On Saturated k-Sperner Systems

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August 1, 2014

Abstract

Given a set X, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be *k-Sperner* if it does not contain a chain of length $k+1$ under set inclusion and it is *saturated* if it is maximal with respect to this property. Gerbner et al. [11] conjectured that, if $|X|$ is sufficiently large with respect to k, then the minimum size of a saturated k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ is 2^{k-1}. We disprove this conjecture by showing that there exists $\varepsilon > 0$ such that for every k and $|X| \geq n_0(k)$ there exists a saturated k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ with cardinality at most $2^{(1-\varepsilon)k}$.

A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be an *oversaturated k-Sperner system* if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, $\mathcal{F} \cup \{S\}$ contains more chains of length $k+1$ than $\mathcal{F}$. Gerbner et al. [11] proved that, if $|X| \geq k$, then the smallest such collection contains between $2^{k/2-1}$ and $O\left(\frac{\log k}{k} 2^k\right)$ elements. We show that if $|X| \geq k^2 + k$, then the lower bound is best possible, up to a polynomial factor.

Keywords: minimum saturation; set systems; antichains

1 Introduction

Given a set X, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *Sperner system* or an *antichain* if there do not exist $A, B \in \mathcal{F}$ such that $A \subsetneq B$. More generally, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *k-Sperner system* if there does not exist a subcollection $\{A_1, \ldots, A_{k+1}\} \subseteq \mathcal{F}$ such that $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Such a subcollection $\{A_1, \ldots, A_{k+1}\}$ is called a *$(k+1)$-chain*. We say that a k-Sperner system is *saturated* if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, we have that $\mathcal{F} \cup \{S\}$ contains a $(k+1)$-chain. A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is an *oversaturated k-Sperner system*[1] if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, we have that the number of $(k+1)$-chains in $\mathcal{F} \cup \{S\}$ is greater than the number of $(k+1)$-chains

[1] In [11], this is called a *weakly saturated k-Sperner system*. Since there is another notion of weak saturation in the literature (see, for instance, Bollobás [3]), we have chosen to use a different term to avoid possible confusion.

in $\mathcal{F}$. Thus, $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated k-Sperner system if and only if it is an oversaturated k-Sperner system that does not contain a $(k+1)$-chain.

For a set X of cardinality n, the problem of determining the maximum size of a saturated k-Sperner system in $\mathcal{P}(X)$ is well understood. In the case $k = 1$, Sperner's Theorem [17] (see also [4]), says that every antichain in $\mathcal{P}(X)$ contains at most $\binom{n}{\lfloor n/2 \rfloor}$ elements, and this bound is attained by the collection consisting of all subsets of X with cardinality $\lfloor n/2 \rfloor$. Erdős [6] generalised Sperner's Theorem by proving that the largest size of a k-Sperner system in $\mathcal{P}(X)$ is the sum of the k largest binomial coefficients $\binom{n}{i}$. In this paper, we are interested in determining the minimum size of a saturated k-Sperner system or an oversaturated k-Sperner system in $\mathcal{P}(X)$. These problems were first studied by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [11].

Given integers n and k, let $\text{sat}(n, k)$ denote the minimum size of a saturated k-Sperner system in $\mathcal{P}(X)$ where $|X| = n$. It was shown in [11] that $\text{sat}(n, k) = \text{sat}(m, k)$ if n and m are sufficiently large with respect to k. We can therefore define

$$\text{sat}(k) := \lim_{n \to \infty} \text{sat}(n, k).$$

We are motivated by the following conjecture of [11].

Conjecture 1 (Gerbner et al. [11]). *For all k,* $\text{sat}(k) = 2^{k-1}$.

Gerbner et al. [11] observed that their conjecture is true for $k = 1, 2, 3$. They also proved that $2^{k/2-1} \leq \text{sat}(k) \leq 2^{k-1}$ for all k, where the upper bound is implied by the following construction.

Construction 2 (Gerbner et al. [11]). Let Y be a set such that $|Y| = k - 2$ and let H be a non-empty set disjoint from Y. Let $X = Y \cup H$ and define

$$\mathcal{G} := \mathcal{P}(Y) \cup \{S \cup H : S \in \mathcal{P}(Y)\}.$$

It is easily verified that $\mathcal{G} \subseteq \mathcal{P}(X)$ is a saturated k-Sperner system of cardinality 2^{k-1}.

In this paper, we disprove Conjecture 1 by establishing the following:

Theorem 3. *There exists $\varepsilon > 0$ such that, for all k,* $\text{sat}(k) \leq 2^{(1-\varepsilon)k}$.

We remark that the value of ε that can be deduced from our proof is approximately $\left(1 - \frac{\log_2(15)}{4}\right) \approx 0.023277$. The proof of Theorem 3 comes in two parts. First, we give an infinite family of saturated 6-Sperner systems of cardinality 30 which shows that $\text{sat}(6) \leq 30 < 2^5$. We then provide a method which, under certain conditions, allows us to combine a saturated k_1-Sperner system of small order and a saturated k_2-Sperner system of small order to obtain a saturated (k_1+k_2-2)-Sperner system of small order. By repeatedly applying this method, we are able to prove Theorem 3 for general k. As it turns out, our method yields the bound $\text{sat}(k) < 2^{k-1}$ for every $k \geq 6$. For completeness, we will prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$, and so $k = 6$ is the first value of k for which Conjecture 1 is false.

Similar techniques show that $\text{sat}(k)$ satisfies a submultiplicativity condition, which leads to the following result.

Theorem 4. *For ε as in Theorem 3, there exists $c \in [1/2, 1-\varepsilon]$ such that* $\mathrm{sat}(k) = 2^{(1+o(1))ck}$.

Naturally, we wonder about the correct value of c in Theorem 4.

Problem 5. *Determine the constant c for which* $\mathrm{sat}(k) = 2^{(1+o(1))ck}$.

We are also interested in oversaturated k-Sperner systems. Given integers n and k, let $\mathrm{osat}(n,k)$ denote the minimum size of an oversaturated k-Sperner system in $\mathcal{P}(X)$ where $|X| = n$. As we will prove in Lemma 7, $\mathrm{osat}(n,k) = \mathrm{osat}(m,k)$ provided that n and m are sufficiently large with respect to k. Similarly to $\mathrm{sat}(k)$, we define $\mathrm{osat}(k) := \lim_{n\to\infty} \mathrm{osat}(n,k)$. Gerbner et al. [11] proved that if $|X| \geq k$, then an oversaturated k-Sperner system in $\mathcal{P}(X)$ of minimum size has between $2^{k/2-1}$ and $O\left(\frac{\log(k)}{k}2^k\right)$ elements. Together with Lemma 7, this implies

$$2^{k/2-1} \leq \mathrm{osat}(k) \leq O\left(\frac{\log(k)}{k}2^k\right).$$

We show that the lower bound gives the correct asymptotic behaviour, up to a polynomial factor.

Theorem 6. *For every integer k and set X with $|X| \geq k^2 + k$ there exists an oversaturated k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $|\mathcal{F}| = O\left(k^5 2^{k/2}\right)$. In particular,*

$$\mathrm{osat}(k) = 2^{(1/2+o(1))k}.$$

In Section 2, we prove some preliminary results which will be used throughout the paper. In particular, we provide conditions under which a saturated k-Sperner system can be decomposed into or constructed from a sequence of k disjoint saturated antichains. In Section 3 we show that certain types of saturated k_1-Sperner and k_2-Sperner systems can be combined to produce a saturated (k_1+k_2-2)-Sperner system, and use this to prove Theorems 3 and 4. Finally, in Section 4, we give a probabilistic construction of oversaturated k-Sperner systems of small cardinality, thereby proving Theorem 6.

