Pure pairs. VII. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix

Alex Scott
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour[1]
Princeton University, Princeton, NJ 08544

Sophie Spirkl[2]
University of Waterloo, Ontario, Canada N2L3G1

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Abstract

For integer $n > 0$, let $f(n)$ be the number of rows of the largest all-0 or all-1 square submatrix of M, minimized over all $n \times n$ 0/1-matrices M. Thus $f(n) = O(\log n)$. But let us fix a matrix H, and define $f_H(n)$ to be the same, minimized over over all $n \times n$ 0/1-matrices M such that neither M nor its complement (that is, change all 0's to 1's and vice versa) contains H as a submatrix. It is known that $f_H(n) \geq \varepsilon n^c$, where $c, \varepsilon > 0$ are constants depending on H.

When can we take $c = 1$? If so, then one of H and its complement must be an acyclic matrix (that is, the corresponding bipartite graph is a forest). Korándi, Pach, and Tomon [4] conjectured the converse, that $f_H(n)$ is linear in n for every acyclic matrix H; and they proved it for certain matrices H with only two rows.

Their conjecture remains open, but we show $f_H(n) = n^{1-o(1)}$ for every acyclic matrix H; and indeed there is a 0/1-submatrix that is either $\Omega(n) \times n^{1-o(1)}$ or $n^{1-o(1)} \times \Omega(n)$.

1 Introduction

A 0/1-matrix can be regarded as a bipartite graph, with a distinguished bipartition (V_1, V_2) say, in which there are linear orders imposed on V_1 and on V_2. Submatrix containment corresponds, in graph theory terms, to induced subgraph containment, respecting the two bipartitions and preserving the linear orders. In two earlier papers [1, 5] (one with with Maria Chudnovsky), we proved some results about excluding induced subgraphs, in a general graph and in a bipartite graph respectively. Now we impose orders on the vertex sets, and only consider induced subgraph containment that respects the orders; and we ask how far our earlier theorems remain true under this much weaker hypothesis.

In this paper, all graphs are finite and with no loops or parallel edges. Two disjoint sets are *complete* to each other if every vertex of the first is adjacent to every vertex of the second, and *anticomplete* if there are no edges between them. A pair (Z_1, Z_2) of subsets of $V(G)$ is *pure* if Z_1 is either complete or anticomplete to Z_2. Let us state the earlier theorems that we want to extend to ordered graphs. First, we proved the following, with Chudnovsky [1]:

1.1 *For every forest T, there exists $\varepsilon > 0$ such that if G is a graph with $n \geq 2$ vertices, and no induced subgraph is isomorphic to T or its complement, then there is a pure pair (Z_1, Z_2) of subsets of $V(G)$ with $|Z_1|, |Z_2| \geq \varepsilon n$.*

This theorem characterizes forests: if T is a graph that is not a forest or the complement of one, then there is no $\varepsilon > 0$ as in 1.1.

Second, we proved a similar theorem about bipartite graphs, but for this we need some more definitions. A *bigraph* is a graph together with a bipartition $(V_1(G), V_2(G))$ of G. A bigraph G *contains* a bigraph H if there is an isomorphism from H to an induced subgraph of G that maps $V_i(H)$ into $V_i(G)$ for $i = 1, 2$. The *bicomplement* of a bigraph H is the bigraph obtained by reversing the adjacency of v_1, v_2 for all $v_i \in V_i(G)$ $(i = 1, 2)$. We proved the following in [5]:

1.2 *For every forest bigraph T, there exists $\varepsilon > 0$ such that if G is a bigraph that does not contain T or its bicomplement, then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon |V_i(G)|$ for $i = 1, 2$.*

Again, this characterizes forests, in that if H is a bigraph that is not a forest or the bicomplement of a forest, there is no $\varepsilon > 0$ as in 1.2.

