# Polynomial bounds for chromatic number
# II. Excluding a star-forest

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**Abstract**

The Gyárfás-Sumner conjecture says that for every forest $H$, there is a function $f_H$ such that if $G$ is $H$-free then $\chi(G) \leq f_H(\omega(G))$ (where $\chi, \omega$ are the chromatic number and the clique number of $G$). Louis Esperet conjectured that, whenever such a statement holds, $f_H$ can be chosen to be a polynomial. The Gyárfás-Sumner conjecture is only known to be true for a modest set of forests $H$, and Esperet's conjecture is known in to be true for almost no forests. For instance, it is not known when $H$ is a five-vertex path. Here we prove Esperet's conjecture when each component of $H$ is a star.

## 1 Introduction

The Gyárfás-Sumner conjecture [6, 20] asserts:

**1.1 Conjecture:** *For every forest $H$, there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $H$-free graph $G$.*

(We use $\chi(G)$ and $\omega(G)$ to denote the chromatic number and the clique number of a graph $G$, and a graph is *$H$-free* if it has no induced subgraph isomorphic to $H$.) This remains open in general, though it has been proved for some very restricted families of trees (see, for example, [1, 7, 8, 9, 11, 13, 14]).

A class $\mathcal{C}$ of graphs is *$\chi$-bounded* if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G$ that is an induced subgraph of a member of $\mathcal{C}$ (see [15] for a survey). Thus the Gyárfás-Sumner conjecture asserts that, for every forest $H$, the class of all $H$-free graphs is $\chi$-bounded. Esperet [5] conjectured that every $\chi$-bounded class is *polynomially $\chi$-bounded*, that is, $f$ can be chosen to be a polynomial. Neither conjecture has been settled in general. There is a survey by Schiermeyer and Randerath [19] on related material.

In particular, what happens to Esperet's conjecture when we exclude a forest? For which forests $H$ can we show the following?

**1.2 Esperet's conjecture:** *There is a polynomial $f_H$ such that $\chi(G) \leq f_H(\omega(G))$ for every $H$-free graph $G$.*

Not for very many forests $H$, far fewer than the forests that we know satisfy 1.1. For instance, 1.2 is not known when $H = P_5$, the five-vertex path. (This case is of great interest, because it would imply the Erdős-Hajnal conjecture [3, 4, 2] for $P_5$, and the latter is currently the smallest open case of the Erdős-Hajnal conjecture.)

We remark that, if in 1.2 we replace $\omega(G)$ by $\tau(G)$, defined to be the maximum $t$ such that $G$ contains $K_{t,t}$ as a subgraph, then all forests satisfy the modified 1.2. More exactly, the following is shown in [16]:

**1.3** *For every forest $H$, there is a polynomial $f_H$ such that $\chi(G) \leq f_H(\tau(G))$ for every $H$-free graph $G$.*

One difficulty with 1.2 is that we cannot assume that $H$ is connected, or more exactly, knowing that each component of $H$ satisfies 1.2 does not seem to imply that $H$ itself satisfies 1.2. For instance, while $H = P_4$ satisfies 1.2, we do not know the same when $H$ is the disjoint union of two copies of $P_4$.

As far as we are aware, the only forests that were already known to satisfy 1.2 are those of the following three results, and their induced subgraphs (a *star* is a tree in which one vertex is adjacent to all the others):

**1.4** *The forest $H$ satisfies 1.2 if either:*

- *$H$ is the disjoint union of copies of $K_2$* (S. Wagon [21])*; or*
- *$H$ is the disjoint union of $H'$ and a copy of $K_2$, and $H'$ satisfies 1.2* (I. Schiermeyer [18])*; or*
- *$H$ is obtained from a star by subdividing one edge once* (X. Liu, J. Schroeder, Z. Wang and X. Yu [12])*.*

In the next paper of this series [17] we will show a strengthening of the third result of 1.4, that is, 1.2 is true when $H$ is a "double star", a tree with two internal vertices, the most general tree that does not contain a five-vertex path. Our main theorem in this paper is a strengthening of the second result of 1.4:

**1.5** *If $H$ is the disjoint union of $H'$ and a star, and $H'$ satisfies 1.2, then $H$ satisfies 1.2.*

A *star-forest* is a forest in which every component is a star. From 1.5 and the result of [17], we deduce

**1.6** *If $H'$ is a double star, and $H$ is the disjoint union of $H'$ and a star-forest, then $H$ satisfies 1.2.*

As far as we know (although it seems unlikely), these might be all the forests that satisfy 1.2.

## 2 The proof

We will need the following well-known version of Ramsey's theorem:

**2.1** *For $k \geq 1$ an integer, if a graph $G$ has no stable subset of size $k$, then*

$$|V(G)| \leq \omega(G)^{k-1} + \omega(G)^{k-2} + \cdots + \omega(G).$$

*Consequently $|V(G)| < \omega(G)^k$ if $\omega(G) > 1$.*

**Proof.** The claim holds for $k \leq 2$, so we assume that $k \geq 3$ and the result holds for $k-1$. Let $X$ be a clique of $G$ of cardinality $\omega(G)$, and for each $x \in X$ let $W_x$ be the set of vertices nonadjacent to $X$. From the inductive hypothesis, $|W_x| \leq \omega(G)^{k-2} + \cdots + \omega(G)$ for each $x$; but $V(G)$ is the union of the sets $W_x \cup \{x\}$ for $x \in X$, and the result follows by adding. This proves 2.1. ∎