Minimum saturation has been studied extensively in the context of graphs [1, 2, 5, 10, 12, 13, 18, 19, 20] and hypergraphs [7, 14, 15, 16]. Such problems are typically of the following form: for a fixed (hyper)graph H, determine the minimum size of a (hyper)graph G on n vertices which does not contain a copy of H and for which adding any edge $e \notin G$, yields a (hyper)graph which contains a copy of H. This line of research was first initiated by Zykov [21] and Erdős, Hajnal and Moon [8]. For more background on minimum saturation problems for graphs, we refer the reader to the survey of Faudree, Faudree and Schmitt [9].

2 Preliminaries

Given a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, we say that a set $A \subseteq X$ is an *atom* for $\mathcal{F}$ if A is maximal with respect to the property that

$$\text{for every set } S \in \mathcal{F},\ S \cap A \in \{\emptyset, A\}. \tag{1}$$

We say that an atom A with $|A| \geq 2$ is *homogeneous* for $\mathcal{F}$. Gerbner et al. [11] proved that if n, m are sufficiently large with respect to k, then $\mathrm{sat}(n,k) = \mathrm{sat}(m,k)$. Using a similar approach, we extend this result to $\mathrm{osat}(n,k)$.

Lemma 7. *Fix k. If $n, m > 2^{2^{k-1}}$, then $\mathrm{sat}(n,k) = \mathrm{sat}(m,k)$ and $\mathrm{osat}(n,k) = \mathrm{osat}(m,k)$.*

Proof. Fix $n > 2^{2^{k-1}}$ and let X be a set of cardinality n. Suppose that $\mathcal{F} \subseteq \mathcal{P}(X)$ is an oversaturated k-Sperner system of cardinality at most 2^{k-1}. We know that such a family exists by Construction 2. We will show that, for sets X_1 and X_2 such that $|X_1| = n-1$ and $|X_2| = n+1$, there exists $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ such that

(a) $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}|$,

(b) $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same number of $(k+1)$-chains as $\mathcal{F}$,

(c) $\mathcal{F}_1$ and $\mathcal{F}_2$ are oversaturated k-Sperner systems.

We observe that this is enough to prove the lemma. Indeed, by taking $\mathcal{F}$ to be a saturated k-Sperner system or an oversaturated k-Sperner system in $\mathcal{P}(X)$ of minimum order, we will have that

$$\max\{\mathrm{sat}(n-1,k), \mathrm{sat}(n+1,k)\} \leq \mathrm{sat}(n,k) \text{ and}$$

$$\max\{\mathrm{osat}(n-1,k), \mathrm{osat}(n+1,k)\} \leq \mathrm{osat}(n,k).$$

Since n was an arbitrary integer greater than $2^{2^{k-1}}$, the result will follow by induction.

We prove the following claim.

Claim 8. *Given a set X and a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, if $|X| > 2^{|\mathcal{F}|}$, then there is a homogeneous set for $\mathcal{F}$.*

Proof. We observe that every atom A for $\mathcal{F}$ corresponds to a subcollection $\mathcal{F}_A := \{S \in \mathcal{F} : A \subseteq S\}$ of $\mathcal{F}$ such that $\mathcal{F}_A \neq \mathcal{F}_{A'}$ whenever $A \neq A'$. This implies that the number of atoms for $\mathcal{F}$ is at most $2^{|\mathcal{F}|}$. Therefore, since $|X| > 2^{|\mathcal{F}|}$, there must be a homogeneous set H for $\mathcal{F}$. $\square$

By Claim 8 and the fact that $|X| > 2^{2^{k-1}} \geq 2^{|\mathcal{F}|}$, there exists a homogeneous set H for $\mathcal{F}$. Let $x_1 \in H$ and $x_2 \notin X$ and define $X_1 := X \setminus \{x_1\}$ and $X_2 := X \cup \{x_2\}$. Let

$$\mathcal{F}_1 := \{S \in \mathcal{F} : S \cap H = \emptyset\} \cup \{S \setminus \{x_1\} : S \in \mathcal{F}_H\}, \text{ and}$$

$$\mathcal{F}_2 := \{S \in \mathcal{F} : S \cap H = \emptyset\} \cup \{S \cup \{x_2\} : S \in \mathcal{F}_H\}.$$

Since H is homogeneous for $\mathcal{F}$, there does not exist a pair of sets in $\mathcal{F}$ which differ only on x_1. Thus, for $i \in \{1,2\}$ there is a natural bijection from $\mathcal{F}_i$ to $\mathcal{F}$ which preserves set inclusion. Hence, (a) and (b) hold. Now, let $i \in \{1,2\}$ and $T_i \in \mathcal{P}(X_i) \setminus \mathcal{F}_i$ and define

$$T := (T_i \setminus (H \cup \{x_2\})) \cup \{x_1\}.$$

Then $T \in \mathcal{P}(X) \setminus \mathcal{F}$ since H is a non-singleton atom and $T \cap H = \{x_1\}$, and so there exists $A_1, \dots, A_k \in \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_1 \subsetneq \cdots \subsetneq A_t \subsetneq T \subsetneq A_{t+1} \subsetneq \cdots \subsetneq A_k.$$

Since $T \cap H \neq H$, we must have $A_j \cap H = \emptyset$ for $j \leq t$ and so $A_1, \dots, A_t \in \mathcal{F}_i$ and $A_1 \subsetneq \cdots \subsetneq A_t \subsetneq T_i$. Also, since $T \cap H \neq \emptyset$, we have $A_j \cap H = H$ for $j \geq t+1$. Setting $A'_j := (A_j \cup \{x_2\}) \cap X_i$, we see that $A'_j \in \mathcal{F}_i$ for $j \geq t+1$ and that $T_i \subsetneq A'_{t+1} \subsetneq \cdots \subsetneq A'_k$. Thus, (c) holds. □

The rest of the results of this section are concerned with the structure of saturated k-Sperner systems. The next lemma, which is proved in [11], implies that for any saturated k-Sperner system there can be at most one homogeneous set. We include a proof for completeness.

Lemma 9 (Gerbner et al. [11])**.** *If $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated k-Sperner system and H_1 and H_2 are homogeneous for $\mathcal{F}$, then $H_1 = H_2$.*

Proof. Suppose to the contrary that H_1 and H_2 are homogeneous for $\mathcal{F}$ and that $H_1 \neq H_2$. Then, since each of H_1 and H_2 are maximal with respect to (1), we have that $H_1 \cup H_2$ is not homogeneous for $\mathcal{F}$. Therefore, there is a set $S \in \mathcal{F}$ which contains some, but not all, of $H_1 \cup H_2$. Without loss of generality, we have $S \cap H_1 = H_1$ and $S \cap H_2 = \emptyset$ since H_1 and H_2 are homogeneous for $\mathcal{F}$. Now, pick $x \in H_1$ and $y \in H_2$ arbitrarily and define

$$T := (S \setminus \{x\}) \cup \{y\}.$$

Clearly T cannot be in $\mathcal{F}$ since $T \cap H_1 = H_1 \setminus \{x\}$ and H_1 is homogeneous for $\mathcal{F}$. Since $\mathcal{F}$ is saturated, there must exist sets $A_1, \dots, A_k \in \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_1 \subsetneq \cdots \subsetneq A_t \subsetneq T \subsetneq A_{t+1} \subsetneq \cdots \subsetneq A_k.$$

Since H_1 and H_2 are homogeneous for $\mathcal{F}$, and neither H_1 nor H_2 is contained in T, we get that $A_t \subsetneq T \setminus (H_1 \cup H_2) \subseteq S$. Similarly, $A_{t+1} \supsetneq S$. However, this implies that $\{A_1, \dots, A_k\} \cup \{S\}$ is a $(k+1)$-chain in $\mathcal{F}$, a contradiction. □

By Lemma 9, if $\mathcal{F}$ is a saturated k-Sperner system for which there exists a homogeneous set, then the homogeneous set must be unique. Throughout the paper, it will be useful to distinguish the elements of $\mathcal{F}$ which contain the homogeneous set from those that do not.