What if we impose an order on the vertex set, and ask for the induced subgraph containment to respect the order? Let us say an *ordered graph* is a graph with a linear order on its vertex set. Every induced subgraph inherits an order on its vertex set in the natural way: let us say an ordered graph G *contains* an ordered graph H if H is isomorphic to an induced subgraph H' of G, where the isomorphism carries the order on $V(H)$ to the inherited order on $V(H')$. One could ask for an analogue of 1.1 for ordered graphs, but it is false. Fox [3] showed:

1.3 *Let H be the three-vertex path with vertices h_1, h_2, h_3 in order, and make H an ordered graph using the same order. For all sufficiently large n, there is an ordered graph G with n vertices, that does not contain H, and such that there do not exist two disjoint subsets of $V(G)$, both of size at least $n/\log(n)$, and complete or anticomplete.*

To deduce that 1.1 does not extend to ordered graphs, let T be an ordered tree such that both T and its bicomplement contain H, and use the construction from 1.3. But something like 1.1 *is* true: we proved in [6] that:

1.4 *For every ordered forest T and all $c > 0$, there exists $\varepsilon > 0$ such that if G is an ordered graph with $|G| \geq 2$ that does not contain T or its complement, there is a pure pair (Z_1, Z_2) in G with $|Z_1|, |Z_2| \geq \varepsilon |G|^{1-c}$.*

Perhaps the situation is better for ordered bipartite graphs: certainly we are not so well-supplied with counterexamples, and there are some positive results about ordered bipartite graphs, proved recently by Korándi, Pach, and Tomon [4]. Let us say an *ordered bigraph* is a bigraph with linear orders on $V_1(G)$ and on $V_2(G)$. This is just a 0/1 matrix in disguise, but graph theory language is convenient for us. (Note that we are not giving a linear order of $V(G)$: that is much too strong and trivially does not work.) An ordered bigraph G *contains* an ordered bigraph H if there is an induced subgraph H' of G and an isomorphism from H to H' mapping $V_i(H)$ to $V_i(H')$ and mapping the order on $V_i(H)$ to the inherited order on $V_i(H')$, for $i = 1, 2$. (In matrix language, this is just submatrix containment.) Korándi, Pach, and Tomon [4] showed:

1.5 *Let H be an ordered bigraph with $|V_1(H)| \leq 2$, such that either*

- *$|V_2(H)| \leq 2$ and both H and its bicomplement are forests, or*
- *every vertex in $V_2(H)$ has degree exactly one.*

Then there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.

In both cases of 1.5, the bigraph H is a forest and so is its bicomplement. Korándi, Pach, and Tomon asked which other ordered bigraphs H satisfy the conclusion of 1.5. They observed that every such bigraph must be a forest and the bicomplement of a forest, and conjectured that this was sufficient as well as necessary, that is:

1.6 Conjecture: *Let H be an ordered bigraph such that both H and its bicomplement are forests. Then there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)|, |V_2(G)| \geq n$; then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.*

We have not been able to decide this conjecture, and indeed have not even been able to prove it for the forest H consisting of a five-vertex path and an isolated vertex with $|V_1(H)| = |V_2(H)|$ (under any ordering of $V_1(H)$ and $V_2(H)$). But we will prove in 1.12 that, for a much more general class of ordered bigraphs, it is possible to find pairs of almost linear size.

Korándi, Pach, and Tomon also proposed an even stronger conjecture (to see that it implies 1.6, let H be as in 1.6, let H' be an ordered forest that contains both H and its bicomplement, and apply 1.7 for H'):

1.7 Conjecture: *For every ordered forest bigraph H, there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph that does not contain H or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$: then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq \varepsilon n$ for $i = 1, 2$.*

This seems a natural extension of 1.6, in analogy with 1.1 and 1.2; but, for what it is worth, our guess is that 1.7 is false. Perhaps $n/\operatorname{polylog}(n)$ might be true?

There is another result of Korándi, Pach, and Tomon, in the same paper [4]:

1.8 *Let H be an ordered forest bigraph such that $|V_1(H)| = 2$ and $|V_2(H)| = k$. For every $\tau > 0$, there exists $\delta > 0$ with the following property. Let G be an ordered bigraph that does not contain H, with $|V_1(G)| = |V_2(G)| = n$, and such that its bicomplement has at least τn^2 edges. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$ with $|Z_1| \geq \delta n 2^{-(1+o(1))(\log\log(\delta n))^k}$ and $|Z_2| \geq \delta n$, such that Z_1, Z_2 are anticomplete.*

So, $|Z_1|$ is not quite linear, but there is more of significance. There is nothing here about forbidding G to contain the bicomplement of a forest, since the bicomplement of H need not be a forest; and the "Z_1 complete to Z_2" outcome is gone. In compensation they have the assumption that the bicomplement of G is not too sparse.

