = H_i^j$ and $Y^j = Y_i^j$, since we will have no need for the other terms of the sequence of \mathcal{T}^j .

(4) *There exists $J_3 \subseteq J_2$ with $|J_3| = |J_2|/(\alpha\tau 2^{\beta\zeta} + 1) = t_3$ such that for all distinct $j, j' \in J_2$, v_j is not dense to $Y^{j'}$.*

Let D be the digraph with vertex set J_1 in which for all distinct $j, j' \in J_1$, j' is adjacent from j if v_j is dense to $Y^{j'}$. Since v_j is not dense to $Y^{j'}$ if $j > j'$, it follows that D is acyclic, and has maximum indegree at most $\alpha\tau 2^{\beta\zeta}$, by 2.6, and so by 3.3 the graph underlying D has chromatic number at most $\alpha\tau 2^{\beta\zeta} + 1$. This proves (4).

(5) *For each $j_0 \in J_3$ there are fewer than δ values of $j \in J_3 \setminus \{j_0\}$ such that v_{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j .*

For suppose that there exists $J_4 \subseteq J_3 \setminus \{j\}$ with $|J_4| = \delta$ such that for each $j \in J_4$, v_{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j . For each $j \in J_4$, since there is a part of Y^j in which v_{j_0} has fewer than α neighbours (because v_{j_0} is not dense to Y^j), it follows that there are distinct parts A, A' of Y^j , such that v_{j_0} has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in A and at least $\zeta - \alpha + 1 \geq \delta(\delta + 1)\varepsilon + s_3\varepsilon$ non-neighbours in A' . Since at most $s_3\varepsilon$ vertices of Y^j have a neighbour in $\{u_i : i \in I_3\}$, there is a subset $P^j \subseteq A$ with cardinality $\delta(\delta + 1)\varepsilon$, such that all vertices in P^j are adjacent to v_{j_0} and have no neighbours in $\{u_i : i \in I_3\}$; and there is a subset $Q^j \subseteq A'$ with cardinality $\delta(\delta + 1)\varepsilon$, such that all vertices in Q^j are nonadjacent to v_{j_0} and have no neighbours in $\{u_i : i \in I_3\}$.

For each $j \in J_4$, we choose a $(1, \delta)$ -broom C^j of $G[Y^j \cup \{v_{j_0}\}]$ with handle v_{j_0} , inductively as follows. Let $j \in J_4$, and assume that $C^{j'}$ is defined for all $j' \in J_4$ with $j' > j$. Let S be the union of all the sets $V(C^{j'}) \setminus \{v_{j_0}\}$ for $j' \in J_4$ with $j' > j$. Then $|S| \leq (\delta - 1)(\delta + 1)$, and since each vertex in S has at most ε neighbours in H^j , it follows that at most $(\delta - 1)(\delta + 1)\varepsilon$ vertices in H^j have a neighbour in S . Since

$$|P^j|, |Q^j| = \delta(\delta + 1)\varepsilon > (\delta - 1)(\delta + 1)\varepsilon + \delta$$

there exist a vertex in P^j , and a set of δ vertices in Q^j , each with no neighbours in S . Hence there is a $(1, \delta)$ -broom C^j of $G[Y^j \cup \{v_{j_0}\}]$ containing no neighbours of S different from v_{j_0} . This completes the inductive definition of C^j for $j \in J_4$.

Now let S be the union of the sets $V(C^j)$ for $j \in J_4$. Then $|S| = \delta(\delta + 1) + 1$, and so there are at most $(\delta(\delta + 1) + 1)\varepsilon$ vertices in H^1 with neighbours in S . Since $s_3 \geq (\delta(\delta + 1) + 1)\varepsilon + \delta$, there are at

least δ values of $i \in I_3$ such that the edge $u_i v_{j_0}$ is the only edge between $V(B_i)$ and $V(C^j)$ for each $j \in J_4$; and so G contains $T(\delta)$ (with handle v_{j_0}), a contradiction. This proves (5).

(6) *There exists $J_4 \subseteq J_3$ with $|J_4| = t_4$ such that for all distinct $j, j' \in J_4$, v_j has fewer than $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in each part of Y^j .*

Let D be the digraph with vertex set J_3 in which for all distinct $j, j' \in J_3$, j' is adjacent from j if v_j has at least $\delta(\delta + 1)\varepsilon + s_3\varepsilon$ neighbours in some part of Y^j . By (5), D has maximum outdegree at most $\delta - 1$, and so by 3.3 the graph underlying D has chromatic number at most 2δ , and the claim follows. This proves (6).