Definition 10. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k-Sperner system and let H be homogeneous for $\mathcal{F}$. We say that a set $S \in \mathcal{F}$ is *large* if $H \subseteq S$ or *small* if $S \cap H = \emptyset$. Let $\mathcal{F}^{\text{large}}$ and $\mathcal{F}^{\text{small}}$ denote the collection of large and small sets of $\mathcal{F}$, respectively. Thus, $\mathcal{F} = \mathcal{F}^{\text{small}} \cup \mathcal{F}^{\text{large}}$.

Lemma 11. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a saturated antichain with homogeneous set H. Then every set $S \in \mathcal{P}(X) \setminus \mathcal{A}$ either contains a set in $\mathcal{A}^{\text{small}}$ or is contained in a set of $\mathcal{A}^{\text{large}}$.*

Proof. Suppose, to the contrary, that $S \in \mathcal{P}(X) \setminus \mathcal{A}$ does not contain a set of $\mathcal{A}^{\text{small}}$ and is not contained in a set of $\mathcal{A}^{\text{large}}$. Since $\mathcal{A}$ is saturated, we get that either

(a) there exists $A \in \mathcal{A}^{\text{large}}$ such that $A \subsetneq S$, or

(b) there exists $B \in \mathcal{A}^{\text{small}}$ such that $S \subsetneq B$.

Suppose that (a) holds. Let $y \in S \setminus A$ and $x \in H$ and define $T := (A \setminus \{x\}) \cup \{y\}$. Since H is homogeneous for $\mathcal{A}$ and $T \cap H = H \setminus \{x\}$, we must have $T \notin \mathcal{A}$. Also, since H is homogeneous for $\mathcal{A}$, any set $T' \in \mathcal{A}$ containing T would have to contain $T \cup \{x\} \supsetneq A$. Therefore, since $\mathcal{A}$ is an antichain, no such set T' can exist. Thus, there is a set $T'' \in \mathcal{A}$ such that $T'' \subsetneq T \subseteq S$. Since H is homogeneous for $\mathcal{A}$ and $T \cap H \neq H$, we get that $T'' \in \mathcal{A}^{\text{small}}$, contradicting our assumption on S.

Note that we are also done in the case that (b) holds by considering the saturated antichain $\{X \setminus A : A \in \mathcal{A}\}$ and applying the argument of the previous paragraph. □

2.1 Constructing and Decomposing Saturated k-Sperner Systems

There is a natural way to partition a k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ into a sequence of k pairwise disjoint antichains. Specifically, for $0 \leq i \leq k-1$, let $\mathcal{A}_i$ be the collection of all minimal elements of $\mathcal{F} \setminus \left(\bigcup_{j<i} \mathcal{A}_j\right)$ under inclusion. We say that $(\mathcal{A}_i)_{i=0}^{k-1}$ is the *canonical decomposition* of $\mathcal{F}$ into antichains.

In this section we provide conditions under which a sequence of k pairwise disjoint saturated antichains can be united to obtain a saturated k-Sperner system. Later we will prove a partial converse: if $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated k-Sperner system with a homogeneous set, then every antichain of the canonical decomposition of $\mathcal{F}$ is saturated. We also provide an example which shows that this is not necessarily the case if we remove the condition that $\mathcal{F}$ has a homogeneous set.

Definition 12. We say that a sequence $(\mathcal{D}_i)_{i=0}^t$ of subsets of $\mathcal{P}(X)$ is *layered* if, for $1 \leq i \leq t$, every $D \in \mathcal{D}_i$ strictly contains some $D' \in \mathcal{D}_{i-1}$ as a subset.

Note that the canonical decomposition of any set system is layered.

Lemma 13. *If $(\mathcal{A}_i)_{i=0}^t$ is a layered sequence of pairwise disjoint saturated antichains, then every $A \in \mathcal{A}_i$ is strictly contained in some $B \in \mathcal{A}_{i+1}$*

Proof. Let $A \in \mathcal{A}_i$. Since $\mathcal{A}_{i+1}$ is a saturated antichain disjoint from $\mathcal{A}_i$, there exists some $B \in \mathcal{A}_{i+1}$ such that either $B \subsetneq A$ or $A \subsetneq B$. In the latter case we are done, so suppose $B \subsetneq A$. Since $(\mathcal{A}_i)_{i=0}^t$ is layered, there exists some $A' \in \mathcal{A}_i$ such that $A' \subsetneq B$. Hence we have $A' \subsetneq B \subsetneq A$, contradicting the fact that $\mathcal{A}_i$ is an antichain and completing the proof. □

Lemma 14. *If $(\mathcal{A}_i)_{i=0}^{k-1}$ is a layered sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, then $\mathcal{F} := \bigcup_{i=0}^{k-1} \mathcal{A}_i$ is a saturated k-Sperner system.*

Proof. Clearly, $\mathcal{F}$ is a k-Sperner system since $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ are antichains. Let $S \in \mathcal{P}(X) \setminus \mathcal{F}$ be arbitrary and define $t = \max\{i : S \supsetneq A \text{ for some } A \in \mathcal{A}_i\}$. If $t \geq 0$, then S strictly contains some set $A_t \in \mathcal{A}_t$. As $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, for $0 \leq i \leq t-1$, there exist sets $A_i \in \mathcal{A}_i$ such that

$$A_0 \subsetneq \cdots \subsetneq A_t \subsetneq S.$$

Now, if $t \geq k-2$, then since $\mathcal{A}_{t+1}$ is a saturated antichain and S does not contain a set of $\mathcal{A}_{t+1}$, there must exist $A_{t+1} \in \mathcal{A}_{t+1}$ such that $S \subsetneq A_{t+1}$. By Lemma 13, we see that for $t+2 \leq i \leq k-1$ there exists $A_i \in \mathcal{A}_i$ such that

$$S \subsetneq A_{t+1} \subsetneq \cdots \subsetneq A_{k-1}.$$

Thus $\{A_0, \ldots, A_{k-1}\} \cup \{S\}$ is a $(k+1)$-chain, as desired. □

In Lemma 14, we require the sequence $(\mathcal{A}_i)_{i=0}^{k-1}$ of saturated antichains to be layered. As it turns out, if each antichain $\mathcal{A}_i$ has a homogeneous set, then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $\left(\mathcal{A}_i^{\text{small}}\right)_{i=0}^{k-1}$ is layered.

Lemma 15. *Let $(\mathcal{A}_i)_{i=0}^{k-1}$ be a sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, each of which has a homogeneous set. Then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $\left(\mathcal{A}_i^{\text{small}}\right)_{i=0}^{k-1}$ is layered.*

Proof. Suppose that $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered and, for some $i \geq 0$, let $A \in \mathcal{A}_{i+1}^{\text{small}}$ be arbitrary. We show that A contains a set of $\mathcal{A}_i^{\text{small}}$. Otherwise, since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, we get that there is some $B \in \mathcal{A}_i^{\text{large}}$ such that $B \subsetneq A$. Therefore, since $\mathcal{A}_i$ is an antichain, A cannot be contained in an element of $\mathcal{A}_i^{\text{large}}$. By Lemma 11 and the fact that $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are disjoint, we get that A contains a set of $\mathcal{A}_i^{\text{small}}$, as desired.