Our objective in this paper is essentially to generalize 1.8 to all ordered forest bigraphs H. We will give two results. Both prove the existence of anticomplete sets Z_1, Z_2 of cardinalities at least $n^{1-o(1)}$, but neither implies the other. One result (the second) gives a linear lower bound for one of the sets and a sublinear bound for the other; and the other result (the first) gives a sublinear (but better) bound for both sets.

Every forest is an induced subgraph of a tree, so we will assume H is a tree, for convenience. The *radius* of a tree T is the minimum r such that for some vertex v, every vertex of T can be joined to v by a path with at most r edges. In the first half of the paper, we will show:

1.9 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. Let G be a bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq n$, such that every vertex of G has degree at most $n/(4t^2)$. Choose K such that $t^{K^r} = n$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq nt^{-5K^{r-1}}$, such that Z_1, Z_2 are anticomplete.*

In the second half of the paper we will show:

1.10 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\varepsilon > 0$ with the following property. Let G be a bigraph not containing T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \varepsilon|V_1(G)|$ and $|Z_2| \geq \varepsilon|V_2(G)|^{1-c}$, or with $|Z_1| \geq \varepsilon|V_1(G)|^{1-c}$ and $|Z_2| \geq \varepsilon|V_2(G)|$, such that Z_1, Z_2 are anticomplete.*

The first result, 1.9, implies:

1.11 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. For all $\tau > 0$ there exists $\delta > 0$ with the following property. Let G be a bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq n$, and with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Choose K such that $t^{K^r} = n$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, with $|Z_1|, |Z_2| \geq \delta nt^{-5K^{r-1}}$, such that Z_1, Z_2 are anticomplete.*

It also implies:

1.12 *Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. Let G be an ordered bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. Choose K such that $t^{K^r} = n$. Then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ and $|Z_i| \geq (16t^2)^{-t} nt^{-5K^{r-1}}$ for $i = 1, 2$.*

Similarly, the second result, 1.10, implies:

1.13 *Let T be an ordered tree bigraph. For all $c, \tau > 0$ there exists $\delta > 0$ with the following property. Let G be a bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq n$, and with at most $(1-\tau)|V_1(G)|\cdot|V_2(G)|$ edges. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \delta n$ and $|Z_2| \geq \delta n^{1-c}$, or with $|Z_1| \geq \delta n^{1-c}$ and $|Z_2| \geq \delta n$, such that Z_1, Z_2 are anticomplete.*

It also implies:

1.14 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\delta > 0$ with the following property. Let G be an ordered bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. Then there is a pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \delta n$ and $|Z_2| \geq \delta n^{1-c}$, or with $|Z_1| \geq \delta n^{1-c}$ and $|Z_2| \geq \delta n$.*

The last six theorems all imply that $|Z_1|, |Z_2| = n^{1-o(1)}$. It is easy to show, with a random graph argument, that this characterizes forests and their bicomplements. Indeed, if T is an (unordered) bigraph such that neither T nor its bicomplement are forests, then the conclusions of 1.12 and 1.14 (with "ordered" deleted) are far from true; and therefore for ordered bigraphs they are at least as far from true. More exactly, here is a standard example:

1.15 *Let T be a bigraph, such that both T and its bicomplement have a cycle of length at most g, and let $c > 1 - 1/g$. Then there is a bigraph G, with $|V_1(G)|, |V_2(G)| = n$, which contains neither T nor its bicomplement, and such that $\min(|Z_1|, |Z_2|) \leq n^c$ for every pure pair (Z_1, Z_2) with $Z_i \subseteq V_i(G)$ for $i = 1, 2$.*

Proof. Take n large, and let V_1, V_2 be disjoint sets of cardinality $2n$; and for each $v_1 \in V_1$ and $v_2 \in V_2$, make v_1, v_2 adjacent independently, with probability $\frac{1}{2}n^{1/g-1}$. Then with high probability, there are fewer than $n/2$ cycles of length at most g, and no pure pair Z_1, Z_2 with $Z_i \subseteq V_i$ and $|Z_i| \geq n^c$ for $i = 1, 2$. By deleting half of V_1, V_2 appropriately, we obtain a bigraph G with girth more than g, which therefore does not contain T or its bicomplement. This proves 1.15. ∎

2 Reduction to the sparse case

In this section we do two things. First, we deduce 1.12 assuming 1.9, and will prove the latter in the next section; and second, we deduce 1.11 and 1.13 from 1.9 and 1.10.