If $X \subseteq V(G)$, we denote the subgraph induced on $X$ by $G[X]$. When we are working with a graph $G$ and its induced subgraphs, it is convenient to write $\chi(X)$ for $\chi(G[X])$. Now we prove 1.5, which we restate:

**2.2** *If $H'$ satisfies 1.2, and $H$ is the disjoint union of $H'$ and a star, then $H$ satisfies 1.2.*

**Proof.** $H$ is the disjoint union of $H'$ and some star $S$; let $S$ have $k+1$ vertices. Since $H'$ satisfies 1.2, and $\chi(G) = \omega(G)$ for all graphs with $\omega(G) \leq 1$, there exists $c'$ such that $\chi(G) \leq \omega(G)^{c'}$ for every $H'$-free graph $G$. Choose $c \geq k+2$ such that

$$x^c - (x-1)^c \geq 1 + x^{k+2} + x^{k(k+2)+c'}$$

for all $x \geq 2$ (it is easy to see that this is possible).

Let $G$ be an $H$-free graph, and write $\omega(G) = \omega$; we will show that $\chi(G) \leq \omega^c$ by induction on $\omega$. If $\omega = 1$ then $\chi(G) = 1$ as required, so we assume that $\omega \geq 2$. Let $n = \omega^{k+1}$. If every vertex of $G$ has degree less than $\omega^c$, then the result holds as we can colour greedily, so we assume that some vertex $v$ has degree at least $\omega^c$. Let $N$ be the set of all neighbours of $v$ in $G$. Let $X_1$ be the largest clique contained in $N$; let $X_2$ be the largest clique contained in $N \setminus X_1$; and in general, let $X_i$ be the largest clique contained in $N \setminus (X_1 \cup \cdots \cup X_{i-1})$. Since $|N| \geq \omega^c \geq n\omega$ (because $c \geq k+2$), it follows that $X_1, \ldots, X_n \neq \emptyset$. Let $X = X_1 \cup \cdots \cup X_n$, and $X_0 = N \setminus X$, and $t = |X_n|$. Thus $1 \leq t \leq \omega - 1$ (because $\omega(G[N]) < \omega$).

(1) $\chi(N \cup \{v\}) \leq t^c + n\omega$.

From the choice of $X_n$, it follows that the largest clique of $G[X_0]$ has cardinality at most $t < \omega$. From the inductive hypothesis, $\chi(X_0) \leq t^c$, and since $X \cup \{v\}$ has cardinality at most $n\omega$, it follows that $\chi(N \cup \{v\}) \leq t^c + n\omega$. This proves (1).

For each stable set $Y \subseteq X$ with $|Y| = k$, let $A_Y$ be the set of vertices in $V(G) \setminus (N \cup \{v\})$ that have no neighbour in $Y$. Let $A$ be the union of all the sets $A_Y$, and $B = V(G) \setminus (A \cup N \cup \{v\})$.

(2) $\chi(A) \leq (n\omega)^k \omega^{c'}$.

For each choice of $Y$, $G[A_Y]$ is $H'$-free (because $Y \cup \{v\}$ induces a copy of $S$ with no edges to $A_Y$), and so $\chi(A_Y) \leq \omega^{c'}$. Since there are at most $|X|^k \leq (n\omega)^k$ choices of $Y$, it follows that the union $A$ of all the sets $A_Y$ has chromatic number at most $(n\omega)^k \omega^{c'}$. This proves (2).

(3) *For each $b \in B$, $b$ has fewer than $\omega^k$ non-neighbours in $X$.*

Let $Z$ be the set of vertices in $X$ nonadjacent to $b$. Since $b \notin A$, $G[Z]$ has no stable set of cardinality $k$; and since it also has no clique of cardinality $\omega$, 2.1 implies that $|Z| \leq (\omega - 1)^k < \omega^k$. This proves (3).

(4) $\chi(B) \leq (\omega - t)^c$.

Suppose that $C \subseteq B$ is a clique with $|C| = \omega - t + 1$. For each $c \in C$, (3) implies that $c$ has a nonneighbour in fewer than $\omega^k$ of the cliques $X_1, \ldots, X_n$; and so fewer than $(\omega - t + 1)\omega^k$ of the cliques $X_1, \ldots, X_n$ contain a vertex with a non-neighbour in $C$. Since $(\omega - t + 1)\omega^k \leq \omega^{k+1} = n$, there exists $i \in \{1, \ldots, n\}$ such that every vertex in $X_i$ is adjacent to every vertex of $C$, and so $C \cup X_i$ is a clique. Since $|X_i| \geq |X_n| = t$, it follows that $|C \cup X_i| > \omega$, a contradiction. Thus there is no such clique $C$, and so $\omega(G[B]) \leq \omega - t$; and from the inductive hypothesis (since $t > 0$) it follows that $\chi(B) \leq (\omega - t)^c$. This proves (4).

From (1), (2), (4) we deduce that

$$\chi(G) \leq \chi(N \cup \{v\}) + \chi(A) + \chi(B) \leq t^c + n\omega + (n\omega)^k \omega^{c'} + (\omega - t)^c.$$

Since $1 \leq t \leq \omega - 1$ and $c \geq 1$, it follows that $t^c + (\omega - t)^c \leq 1 + (\omega - 1)^c$, and so

$$\chi(G) \leq 1 + \omega^{k+2} + (n\omega)^k \omega^{c'} + (\omega - 1)^c \leq \omega^c$$

from the choice of $c$ and since $\omega \geq 2$. This proves 1.5. ∎

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