Now, suppose that $\left(\mathcal{A}_i^{\text{small}}\right)_{i=0}^{k-1}$ is layered. Given $i \geq 0$ and $S \in \mathcal{A}_{i+1}^{\text{large}}$, we show that S contains a set of $\mathcal{A}_i$, which will complete the proof. If not, then since $\mathcal{A}_i$ is saturated and disjoint from $\mathcal{A}_{i+1}$, there must exist $T \in \mathcal{A}_i$ such that $S \subsetneq T$. Since $\mathcal{A}_{i+1}$ is an antichain, S cannot be strictly contained in a set of $\mathcal{A}_{i+1}^{\text{large}}$, and so neither can T. Therefore, by Lemma 11, there is a set $A \in \mathcal{A}_{i+1}^{\text{small}}$ contained in T. However, since $\left(\mathcal{A}_i^{\text{small}}\right)_{i=0}^{k-1}$ is layered, there exists $A' \in \mathcal{A}_i^{\text{small}}$ such that $A' \subsetneq A$. But then, $A' \subsetneq T$, which contradicts the assumption that $\mathcal{A}_i$ is an antichain. The result follows. □

It is natural to wonder whether a converse to Lemma 14 is true. That is: *if $\mathcal{F}$ is a saturated k-Sperner system, can we decompose $\mathcal{F}$ into a layered sequence of k pairwise disjoint saturated antichains?* The following example shows that this is not always the case.

Example 16. Let $X := \{x_1, x_2, x_3\}$, $Y := \{y_1, y_2, y_3\}$ and $Z := X \cup Y$. We define

$$\mathcal{B}_0 := \{\{x_i, x_j\} : i \neq j\} \cup \{\{x_i, y_i\} : i \in \{1,2,3\}\} \cup \{\{x_k, y_i, y_j\} : i, j, k \text{ distinct }\} \cup \{Y\},$$

$$\mathcal{B}_1 := \{X, \{x_1, x_2, y_1\}, \{x_1, x_3, y_3\}, \{x_2, x_3, y_2\}, \{x_1, y_1, y_3\}, \{x_2, y_1, y_2\}, \{x_3, y_2, y_3\},$$
$$\{x_1, x_2, y_2, y_3\}, \{x_1, x_3, y_1, y_2\}, \{x_2, x_3, y_1, y_3\}\}.$$

Then $(\mathcal{B}_i)_{i=0}^1$ is a layered sequence of disjoint antichains. In fact, $(\mathcal{B}_i)_{i=0}^1$ is the canonical decomposition of $\mathcal{F} := \mathcal{B}_0 \cup \mathcal{B}_1$. Clearly $\mathcal{B}_1$ is not saturated as $\mathcal{B}_1 \cup \{Y\}$ is an antichain. We claim that $\mathcal{F}$ is a saturated 2-Sperner system.

Consider any $S \in \mathcal{P}(Z) \setminus \mathcal{F}$. We will show that $\mathcal{F} \cup \{S\}$ contains a 3-chain. It is easy to check that every element of $\mathcal{B}_0 \setminus \{Y\}$ is contained in a set of $\mathcal{B}_1$. Hence if S is contained in some set $B \in \mathcal{B}_0 \setminus \{Y\}$, then $\mathcal{F} \cup \{S\}$ contains a 3-chain. In particular, this completes the proof when $|S| \in \{0, 1, 2\}$. Similarly, since $(\mathcal{B}_i)_{i=0}^1$ is layered, if S contains some set $B \in \mathcal{B}_1$, then $\mathcal{F} \cup \{S\}$ contains a 3-chain. Therefore, we are done if $|S| \in \{4, 5, 6\}$.

It remains to consider the case that $|S| = 3$. Since $X, Y \in \mathcal{F}$, we must have $|S \cap Y| = 2$, or $|S \cap X| = 2$. If $|S \cap Y| = 2$, we have $S \in \{\{x_1, y_1, y_2\}, \{x_2, y_2, y_3\}, \{x_3, y_1, y_3\}\}$. This implies that S is contained in a set $B \in \mathcal{B}_1$ and contains a set $B' \in \mathcal{B}_0 \cap \mathcal{P}(X)$. If $|S \cap X| = 2$, then S contains some set $\{x_i, x_j\} \in \mathcal{B}_0$. Also, it is easily verified that S is contained in a set of $\mathcal{B}_1$. Thus, $\mathcal{F}$ is a saturated 2-Sperner system.

However, for saturated k-Sperner systems with a homogeneous set, the converse to Lemma 14 does hold; we can partition $\mathcal{F}$ into a layered sequence of k pairwise disjoint saturated antichains.

Lemma 17. *Let $\mathcal{F} \in \mathcal{P}(X)$ be a saturated k-Sperner system with homogeneous set H and canonical decomposition $(\mathcal{A}_i)_{i=0}^{k-1}$. Then $\mathcal{A}_i$ is saturated for all i.*

Proof. Fix i and let $S \in \mathcal{P}(X) \setminus \mathcal{A}_i$. Let $x \in H$ and define

$$T := (S \setminus H) \cup \{x\}.$$

Then $T \notin \mathcal{F}$ since $T \cap H = \{x\}$ and H is homogeneous for $\mathcal{F}$. Therefore, there exists $\{A_0, \dots, A_{k-1}\} \subseteq \mathcal{F}$ and $t \in \{0, \dots, k\}$ such that

$$A_0 \subsetneq \cdots \subsetneq A_{t-1} \subsetneq T \subsetneq A_t \subsetneq \cdots \subsetneq A_{k-1}.$$

By definition of the canonical decomposition, we must have $A_j \in \mathcal{A}_j$ for all j. Also, since H is homogeneous for $\mathcal{F}$ and $T \cap H \notin \{\emptyset, H\}$, we must have $A_{t-1} \subseteq T \setminus H \subseteq S$ and $A_t \supseteq T \cup H \supseteq S$. Therefore,

$$A_0 \subsetneq \cdots \subsetneq A_{t-1} \subseteq S \subseteq A_t \subsetneq \cdots \subsetneq A_{k-1}.$$

Since $S \neq A_i$, we must have either $A_i \subsetneq S$ or $S \subsetneq A_i$ depending on whether or not $i < t$. Therefore, $\mathcal{A}_i$ is saturated for all i. □

3 Combining Saturated k-Sperner Systems

Our first goal in this section is to prove that, under certain conditions, a saturated k_1-Sperner system $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and a saturated k_2-Sperner system $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ can be combined to yield a saturated $(k_1 + k_2 - 2)$-Sperner system in $\mathcal{P}(X_1 \cup X_2)$. We apply this result to prove Theorem 3. Afterwards, we prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$. We conclude the section with a proof of Theorem 4.

Lemma 18. *Let X_1 and X_2 be disjoint sets. For $i \in \{1,2\}$, let $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$ be a saturated k_i-Sperner system which contains $\{\emptyset, X_i\}$ and let $H_i \subseteq X_i$ be homogeneous for $\mathcal{F}_i$. If $\mathcal{G}$ is the set system on $\mathcal{P}(X_1 \cup X_2)$ defined by*

$$\mathcal{G} := \left\{A \cup B : A \in \mathcal{F}_1^{\text{small}}, B \in \mathcal{F}_2^{\text{small}}\right\} \cup \left\{S \cup T : S \in \mathcal{F}_1^{\text{large}}, T \in \mathcal{F}_2^{\text{large}}\right\},$$

then $\mathcal{G}$ is a saturated (k_1+k_2-2)-Sperner system which contains $\{\emptyset, X_1 \cup X_2\}$ and $H_1 \cup H_2$ is homogeneous for $\mathcal{G}$.