We need the following lemma, a version of a theorem of Erdős, Hajnal and Pach [2] adapted for ordered bipartite graphs. (It is similar to a result of [5] but with different parameters).

2.1 *Let H be an ordered bigraph, let $|V(H_i)| = h_i$ for $i = 1, 2$, let $0 < \varepsilon < 1/8$, let $d = \lceil 1/(4\varepsilon) \rceil$, and let $m_1, m_2 > 0$ be integers. Let G be an ordered bigraph not containing H, with $|V_1(G)| \geq h_1 d^{h_2} m_1$ and $|V_2(G)| \geq 2h_1h_2m_2$. Then there are subsets $Y_i \subseteq V_i(G)$ with $|Y_i| = m_i$ for $i = 1, 2$, such that either*

- *every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1, or*
- *every vertex in Y_1 has at most $\varepsilon|Y_2|$ non-neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ non-neighbours in Y_1.*

Proof. Divide $V_1(G)$ into h_1 disjoint intervals, each of cardinality at least $m_1 d^{h_2}$, numbered B_u ($u \in V_1(H)$) in order. Divide $V_2(G)$ into disjoint intervals B_u ($u \in V_2(H)$) of cardinality at least $m_2 h_1$. Choose $W \subseteq V_2(H)$ maximal such that for each $v \in W$ there exists $x_v \in B_v$, and for each $u \in V_1(H)$ there exist $Q_u \subseteq B_u$, with the following properties:

- $|Q_u| \geq m_1 d^{h_2 - |W|}$ for each $u \in V_1(H)$;
- for each $u \in V_1(H)$ and each $v \in W$, if u, v are H-adjacent then x_v is complete to Q_u, and if u, v are not H-adjacent then x_v is anticomplete to Q_u.

This is possible since we may take $W = \emptyset$ and $Q_u = B_u$ for each $u \in V_1(H)$. Since G does not contain H, it follows that $W \neq V_2(H)$. Choose $v \in V_2(H) \setminus W$. Say $u \in V_1(H)$ is a *problem* for $x \in B_v$ if either u, v are H-adjacent and x has fewer than $|Q_u|/d$ neighbours in Q_u, or u, v are not H-adjacent and x has fewer than $|Q_u|/d$ non-neighbours in Q_u. From the maximality of W, for each $x \in B_v$ there exists $u \in V_1(H)$ that is a problem for x. Since there are only h_1 possible problems, there exist $u \in V_1(H)$, and $C \subseteq B_v$ with $|C| \geq |B_v|/h_1$, such that for every $x \in C$, u is a problem for x. By moving to the bicomplement if necessary, we may assume that u, v are H-adjacent; and so every vertex in C has fewer than $|Q_u|/d$ neighbours in Q_u. Since $|Q_u| \geq m_1 d^{h_2 - |W|} \geq m_1 d \geq 2m_1$ and $|C| \geq |B_v|/h_1 \geq 2m_2$, it follows by averaging that there are subsets $X_1 \subseteq Q_u$ and $X_2 \subseteq C$, of cardinality exactly $2m_1, 2m_2$ respectively, such that there are at most $|X_1| \cdot |X_2|/d = 4m_1 m_2/d$ edges joining them. Let Y_1 be the set of the m_1 vertices in X_1 that have fewest neighbours in X_2; then they each have at most $4m_2/d$ neighbours in X_2. Define Y_2 similarly; then $|Y_i| = m_i$ for $i = 1, 2$, and every vertex in Y_1 has at most $4m_2/d \leq \varepsilon m_2$ neighbours in Y_2 and vice versa. This proves 2.1. ∎

Proof of 1.12, assuming 1.9. Let T be an ordered tree bigraph, of radius r, and with t vertices. Let $\varepsilon = 1/(4t^2)$, and let $d = \lceil 1/(4\varepsilon) \rceil$. Let $c = (16t^2)^{-t}$. Let G be a bigraph not containing T or its bicomplement, with $|V_1(G)|, |V_2(G)| \geq n$. We may assume that $cnt^{-5K^{r-1}} > 1$, for otherwise the result is true, taking $|Z_1| = |Z_2| = 1$. Hence $2cn \geq 2t^{5K^{r-1}} \geq 1$. Let m be the largest integer such that $m \leq 2cn$. Thus $m \geq cn$, since $2cn \geq 1$.