Fix $i \in I_3$; and partition J_4 into two sets J'_4, J''_4 , both of cardinality δ . For each $j \in J'_4$, we define a $(2, \delta)$ -broom C^j of $G[Y^j \cup \{v_j, u_i\}]$ with handle u_i , and for each $j \in J''_4$, we define a $(1, \delta)$ -broom C^j of $G[Y^j \cup \{v_j, u_i\}]$ with handle u_i , inductively as follows. Let $j \in J_4$, and assume that $C^{j'}$ is defined for all $j' \in J_4$ with $j' > j$. Let S be the union of $\{u_i\}$ and all the sets $V(C^{j'}) \setminus \{v^{j'}\}$ for $j' \in J_4$ with $j' > j$. Then $|S| \leq (\delta + 1)(t_4 - 1)$. There are distinct parts A, A' of Y^j such that v_j has at least η neighbours in A and at most $\alpha - 1$ neighbours in A' . But at most $\varepsilon(\delta + 1)(t_4 - 1)$ vertices of A have a neighbour in S , and at most $(t_4 - 1)(\delta(\delta + 1)\varepsilon + s_3\varepsilon)$ vertices in A have a neighbour in $\{v_{j'} : j' \in J_4 \setminus \{j\}\}$, by (6) and the definition of J_4 , and the same for A' . Since

$$\eta, \zeta - \alpha + 1 \geq (\delta + 1)(t_4 - 1)\varepsilon + (t_4 - 1)(\delta(\delta + 1)\varepsilon + s_3\varepsilon) + \delta,$$

there exist a set of δ neighbours of v_j in A , and a set of δ non-neighbours of v_j in A' , all with no neighbours in $S \cup \{v_{j'} : j' \in J_4 \setminus \{j\}\}$. Consequently the desired broom C_j can be chosen as specified. This completes the inductive definition. But by taking the union of all the $C^j (j \in J_4)$, we see that G contains $T(\delta)$, a contradiction. This proves 8.1. ■

8.2 *Let $\zeta, \eta, s \geq 0$. Let \mathcal{T} be a 2-cleaned (ζ, η) -template array in G , with sequence $(Y_i, H_i) (1 \leq i \leq n)$, that admits a privatization Π . Suppose that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i . Let $q = 2\delta + s$ and*

$$r = (4(\delta + 1)s + 1)q\zeta\beta\tau(1 + \tau((q + s)\delta^2 + (2s(\delta + 1) + 1)\delta\tau)).$$

Then

$$\chi(U(\mathcal{T}) \setminus \Pi) \leq 3rs\beta\delta\zeta\tau^2.$$

Proof. It follows that every shadowing has degree less than s . For $1 \leq i \leq n$, let B_i be the set of vertices in $U(\mathcal{T})$ with a neighbour in H_i and with no neighbour in $H_1 \cup \dots \cup H_{i-1}$. It follows that (B_1, \dots, B_n) is a shadowing, and hence has degree less than s .

For distinct $u, v \in U(\mathcal{T})$, we say that v is *later* than u if $u \in B_i$ and $v \in B_j$ where $j > i$. For $i, a, b, c \in \{1, \dots, n\}$, we say that i is *strong* to (a, b, c) if

- $i < \min(a, b, c)$ and $a < c$;
- there exist $u \in B_i \setminus \Pi$ and $v \in B_a \setminus \Pi$, adjacent;

- there exist δ vertices in $B_b \setminus \Pi$, pairwise nonadjacent, and all adjacent to u (and possibly also adjacent to v); and
- there exist δ vertices in $B_c \setminus \Pi$, pairwise nonadjacent, and all adjacent to v and not to u .

(1) For $1 \leq i \leq n$, there do not exist r triples $(a_1, b_1, c_1), \dots, (a_r, b_r, c_r)$, such that i is strong to them all, and a_j, b_j, c_j are different from $a_{j'}, b_{j'}, c_{j'}$ for $1 \leq j < j' \leq r$.