Proof. It is clear that $\mathcal{G}$ contains $\{\emptyset, X_1 \cup X_2\}$ and that $H_1 \cup H_2$ is homogeneous for $\mathcal{G}$. We show that $\mathcal{G}$ is a saturated (k_1+k_2-2)-Sperner system.

First, let us show that $\mathcal{G}$ does not contain a chain of length k_1+k_2-1. Suppose that $\{A_1, \dots, A_r\}$ is an r-chain in $\mathcal{G}$. We can assume that $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. Define

$$I_1 := \{i : A_i \cap X_1 \subsetneq A_{i+1} \cap X_1\}, \text{ and}$$
$$I_2 := \{i : A_i \cap X_2 \subsetneq A_{i+1} \cap X_2\}.$$

Clearly, $I_1 \cup I_2 = \{1, \dots, r-1\}$. Also, for $i \in \{1,2\}$, since $\mathcal{F}_i$ is a k_i-Sperner system, we must have $|I_i| \le k_i - 1$. Let t be the maximum index such that $A_t \cap X_1 \in \mathcal{F}_1^{\text{small}}$. Note that t exists and is less than r since $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. By construction of $\mathcal{G}$, $A_t \cap X_2$ is a small set for $\mathcal{F}_2$ and, for $i \in \{1,2\}$, $A_{t+1} \cap X_i$ is a large set for $\mathcal{F}_i$. This implies that $t \in I_1 \cap I_2$ and so

$$r - 1 = |I_1 \cup I_2| = |I_1| + |I_2| - |I_1 \cap I_2| \le k_1 + k_2 - 3$$

as required.

Now, let $S \in \mathcal{P}(X_1 \cup X_2) \setminus \mathcal{G}$. We show that $\mathcal{G} \cup \{S\}$ contains a (k_1+k_2-1)-chain. Fix $x_1 \in H_1$ and $x_2 \in H_2$ and define

$$T := (S \setminus (H_1 \cup H_2)) \cup \{x_1, x_2\}.$$

For $i \in \{1,2\}$, let $T_i := T \cap X_i$. Then $T_i \notin \mathcal{F}_i$ since $T_i \cap H_i = \{x_i\}$. Therefore, there exists $A_1^i, \dots, A_{k_i}^i \in \mathcal{F}_i$ and $t_i \in \{1, \dots, k_i - 1\}$ such that

$$\emptyset = A_1^i \subsetneq \cdots \subsetneq A_{t_i}^i \subsetneq T_i \subsetneq A_{t_i+1}^i \subsetneq \cdots \subsetneq A_{k_i}^i = X_i$$

Note that $A_j^i \in \mathcal{F}_i^{\text{small}}$ for $j \le t_i$ and $A_j^i \in \mathcal{F}_i^{\text{large}}$ for $j \ge t_i+1$. This implies that $A_{t_1}^1 \cup A_{t_2}^2 \subsetneq S$ and $A_{t_1+1}^1 \cup A_{t_2+1}^2 \supsetneq S$. Therefore,

$$A_1^1 \cup A_1^2 \subsetneq A_1^1 \cup A_2^2 \subsetneq \cdots \subsetneq A_1^1 \cup A_{t_2}^2 \subsetneq A_2^1 \cup A_{t_2}^2 \subsetneq \cdots \subsetneq A_{t_1}^1 \cup A_{t_2}^2 \subsetneq S$$
$$\subsetneq A_{t_1+1}^1 \cup A_{t_2+1}^2 \subsetneq A_{t_1+1}^1 \cup A_{t_2+2}^2 \subsetneq \cdots \subsetneq A_{t_1+1}^1 \cup A_{k_2}^2 \subsetneq A_{t_1+2}^1 \cup A_{k_2}^2 \subsetneq \cdots \subsetneq A_{k_1}^2 \cup A_{k_2}^2$$

and so $\mathcal{G} \cup \{S\}$ contains a (k_1+k_2-1)-chain. The result follows. □

Remark 19. If $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ are as in Lemma 18, then

$$|\mathcal{G}| = \left|\mathcal{F}_1^{\text{small}}\right| \left|\mathcal{F}_2^{\text{small}}\right| + \left|\mathcal{F}_1^{\text{large}}\right| \left|\mathcal{F}_2^{\text{large}}\right|.$$

3.1 Proof of Theorem 3

We apply Lemma 18 to prove Theorem 3. The first part of the proof of Theorem 3 is to exhibit an infinite family of saturated 6-Sperner systems with cardinality $30 < 2^5$.

Proposition 20. *For any set X such that $|X| \geq 8$, there is a saturated 6-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ with a homogeneous set such that $\left|\mathcal{F}^{\text{small}}\right| = \left|\mathcal{F}^{\text{large}}\right| = 15$.*

Proof. Let X be a set such that $|X| \geq 8$. Let x_1, x_2, y_1, y_2, w and z be distinct elements of X and define $H := X \setminus \{x_1, x_2, y_1, y_2, w, z\}$. We apply Lemma 14 to construct a saturated 6-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ of order 30. Naturally, we define $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_5 := \{X\}$. Also, define

$$\mathcal{A}_1 := \{\{x_1\}, \{x_2\}, \{y_1\}, \{w\}, H \cup \{y_2, z\}\}, \text{ and}$$
$$\mathcal{A}_4 := \{X \setminus A : A \in \mathcal{A}_1\}.$$

It is easily observed that $\mathcal{A}_1$ and $\mathcal{A}_4$ are saturated antichains. We define $\mathcal{A}_2$ and $\mathcal{A}_3$ by first specifying their small sets. Define

$$\mathcal{A}_2^{\text{small}} := \{\{x_i, y_j\} : 1 \leq i, j \leq 2\} \cup \{\{w, z\}\}, \text{ and}$$
$$\mathcal{A}_3^{\text{small}} := \{\{x_1, y_1, w\}, \{x_1, y_1, z\}, \{x_2, y_2, w\}, \{x_2, y_2, z\}\}.$$

Given any collection $\mathcal{B} \subseteq \mathcal{P}(X)$, a set $S \subseteq X$ is said to be *stable* for $\mathcal{B}$ if S does not contain an element of $\mathcal{B}$. For $i = 2, 3$, define $\mathcal{A}_i^{\text{large}}$ to be the collection consisting of all maximal stable sets of $\mathcal{A}_i^{\text{small}}$ and let $\mathcal{A}_i := \mathcal{A}_i^{\text{small}} \cup \mathcal{A}_i^{\text{large}}$. Note that every element of $\mathcal{A}_i^{\text{large}}$ contains H. It is clear that $\mathcal{A}_i$ is an antichain for $i = 2, 3$. Moreover, $\mathcal{A}_i$ is saturated since every set $A \in \mathcal{P}(X)$ either contains an element of $\mathcal{A}_i^{\text{small}}$ or is contained in an element of $\mathcal{A}_i^{\text{large}}$.

One can easily verify that $\left(\mathcal{A}_i^{\text{small}}\right)_{i=0}^5$ is layered. Therefore, by Lemma 15, $(\mathcal{A}_i)_{i=0}^5$ is a layered sequence of pairwise disjoint saturated antichains. By Lemma 14, $\mathcal{F} := \bigcup_{i=0}^5 \mathcal{A}_i$ is a saturated 6-Sperner system. Also,

$$\left|\mathcal{F}^{\text{small}}\right| = \sum_{i=0}^{5} \left|\mathcal{A}_i^{\text{small}}\right| = (1 + 5 + 9 + 0) = 15, \text{ and}$$

$$\left|\mathcal{F}^{\text{large}}\right| = \sum_{i=0}^{5} \left|\mathcal{A}_i^{\text{large}}\right| = (0 + 9 + 5 + 1) = 15,$$

as desired. □

We remark that the construction in Proposition 20 is similar to one which was used in [11] to prove that $\text{sat}(k, k) \leq \frac{15}{16} 2^{k-1}$ for every $k \geq 6$.