Let $|V_i(T)| = h_i$ for $i = 1, 2$. Now $n \geq \max(h_1 d^{h_2} m, 2h_1 h_2 m)$. By 2.1, and moving to the bicomplement if necessary, we may assume that there exist $Y_i \subseteq V_i(G)$ with $|Y_i| = m$ for $i = 1, 2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1. Choose J with $m = t^{J^r}$. By 1.9 applied to the ordered bigraph induced on $Y_1 \cup Y_2$, there exist $Z_i \subseteq Y_i$ with $|Z_1|, |Z_2| \geq mt^{-5J^{r-1}}$, such that Z_1, Z_2 are anticomplete, where $m = t^{J^r}$. Since $J \leq K$, it follows that $|Z_1|, |Z_2| \geq cnt^{-5K^{r-1}}$. This proves 1.12. ∎

The proof that 1.10 implies 1.14 is similar and we omit it.

The result 1.8 of Korándi, Pach, and Tomon [4] has as a hypothesis that the bicomplement of G has at least τn^2 edges. This is apparently much weaker than the hypothesis that every vertex of G has degree at most εn, but in fact the "not very dense" hypothesis is as good as the "very sparse" hypothesis, because of the next result, proved in [5].

2.2 *For all $c, \varepsilon, \tau > 0$ with $\varepsilon < \tau$, there exists $\delta > 0$ with the following property. Let G be a bigraph with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges and with $V_1(G), V_2(G) \neq \emptyset$. Then there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta|V_i(G)|$ for $i = 1, 2$, such that there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1, 2$.*

We deduce:

2.3 *For every ordered bigraph H, and for all $\varepsilon, \tau > 0$, there exists $\delta > 0$ with the following property. Let G be an ordered bigraph not containing H, with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. Then there exist $Y_i \subseteq V_i(G)$ with $|Y_i| \geq \delta|V_i(G)|$ for $i = 1,2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1.*

Proof. We may assume that $\varepsilon < 1/8$ and $\varepsilon < \tau$, by reducing ε, and $|V_1(H)|, |V_2(H)| \neq \emptyset$, by adding vertices to H. Let $h_i = |V(H_i)|$ for $i = 1,2$, let let $d = \lceil 1/(4\varepsilon) \rceil$, and let $1/c = \max(h_1 d^{h_2}, 2h_1h_2)$. Choose δ' such that 2.2 holds with δ replaced by δ', and let $\delta = c\delta'$. Now let G be an ordered bigraph not containing H, with at most $(1-\tau)|V_1(G)| \cdot |V_2(G)|$ edges. We may assume that $V_1(G), V_2(G) \neq \emptyset$. By 2.2, there exist $Z_i \subseteq V_i(G)$ with $|Z_i| \geq \delta'|V_i(G)|$ for $i = 1,2$, such that there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2 for all subsets $Y_i \subseteq Z_i$ with $|Y_i| \geq c|Z_i|$ for $i = 1,2$. By 2.1, applied to the ordered sub-bigraph induced on $Z_1 \cup Z_2$, there exist $Y_i \subseteq Z_i$ with $|Y_i| \geq c|V_i(G)|$ for $i = 1,2$, such that either

- every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1, or
- every vertex in Y_1 has at most $\varepsilon|Y_2|$ non-neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ non-neighbours in Y_1.

Since there are fewer than $(1-\varepsilon)|Y_1| \cdot |Y_2|$ edges between Y_1, Y_2, the second is impossible, and so the first holds. Then for $i = 1,2$, $|Y_i| \geq c|Z_i| \geq c\delta'|V_i(G)| = \delta|V_i(G)|$. This proves 2.3. ∎

Proof of 1.11, assuming 1.9. Let T be an ordered tree bigraph, of radius r, and with $t \geq 2$ vertices, and let $\tau > 0$. Let $\varepsilon = 1/(4t^2)$, and choose $\delta > 0$ as in 2.2, with H replaced by T.