Suppose that r such triples exist. For $1 \leq j \leq r$, choose $u_j \in B_i \setminus \Pi$ and $v_j \in B_{a_j} \setminus \Pi$ adjacent to u_j , and a stable set P_j of δ vertices in $B_{b_j} \setminus \Pi$, all adjacent to u_j , and a stable set Q_j of δ vertices in $B_{c_j} \setminus \Pi$, all adjacent to v_j and not to u_j . Let D be the digraph with vertex set $\{1, \dots, r\}$ in which for all distinct $j, j' \in \{1, \dots, r\}$, j' is adjacent from j if some vertex $u \in \{u_j, v_j\} \cup P_j \cup Q_j$ is adjacent in G to a vertex $v \in \{v_{j'}\} \cup P_{j'} \cup Q_{j'}$ and u is earlier than v . Then D has maximum outdegree less than $2(\delta + 1)s$, and so by 3.3 there exists $J \subseteq \{1, \dots, r\}$ with

$$|J| \geq r / (4(\delta + 1)s) = q\zeta\beta(1 + \tau((q + s)\delta^2 + (2s(\delta + 1) + 1)\delta\tau))$$

such that for all distinct $j, j' \in J$, there are no edges between $\{u_j, v_j\} \cup P_j \cup Q_j$ and $\{v_{j'}\} \cup P_{j'} \cup Q_{j'}$. In particular, the vertices $u_j (j \in J)$ are all distinct. Since the set $\{u_j : j \in J\}$ has chromatic number at most τ , there exists $J_1 \subseteq J$ with $|J_1| = |J|/\tau$ such that the vertices $u_j (j \in J_1)$ are pairwise nonadjacent. For each $j \in J_1$ choose $w_j \in H_j$ adjacent to u_j ; then w_j has no neighbours in P_j , from the definition of the shadowing, and so $G[\{w_j, u_j\} \cup P_j]$ is a daisy D_{b_j} say, and $\{D_{b_j} : j \in J_1\}$ is a bunch of daisies. But

$$|J|/\tau \geq 2q\zeta\beta(1 + (q + s)(\delta^2 + 1) + 2\delta + \delta\tau)\tau,$$

and so by 6.3, applied to $\{D_{b_j} : j \in J_1\}$, we deduce that there exist $w \in H_i \cup B_i$ and $J_2 \subseteq J_1$ with $|J_2| = q$, such that for each $j \in J_2$, w is adjacent to the eye u_j of D_{b_j} and nonadjacent to the petals P_j of D_{b_j} . Moreover there are no edges between $\{u_j, v_j\} \cup P_j \cup Q_j$ and $\{u_{j'}, v_{j'}\} \cup P_{j'} \cup Q_{j'}$ for all distinct $j, j' \in J_2$. Now w has neighbours in at most s of the sets $\{v_j\} \cup P_j \cup Q_j (j \in J_2)$, and so there exists $J_3 \subseteq J_2$ with $|J_3| = 2\delta$ such that w has no neighbours in $\{v_j\} \cup P_j \cup Q_j$ for $j \in J_3$. Hence for $j \in J_3$, $G[\{w, u_j\} \cup P_j]$ is a $(1, \delta)$ -broom with handle w , and $G[\{w, u_j, v_j\} \cup Q_j]$ is a $(2, \delta)$ -broom with handle w ; and taking the first for δ choices of $j \in J_3$, and the second for the remaining δ choices of j , and taking their union, we find that G contains $T(\delta)$, a contradiction. This proves (1).

By (1), for each $i \in \{1, \dots, n\}$ there is a subset $J_i \subseteq \{i + 1, \dots, n\}$ with $|J_i| < 3r$, such that for all a, b, c , if i is strong to (a, b, c) then one of a, b, c is in J_i . Let D be the digraph with vertex set $\{1, \dots, n\}$ in which for $i < j$, j is adjacent from i if $j \in J_i$. Thus D has maximum outdegree less than $3r$, and since D is acyclic, by 3.3 the graph underlying D has chromatic number at most $3r$. It follows that there is a subset $I \subseteq \{1, \dots, n\}$ such that

$$3r\chi\left(\bigcup_{i \in I} B_i \setminus \Pi\right) \geq \chi\left(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi\right),$$

with the property that for all $i, a, b, c \in I$, i is not strong to (a, b, c) . Let $W = \bigcup_{i \in I} B_i \setminus \Pi$.

(2) $\chi(G[W]) \leq s^2\delta\zeta\beta\tau^2$.