For the proof of Theorem 3 we require that

$$\text{sat}(k) \leq 2\,\text{sat}(k-1). \tag{2}$$

This was proved in [11] using the fact that if $\mathcal{F} \subseteq \mathcal{P}(X)$ is a saturated $(k-1)$-Sperner system and $y \notin X$, then $\mathcal{F} \cup \{A \cup \{y\} : A \in \mathcal{F}\}$ is a saturated k-Sperner system in $\mathcal{P}(X \cup \{y\})$.

Proof of Theorem 3. First, we prove that the result holds when k is of the form $4j + 2$ for some $j \geq 1$. In this case, we repeatedly apply Lemma 18 and Proposition 20 to obtain a saturated k-Sperner system $\mathcal{F}$ on an arbitrarily large ground set X such that

$$\left|\mathcal{F}^{\text{small}}\right| + \left|\mathcal{F}^{\text{large}}\right| = 15^j + 15^j = 2 \cdot 15^j.$$

Therefore, if $k = 4j + 2$, then $\text{sat}(k) \leq 2 \cdot 15^j$.

For k of the form $4j+2+s$ for $j \geq 1$ and $1 \leq s \leq 3$, apply (2) to obtain $\text{sat}(k) \leq 2^s \,\text{sat}(4j+2) \leq 2^{s+1} \cdot 15^j$. Thus, we are done by setting ε slightly smaller than $\left(1 - \frac{\log_2(15)}{4}\right)$. □

3.2 Bounding sat(k) From Below

One can easily deduce from the proof of Theorem 3 that $\text{sat}(k) < 2^{k-1}$ for all $k \geq 6$. For completeness, we prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$.

Proposition 21. *If $k \leq 5$, then $\text{sat}(k) = 2^{k-1}$.*

Proof. Fix $k \leq 5$. The upper bound follows from Construction 2, and so it suffices to prove that $\text{sat}(k) \geq 2^{k-1}$. Let X be a set with $n := |X| > 2^{2^{k-1}}$ and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k-Sperner system of minimum order. By Claim 8 and the fact that $|X| > 2^{2^{k-1}} \geq 2^{|\mathcal{F}|}$, there is a homogeneous set H for $\mathcal{F}$.

Let $(\mathcal{A}_i)_{i=0}^{k-1}$ be the canonical decomposition of $\mathcal{F}$. By Lemma 17, we get that $\mathcal{A}_i$ is a saturated antichain for each i. Also, since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, by Lemma 13 we see that

$$\text{every element of } \mathcal{A}_i \text{ has cardinality between } i \text{ and } n - k + i + 1. \tag{3}$$

Our goal is to to show that for $k \leq 5$, every saturated antichain $\mathcal{A}_i$ which satisfies (3) must contain at least $\binom{k-1}{i}$ elements. Clearly this is enough to complete the proof of the proposition. Note that it suffices to prove this for $i < \frac{k}{2}$ since $\{X \setminus A : A \in \mathcal{A}_i\}$ is a saturated antichain in which every set has size between $k - i - 1$ and $n - i$. Since $k \leq 5$, this means that we need only check the cases $i = 0, 1, 2$. In the case $i = 0$, we obtain $|\mathcal{A}_0| \geq 1 = \binom{k-1}{0}$ trivially.

Next, consider the case $i = 1$. Let A be the largest set in $\mathcal{A}_1$ such that $H \subseteq A$. Then, by (3), we must have $|A| \leq n - k + 2$ and so $|X \setminus A| \geq k - 2$. Fix an element x of H and, for each $y \in X \setminus A$, define $A_y := (A \setminus \{x\}) \cup \{y\}$. Since $\mathcal{A}_1$ is saturated, H is homogeneous for $\mathcal{F}$, and A is the largest set in $\mathcal{A}_1$ containing H, there must be a set $B_y \in \mathcal{A}_1$ such that $B_y \subsetneq A_y$. However, since $\mathcal{A}_1$ is an antichain, $B_y \nsubseteq A$, and so $B_y \setminus A = \{y\}$. In particular, $B_y \neq B_{y'}$ for $y \neq y'$. Therefore, $|\mathcal{A}_1| \geq |\{A\} \cup \{B_y : y \in X \setminus A\}| \geq 1 + |X \setminus A| \geq k - 1 = \binom{k-1}{1}$, as desired.

Thus, we are finished except for the case $i = 2$ and $k = 5$. Suppose to the contrary that $|\mathcal{A}_2| < \binom{4}{2} = 6$. We begin by proving the following claim.

Claim 22. *For every vertex $y \in X \setminus H$, there is a set $S_y \in \mathcal{A}_2^{\text{large}}$ containing y.*

Proof. Let $x \in H$ be arbitrary and consider the set $T := \{x, y\}$. Then T is not contained in $\mathcal{A}_2$ since H is homogeneous for $\mathcal{F}$. Also, no strict subset of T is in $\mathcal{A}_2$ by (3). Since $\mathcal{A}_2$ is saturated, there must be some $S_y \in \mathcal{A}_2^{\text{large}}$ containing T, which completes the proof. $\square$

Let us argue that $\left|\mathcal{A}_2^{\text{large}}\right| \geq 3$. By (3), each set $A \in \mathcal{A}_2^{\text{large}}$ has at most $n-2$ elements. So, by Claim 22, if $\left|\mathcal{A}_2^{\text{large}}\right| < 3$, then it must be the case that $\mathcal{A}_2^{\text{large}} = \{A_1, A_2\}$ where $A_1 \cup A_2 = X$. Therefore, since each of $|A_1|$ and $|A_2|$ is at most $n-2$, we can pick $\{w_1, w_2\} \subseteq A_1 \setminus A_2$ and $\{z_1, z_2\} \subseteq A_2 \setminus A_1$. Given $x \in H$ and $1 \leq i, j \leq 2$, we have that $\{x, w_i, z_j\} \notin \mathcal{A}_2$ since H is homogeneous for $\mathcal{F}$. Note that $\{x, w_i, z_j\}$ is not contained in either A_1 or A_2, and so by Lemma 11 and (3) we must have $\{w_i, z_j\} \in \mathcal{A}_2$. However, this implies that $|\mathcal{A}_2| \geq |\{\{w_i, z_j\} : 1 \leq i, j \leq 2\} \cup \{A_1, A_2\}| = 6$, a contradiction.

So, we get that $\left|\mathcal{A}_2^{\text{large}}\right| \geq 3$. Note that $\{X \setminus A : A \in \mathcal{A}_2\}$ is also a saturated antichain in which every set has cardinality between 2 and $n-2$. Thus, we can apply the argument of the previous paragraph to obtain $\left|\mathcal{A}_2^{\text{small}}\right| \geq 3$. Therefore, $|\mathcal{A}_2| = \left|\mathcal{A}_2^{\text{small}}\right| + \left|\mathcal{A}_2^{\text{large}}\right| \geq 6$, which completes the proof. $\square$

It is possible that a similar approach may prove fruitful for improving the lower bound on $\text{sat}(k)$ from $2^{k/2-1}$ to $2^{(1+o(1))ck}$ for some $c > 1/2$. That is, one may first decompose a saturated k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ of minimum size into its canonical decomposition $(\mathcal{A}_i)_{i=0}^{k-1}$ and then bound the size of $|\mathcal{A}_i|$ for each i individually. Since there are only k antichains in the decomposition and the bound on $|\mathcal{F}|$ that we are aiming for is exponential in k, one could obtain a fairly tight lower bound on $\text{sat}(k)$ by focusing on a single antichain of the decomposition. Setting $i = \left\lfloor \frac{k}{2} \right\rfloor$ in (3), we see that it would be sufficient to prove that there exists $c > 1/2$ such that every saturated antichain $\mathcal{A}$ with a homogeneous set such that every element of $\mathcal{A}$ has cardinality between $\left\lfloor \frac{k}{2} \right\rfloor$ and $n - \left\lceil \frac{k}{2} \right\rceil + 1$ must satisfy $|\mathcal{A}| \geq 2^{(1+o(1))ck}$. The problem of determining whether such a c exists is interesting in its own right.