Let G be a bigraph not containing T, with $|V_1(G)|, |V_2(G)| \geq n$, such that the bicomplement of G has at least $\tau|V_1(G)| \cdot |V_2(G)|$ edges. From the choice of δ, there exist $Y_i \subseteq V_i(G)$ with $|Y_i| \geq \delta|V_i(G)|$ for $i = 1,2$, such that every vertex in Y_1 has at most $\varepsilon|Y_2|$ neighbours in Y_2, and every vertex in Y_2 has at most $\varepsilon|Y_1|$ neighbours in Y_1. By 1.9 applied to the sub-bigraph $G[Y_1 \cup Y_2]$, there are subsets $Z_i \subseteq Y_i$ for $i = 1,2$, with $|Z_1|, |Z_2| \geq \delta n t^{-5k^{r-1}}$, such that Z_1, Z_2 are anticomplete, where k satisfies $t^{k^r} = \delta n$. Choose K such that $t^{K^r} = n$; thus $K \geq k$, and so $|Z_1|, |Z_2| \geq \delta n t^{-5K^{r-1}}$. This proves 1.11. ∎

The proof that 1.10 implies 1.13 is similar and we omit it.

3 Proof of the first main theorem

In this section we prove 1.9, which we restate:

3.1 *Let T be a ordered tree bigraph, of radius r, and with $t \geq 2$ vertices. Let G be a bigraph with $|V_1(G)|, |V_2(G)| \geq n$, that does not contain T, and such that every vertex has degree at most $n/(4t^2)$. Choose K such that $t^{K^r} = n$. Then there are two anticomplete subsets $Z_i \subseteq V_i(G)$ for $i = 1,2$, with $|Z_1|, |Z_2| \geq n t^{-5K^{r-1}}$.*

Proof. Let $\varepsilon = 1/(4t^2)$. Since $\varepsilon < 1$, G is not complete bipartite, and so we may assume that $t^{-5K^{r-1}}n > 1$ (or else the theorem holds); that is, $t^{5K^{r-1}} < t^{K^r}$. So $K \geq 5$, and $n \geq t^{5^r}$.

If $r = 1$, then T has a vertex of degree d say, and all other vertices of T are neighbours of d. Let this vertex belong to $V_1(T)$ say. Since G does not contain T, all vertices in $V_1(G)$ have degree at most $d-1$. Choose a set Z_1 of at most n/d vertices in $V_1(G)$; then the set of vertices with neighbours in X has cardinality at most $(d-1)|X| \leq (d-1)n/d$, and so there is a set of at least n/d vertices in $V_2(G)$ anticomplete to Z_1. So to prove 1.9 in this case, we just have to check that $\lfloor n/d \rfloor \geq nt^{-5K^{r-1}}$. But $t = d+1$ and $n \geq t$ (because $n \geq t^{5^r}$), so $\lfloor n/d \rfloor \geq n/(2(t-1))$; and now it remains to check that $n/(2(t-1)) \geq nt^{-5K^{r-1}}$, which is clear. Thus we may assume that $r \geq 2$, and so $t \geq 4$.

Choose a real number $x \geq 0$ with $x \leq K^{r-1}$, maximum such that there exist $A_1 \subseteq V_1(G)$ and $A_2 \subseteq V_2(G)$ with the properties that

- $|A_1|, |A_2| \geq nt^{-x}$;
- every vertex in A_1 has at most εnt^{-Kx} neighbours in A_2 and vice versa.

This is possible since we may take $x = 0$ and $A_i = V_i(G)$ for $i = 1, 2$. Let A_1, A_2 be as above. Let $d = \varepsilon nt^{-Kx}$. Since $|A_1| \geq nt^{-x} \geq nt^{-5K^{r-1}}$, we may assume that A_1 is not anticomplete to A_2 (or else the theorem holds); so $d \geq 1$. For $1 \leq s \leq r-1$, let $k_s = 4(K^{s-1} + K^{s-2} + \cdots + 1)$.

(1) $x < K^{r-1} - k_{r-1}$.