Let D be the digraph obtained from $G[W]$ by deleting all edges uv such that $\{u, v\}$ is a subset of some $B_i (i \in I)$, and directing every remaining edge such that the head of every edge is later than its tail. Let X be the set of all $v \in W$, where $v \in B_i$ say, such that for some $j > i$, there are $s\delta$ neighbours of v in B_j , pairwise nonadjacent. Suppose that $D[X]$ has a directed path with $s + 1$ vertices, say $v_0 - \dots - v_s$ in order. Let $v_i \in B_{j_i}$ where $j_i \in I$; then $j_0 \leq \dots \leq j_s$. For $0 \leq i \leq s$, since $v_i \in X$, there exist $b_i \in I$ with $b_i > j_i$ and a stable subset P_i of B_{b_i} with $|P_i| = s\delta$, all adjacent to v_i . For $1 \leq i \leq s$, let Q_i be the set of vertices in P_s that are adjacent to v_i and not to v_{i-1} . Since $i - 1$ is not strong to (j_i, b_{i-1}, b_i) , it follows that $|Q_i| < \delta$ for $1 \leq i \leq s$, and so there is a vertex $v \in P_s$ that belongs to none of the sets Q_1, \dots, Q_s . Consequently v is adjacent to all of v_0, v_1, \dots, v_s , and hence has a neighbour in B_j for $s + 1$ different values of j , contrary to the hypothesis. This proves that $D[X]$ has no directed path with $s + 1$ vertices, and hence the graph underlying $D[X]$ has chromatic number at most s , from the Gallai-Roy theorem [2, 8].

For each $v \in W \setminus X$, there are fewer than s values of $j \in I$ such that $B_j \setminus \Pi$ contains some vertex adjacent from v in H , by hypothesis. Moreover, since $v \notin X$, if $v \in B_i \setminus \Pi$ and some vertex in $B_j \setminus \Pi$ is adjacent from v in D , then $j > i$, and there do not exist $s\delta$ vertices in $B_j \setminus \Pi$, pairwise nonadjacent in G and all adjacent to v in G . Since the set of neighbours of v is τ -colourable, it follows that fewer than $s\delta\tau$ vertices in $B_j \setminus \Pi$ are adjacent in D from v ; and so $H[W \setminus X]$ has maximum outdegree less than $(s - 1)s\delta\tau$. Thus by 3.3 the graph underlying $D[W \setminus X]$ has chromatic number at most $(s - 1)s\delta\tau$ (since D is acyclic).

Consequently the graph underlying D has chromatic number at most $s + (s - 1)s\delta\tau \leq s^2\delta\tau$. Since for each $i \in I$, $G[B_i \setminus \Pi]$ has chromatic number at most $\beta\zeta\tau$, it follows that $\chi(G[W]) \leq s^2\beta\delta\zeta\tau^2$. This proves (2).

From the choice of I , $3r\chi(W) \geq \chi(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi)$, and so from (2),

$$\chi\left(\bigcup_{1 \leq i \leq n} B_i \setminus \Pi\right) \leq 3rs\beta\delta\zeta\tau^2.$$

This proves 8.2. ■

We deduce, finally:

8.3 *There exists c such that if G satisfies (i)–(v) then $\chi(G) \leq c$.*

Proof. Let $\eta = \alpha + 2(\delta + 1)^3(\varepsilon + 1)^2$ and $\zeta = \eta + \delta$. Let ϕ and s satisfy 8.1. Let r be as in 8.2. Let $c_2 = 3rs\beta\delta\zeta\tau^2$, let $c_1 = c_2 + \delta\tau^2$, and let $c = \phi(c_1)$.

Let G satisfy (i)–(v), and suppose that $\chi(G) > c$. By 8.1, there is a 3-cleaned (ζ, η) -template array \mathcal{T} in G , with sequence (Y_i, H_i) ($1 \leq i \leq n$), such that $\chi(U(\mathcal{T})) > c_1$, and such that for each vertex $v \in U(\mathcal{T})$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T})$ has a neighbour in H_i . By 6.2, there is a 2-cleaned (in fact 3-cleaned) (ζ, η) -template array \mathcal{T}' , with sequence (Y_i, H'_i) ($1 \leq i \leq n$) and a privatization Π for \mathcal{T}' such that

- $H'_i \subseteq H_i$ for $1 \leq i \leq n$;
- $U(\mathcal{T}') \subseteq U(\mathcal{T})$; and
- $\chi(U(\mathcal{T}') \setminus \Pi) \geq \chi(U(\mathcal{T})) - \delta\tau^2 > c_2$.

It follows that for each vertex $v \in U(\mathcal{T}')$, there are fewer than s values of $i \in \{1, \dots, n\}$ such that some neighbour of v in $U(\mathcal{T}')$ has a neighbour in H'_i . By 8.2 applied to \mathcal{T}' ,

$$\chi(U(\mathcal{T}') \setminus \Pi) \leq 3rs\beta\delta\zeta\tau^2,$$

a contradiction. This proves 8.3. ▀

Consequently, this completes the proof of 2.5, and hence of 1.2.

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