3.3 Asymptotic Behaviour of $\text{sat}(k)$

To prove Theorem 4, we require the following fact, which is proved in [11].

Lemma 23 (Gerbner et al. [11])**.** *For any $n \geq k \geq 1$ and set X with $|X| = n$ there is a saturated k-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ such that $|\mathcal{F}| = \text{sat}(n, k)$ and $\{\emptyset, X\} \subseteq \mathcal{F}$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated k-Sperner system such that $|\mathcal{F}| = \text{sat}(n, k)$. We let $(\mathcal{A}_i)_{i=0}^{k-1}$ denote the canonical decomposition of $\mathcal{F}$ and define

$$\mathcal{F}' := (\mathcal{F} \setminus (\mathcal{A}_0 \cup \mathcal{A}_{k-1})) \cup \{\emptyset, X\}.$$

It is clear that $\mathcal{F}' \subseteq \mathcal{P}(X)$ is a saturated k-Sperner system and $|\mathcal{F}'| \leq |\mathcal{F}| = \text{sat}(n, k)$, which proves the result. $\square$

Proof of Theorem 4. We show that, for all k, ℓ,

$$\operatorname{sat}(k+\ell) \leq 4 \operatorname{sat}(k) \operatorname{sat}(\ell). \tag{4}$$

Letting $f(k) := 4\operatorname{sat}(k)$, we see that (4) implies that $f(k+\ell) \leq f(k)f(\ell)$ for every k, ℓ. It follows by Fekete's Lemma that $f(k)^{1/k}$ converges, and so $\operatorname{sat}(k)^{1/k}$ converges as well.

For $n > 2^{2^{k+\ell-2}}$, let X and Y be disjoint sets of size n and let $\mathcal{F}_k \subseteq \mathcal{P}(X)$ and $\mathcal{F}_\ell \subseteq \mathcal{P}(Y)$ be saturated k-Sperner and ℓ-Sperner systems of cardinalities $\operatorname{sat}(k)$ and $\operatorname{sat}(\ell)$, respectively. By Claim 8, we can assume that $\mathcal{F}_k$ and $\mathcal{F}_\ell$ have homogeneous sets and, by Lemma 23, we can assume that $\{\emptyset, X\} \subseteq \mathcal{F}_k$ and $\{\emptyset, Y\} \subseteq \mathcal{F}_\ell$. We apply Lemma 18 and Remark 19 to obtain a saturated $(k+\ell-2)$-Sperner system $\mathcal{G} \subseteq \mathcal{P}(X \cup Y)$ of order at most $|\mathcal{F}_k||\mathcal{F}_\ell| = \operatorname{sat}(k)\operatorname{sat}(\ell)$. Therefore, by (2), we have

$$\operatorname{sat}(k+\ell) \leq 4\operatorname{sat}(k+\ell-2) \leq 4|\mathcal{G}| \leq 4\operatorname{sat}(k)\operatorname{sat}(\ell)$$

as required. □

4 Oversaturated k-Sperner Systems

In this section we construct oversaturated k-Sperner systems of small order. We first state a lemma, from which Theorem 6 follows, and then prove the lemma itself.

Lemma 24. *Given $k \geq 1$, let X be a set of cardinality $k^2 + k$. Then for all t such that $1 \leq t \leq k^2 + k$ there exist non-empty collections $\mathcal{F}_t$, $\mathcal{G}_t \subseteq \mathcal{P}(X)$ that have the following properties:*

(a) For every $F \in \mathcal{F}_t$ and $G \in \mathcal{G}_t$, $|F| + |G| \geq k$,

(b) $|\mathcal{F}_t| + |\mathcal{G}_t| = O\left(k^2 2^{k/2}\right)$,

(c) For every $S \subseteq X$ such that $|S| = t$, there exists some $F \in \mathcal{F}_t$ and some $G \in \mathcal{G}_t$ such that $F \subsetneq S$ and $G \cap S = \emptyset$.

We apply Lemma 24 to prove Theorem 6.

Proof of Theorem 6. First, let X be a set of cardinality $k^2 + k$. For $t \in \{1, \ldots, k^2 + k\}$, let $\mathcal{F}_t$ and $\mathcal{G}_t$ be as in Lemma 24. For each $F \in \mathcal{F}_t \cup \mathcal{G}_t$, choose $F_1, \ldots, F_i \in \mathcal{P}(X)$ such that

$$F_1 \subsetneq \cdots \subsetneq F_i \subsetneq F$$

where $i := \min\{k-1, |F|\}$. We let $\mathcal{C}_F := F \cup \{F_1, \ldots, F_i\}$ and define

$$\mathcal{G} := \bigcup_{1 \leq t \leq k^2+k} \left(\{T : T \in \mathcal{C}_F \text{ for some } F \in \mathcal{F}_t\} \cup \{X \setminus T : T \in \mathcal{C}_G \text{ for some } G \in \mathcal{G}_t\}\right).$$

For each $t \le k^2+k$ and $F \in \mathcal{F}_t \cup \mathcal{G}_t$, we have $|\mathcal{C}_F| \le k$. Thus, by Property (b) of Lemma 24,

$$|\mathcal{G}| \le \sum_{t=1}^{k^2+k} k(|\mathcal{F}_t| + |\mathcal{G}_t|) = O\left(k^5 2^{k/2}\right).$$

We will now show that for any $S \in \mathcal{P}(X) \setminus \mathcal{G}$ there is a $(k+1)$-chain in $\mathcal{G} \cup \{S\}$ containing S, which will imply that $\mathcal{G}$ is an oversaturated k-Sperner system. Let $S \subsetneq X$ and define $t := |S|$. By Property (c) of Lemma 24, there exists $F \in \mathcal{F}_t$ such that $F \subsetneq S$ and $G \in \mathcal{G}_t$ such that $G \cap S = \emptyset$. This implies that $S \subsetneq X \setminus G$. By Property (a) of Lemma 24 we get that

$$\mathcal{C}_F \cup \{X \setminus T : T \in \mathcal{C}_G\} \cup \{S\}$$

contains a $(k+1)$-chain in $\mathcal{G} \cup \{S\}$ containing S.

Now, suppose that $|X| > k^2+k$. Let $Y \subsetneq X$ such that $|Y| = k^2+k$ and define $H := X \setminus Y$. As above, let $\mathcal{G} \subseteq \mathcal{P}(Y)$ be an oversaturated k-Sperner system of cardinality at most $O\left(k^5 2^{k/2}\right)$. Define $\mathcal{G}' \subseteq \mathcal{P}(X)$ as follows:

$$\mathcal{G}' := \{T : T \in \mathcal{G}\} \cup \{T \cup H : T \in \mathcal{G}\}.$$

Consider any set $S \in \mathcal{P}(X) \setminus \mathcal{G}'$. Let $S' = S \cap Y$. We have, by definition of $\mathcal{G}$, that there is a $(k+1)$-chain $\mathcal{C}$ in $\mathcal{G} \cup \{S'\}$ containing S'. Adding H to every superset of S' in $\mathcal{C}$ and replacing S' by S in $\mathcal{C}$ gives us a $(k+1)$-chain in $\mathcal{G}' \cup \{S\}$ containing S. The result follows. $\square$

To prove Lemma 24, we use a probabilistic approach.