Suppose not. Since $|A_1| \geq t^{-x}n \geq t^{-K^{r-1}}n \geq 2t^{-5K^{r-1}}n$, there exists a set $X \subseteq A_1$ of cardinality $\lceil t^{-5K^{r-1}}n \rceil \leq 2t^{-5K^{r-1}}n$. The union of the neighbours in A_2 of vertices in X has cardinality at most $(2t^{-5K^{r-1}}n)(\varepsilon nt^{-Kx})$. Since $|A_2|/2 \geq t^{-5K^{r-1}}n$, there are fewer than $|A_2|/2$ vertices in A_2 anticomplete to X; so $(2t^{-5K^{r-1}}n)(\varepsilon nt^{-Kx}) \geq |A_2|/2$, and hence $4\varepsilon t^{-5K^{r-1}}nt^{-Kx} \geq t^{-x}$. Consequently $4\varepsilon t^{-5K^{r-1}}n \geq t^{(K-1)x}$. But

$$(K-1)x \geq (K-1)(K^{r-1} - k_{r-1}) = K^r - 5K^{r-1} + 4,$$

and so

$$4\varepsilon t^{-5K^{r-1}}n \geq t^{K^r - 5K^{r-1} + 4}.$$

Hence $n \geq t^{K^r+4}$, a contradiction. This proves (1).

Since $x < K^{r-1} - 4$, and $n > t^{5K^{r-1}}$, it follows that $nt^{-x-1} \geq t$. Hence

$$|A_i| \geq nt^{-x} \geq (t-1)nt^{-x-1} + t$$

for $i = 1, 2$. Since $|V(T_1)| \leq t-1$, it follows that we may choose $|V_1(T)|$ disjoint blocks B_u ($u \in V_1(T)$), all intervals of A_1, and of cardinality $\lceil nt^{-x-1} \rceil$, numbered in order. Partition A_2 into blocks B_v ($v \in V_2(T)$) similarly. For $1 \leq s \leq r-1$, let $p_s = dt^{-Kk_s}$. Let $p_r = 1$. For $2 \leq s \leq r$, let $f_s = dt^2/p_{s-1} = t^{Kk_{s-1}+2}$. So $f_s \geq t^{4K+2}$.

Since T has radius at most r, there is a vertex $v_0 \in V(T)$ such that every vertex of T can be joined to v_0 by a path of length at most r. For $1 \leq s \leq r$ let T_s be the subtree of T induced on the vertices with distance at most s from t_0. So $V(T_0) = \{t_0\}$, and $T_r = T$. Let L_s be the set of vertices with distance exactly s from v_0. For $2 \leq s \leq r$, and each edge uv of T with $u \in L_{s-1}$ and $v \in L_s$, choose $X_{uv} \subseteq B_u$ and $Y_{uv} \subseteq B_v$ satisfying the following conditions:

- every vertex in X_{uv} has fewer than p_s neighbours in $B_v \setminus Y_{uv}$;
- $|Y_{uv}| \le f_s|X_{uv}|$ and $|Y_{uv}| \le |B_v|/2$; and
- subject to these conditions, Y_{uv} is maximal.

This is possible since we could take $X(uv) = Y(uv) = \emptyset$ to satify the first two bullets.

(2) *For $2 \le s \le r$, and each edge uv of T with $u \in L_{s-1}$ and $v \in L_s$, we may assume that $|X_{uv}| = \lceil |Y_{uv}|/f_s \rceil$, and $|X_{uv}| \le |B_u|/(2t)$, and $|Y_{uv}| \le |B_v|/2 - td$.*

We may assume that $|X_{uv}| = \lceil |Y_{uv}|/f_s \rceil$, by removing elements from X_{uv} if necessary. Suppose first that $s = r$. Then X_{uv} is anticomplete to $B_v \setminus Y_{uv}$ (because $p_r = 1$), and so either $|X_{uv}| < t^{-5K^{r-1}}n$ or $|B_v \setminus Y_{uv}| < t^{-5K^{r-1}}n$. The second implies that $|B_v| < 2t^{-5K^{r-1}}n$ (since $|Y_{uv}| \le |B_v|/2$), and so $nt^{-x-1} < 2t^{-5K^{r-1}}n$, that is, $t^{5K^{r-1}-x-1} < 2$, a contradiction. So $|X_{uv}| < t^{-5K^{r-1}}n$. Since $t^{-5K^{r-1}}n \le nt^{-x-1}/(2t)$, it follows that $|X_{uv}| \le |B_u|/(2t)$. Also, $|Y_{uv}| \le f_r t^{-5K^{r-1}}n$. We claim that $f_r t^{-5K^{r-1}}n \le |B_v|/2 - td$. Suppose not; then either $f_r t^{-5K^{r-1}}n > |B_v|/4$ or $td > |B_v|/4$. The first implies that $t^{Kk_{r-1}+2}t^{-5K^{r-1}}n > nt^{-x-1}/4$, and so $t^{Kk_{r-1}+x+4}t^{-5K^{r-1}} > 1$. Hence $Kk_{r-1} + x + 4 - 5K^{r-1} > 0$. But $x \le K^{r-1} - k_{r-1}$ by (1), so