Proof of Lemma 24. Throughout the proof, we assume that k is sufficiently large and let X be a set of cardinality k^2+k. Let $1 \le t \le k^2+k$ be given. We can assume that $t \le \frac{k^2+k}{2}$ since, otherwise, we can simply define $\mathcal{F}_t := \mathcal{G}_{k^2+k-t}$ and $\mathcal{G}_t := \mathcal{F}_{k^2+k-t}$. We divide the proof into two cases depending on the size of t.

Case 1: $t \le \frac{k^2+k}{8}$.

We define $\mathcal{F}_t := \{\emptyset\}$ and let $\mathcal{G}_t$ be a uniformly random collection of $2^{k/2}$ subsets of X, each of cardinality k. Given $S \subseteq X$ of cardinality t, the probability that S is not disjoint from any set of $\mathcal{G}_t$ is

$$\left(1 - \prod_{i=0}^{k-1}\left(\frac{k^2+k-t-i}{k^2+k-i}\right)\right)^{2^{k/2}} \le \left(1 - \left(\frac{k^2-t}{k^2}\right)^k\right)^{2^{k/2}} \le \left(1 - \left(\frac{7}{8} - \frac{1}{8k}\right)^k\right)^{2^{k/2}}$$

$$\le e^{-\left(\frac{7}{8}-\frac{1}{8k}\right)^k 2^{k/2}} < e^{-(1.1)^k}.$$

Therefore, the expected number of subsets of X of cardinality t which are not disjoint from any set of $\mathcal{G}_t$ is at most $\binom{k^2+k}{t} e^{-(1.1)^k}$, which is less than 1. Thus, with non-zero probability, every $S \subseteq X$ of cardinality t is disjoint from some set in $\mathcal{G}_t$.

Case 2: $\frac{k^2+k}{8} < t \leq \frac{k^2+k}{2}$.

Define $p := \frac{t}{k^2+k}$ and let a be the rational number such that $ak = \left\lfloor \frac{-k\log\sqrt{2}}{\log(p)} + 1 \right\rfloor$. Then, since $\frac{1}{8} \leq p \leq \frac{1}{2}$, we have

$$1/6 \leq a \leq 1/2 + 1/k < 4/7. \tag{5}$$

Now, let $\mathcal{F}_t$ be a collection of $\lceil 8e^8k^2 2^{k/2} \rceil$ subsets of X, each of cardinality ak, chosen uniformly at random with replacement. Similarly, let $\mathcal{G}_t$ be a collection of $\lceil e^2k^2 2^{k/2} \rceil$ subsets of X, each of cardinality $(1-a)k$, chosen uniformly at random with replacement. We show that, with non-zero probability, every $S \subseteq X$ of size t contains a set of $\mathcal{F}_t$ and is disjoint from a set of $\mathcal{G}_t$.

Given $S \subseteq X$ of size $t = p(k^2+k)$, the probability that S does not contain a set of $\mathcal{F}_t$ is at most

$$\begin{aligned}\left(1 - \prod_{i=0}^{ak-1}\left(\frac{p(k^2+k)-i}{k^2+k-i}\right)\right)^{|\mathcal{F}_t|} &\leq \left(1 - \left(\frac{p(k^2+k)-k}{k^2}\right)^{ak}\right)^{|\mathcal{F}_t|} \\ &= \left(1 - \left(1 - \frac{1-p}{pk}\right)^{ak} p^{ak}\right)^{|\mathcal{F}_t|}. \end{aligned} \tag{6}$$

Observe that $\left(1 - \frac{1-p}{pk}\right) \geq e^{-\frac{2(1-p)}{pk}}$ for large enough k. So, $\left(1 - \frac{1-p}{pk}\right)^{ak} \geq e^{\frac{-2a(1-p)}{p}}$ which is at least e^{-8} since $a < 4/7$ and $p \geq 1/8$. Thus, the expression in (6) is at most

$$\left(1 - e^{-8}p^{ak}\right)^{|\mathcal{F}_t|} \leq e^{-e^{-8}p^{ak}|\mathcal{F}_t|} \leq e^{-e^{-8}p^{ak}\left(8e^8k^2 2^{k/2}\right)} = e^{-p^{ak}8k^2 2^{k/2}}.$$

Using our choice of a and the fact that $p \geq 1/8$, we can bound the exponent by

$$p^{ak}8k^2 2^{k/2} \geq p^{\left(-\frac{\log\sqrt{2}}{\log(p)} + \frac{1}{k}\right)k} 8k^2 2^{k/2} = p8k^2 \geq k^2.$$

Therefore, the expected number of subsets of X of size t which do not contain a set of $\mathcal{F}_t$ is at most

$$\binom{k^2+k}{t} e^{-k^2} < 2^{k^2+k} e^{-k^2}$$

which is less than 1. Thus, with positive probability, every subset of X of cardinality t contains a set of $\mathcal{F}_t$.

The proof that, with positive probability, every set of cardinality t is disjoint from a set of $\mathcal{G}_t$ is similar; we sketch the details. First, let us note that

$$a \geq \frac{-\log\sqrt{2}}{\log(p)} \geq 1 + \frac{\log\sqrt{2}}{\log(1-p)} \tag{7}$$

since $p \leq 1/2$. For a fixed set $S \subseteq X$ of size $t = p(k^2+k)$, the probability that S is not disjoint from any set of $\mathcal{G}_t$ is at most

$$\left(1 - \prod_{i=0}^{(1-a)k-1}\left(\frac{(1-p)(k^2+k)-i}{k^2+k-i}\right)\right)^{|\mathcal{G}_t|} \leq \left(1 - \left(\frac{(1-p)(k^2+k)-k}{k^2}\right)^{(1-a)k}\right)^{|\mathcal{G}_t|}$$

$$= \left(1 - \left(1 - \frac{p}{(1-p)k}\right)^{(1-a)k} (1-p)^{(1-a)k}\right)^{|\mathcal{G}_t|} \tag{8}$$

Now, $\left(1 - \frac{p}{(1-p)k}\right) \geq e^{\frac{-2p}{(1-p)k}}$ for large enough k. So, $\left(1 - \frac{p}{(1-p)k}\right)^{(1-a)k} \geq e^{\frac{-2(1-a)p}{(1-p)}}$, which is at least e^{-2} since $a \geq 1/6$ and $\frac{1}{8} \leq p \leq \frac{1}{2}$. Therefore, the expression in (8) is at most

$$\left(1 - e^{-2}(1-p)^{(1-a)k}\right)^{|\mathcal{G}_t|} \leq e^{-e^{-2}(1-p)^{(1-a)k}|\mathcal{G}_t|} \leq e^{-e^{-2}(1-p)^{(1-a)k}\left(e^2 k^2 2^{k/2}\right)}$$
$$= e^{-(1-p)^{(1-a)k} k^2 2^{k/2}}.$$

By (7), we can bound the exponent by

$$(1-p)^{(1-a)k} k^2 2^{k/2} \geq (1-p)^{\left(\frac{-\log\sqrt{2}}{\log(1-p)}\right)k} k^2 2^{k/2} \geq k^2.$$

As with $\mathcal{F}_t$, we get that the expected number of sets of cardinality t which are not disjoint from a set of $\mathcal{G}_t$ is less than one. The result follows. □

Acknowledgements

The first two authors would like to thank Antonio Girão for many stimulating discussions, one of which lead to the discovery of Example 16.

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