$$Kk_{r-1} + K^{r-1} - k_{r-1} + 4 - 5K^{r-1} > 0,$$

a contradiction (in fact, the left side sums to zero). The second implies that $4\varepsilon t^{-Kx} > t^{-x-2}$ and so $K < 1$ since $4\varepsilon = t^{-2}$, a contradiction. Thus when $s = r$, all three statements of (2) hold.

Now we assume that $2 \le s < r$. We have $|X_{uv}| \le |Y_{uv}|/f_s + 1 \le |B_v|/(2f_s) + 1 \le |B_v|/4$, because $f_s \ge t^{4K+2} \ge 4$ and $|B_v| \ge 8$. There are at most $|X_{uv}|p_s$ edges between X_{uv} and $B_v \setminus Y_{uv}$; and so at most $|X_{uv}|$ vertices in $B_v \setminus Y_{uv}$ have at least p_s neighbours in X_{uv}. Since $|B_v \setminus Y_{uv}| \ge |B_v|/2 \ge 2|X_{uv}|$, the $|X_{uv}|$ vertices in $B_v \setminus Y_{uv}$ with fewest neighbours in X_{uv} each have fewer than p_s neighbours in X_{uv}. Since $p_s = dt^{-Kk_s}$, and $k_s \le 4K^{r-2} + 4K^{r-3} + \cdots + 4$, the maximality of x and (1) imply that $|X_{uv}| < nt^{-x-k_s}$. Hence $|X_{uv}| \le |B_u|/(2t)$, because $nt^{-x-k_s} \le nt^{-x-1}/(2t)$ (because $k_s \ge 3$). Thus the second claim holds. For the third claim, since $|Y_{uv}| \le f_s|X_{uv}|$, it suffices to show that $f_s|X_{uv}| \le |B_v|/2 - td$, and to prove this, it suffices to show that $f_s|X_{uv}| \le |B_v|/4$ and $td \le |B_v|/4$. To show the first, it suffices to show that $nt^{-x-k_s}f_s \le |B_v|/t$, that is, $nt^{-x-k_s}t^{Kk_{s-1}+2} \le nt^{-x-2}$, which simplifies to $4 + Kk_{s-1} \le k_s$, and this holds with equality. To show that $td \le |B_v|/4$, it suffices to show that $t\varepsilon nt^{-Kx} \le nt^{-x-1}/4$, which simplifies to $4\varepsilon t^{2+x-Kx} \le 1$; and this is true since $4\varepsilon t^2 \le 1$, and $K \ge 1$. This proves (2).

For $2 \le s \le r$, and each $u \in L_{s-1}$, let X_u be the union of the sets X_{uv} over all $v \in L_s$ that are T-adjacent to u. Then:

(3) *For $2 \le s \le r$, and each $u \in L_{s-1}$, $|X_u| \le |B_u|/2$.*

For each $v \in L_s$ that is T-adjacent to u, $|X_{uv}| \le |B_u|/(2t)$ by (2), and the claim follows. This proves (3).

Let $P_{v_0} = B_{v_0}$. For $s = 1, \ldots, r-1$ we will choose $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, and $y_v \in P_v$ for each $v \in L_{s-1}$, satisfying the following conditions:

- for all distinct $u, v \in V(T_{s-1})$, u, v are T-adjacent if and only if y_u, y_v are G-adjacent;
- for all $u \in V(T_{s-1})$ and $v \in L_s$, and all $y \in P_v$, u, v are T-adjacent if and only if y_u, y are G-adjacent;
- for each $v \in L_s$, $|P_v| \geq p_s$.

First let us assume $s = 1 < r$. If there exists $y \in B_{v_0}$ with at least p_1 neighbours in $B_v \setminus X_v$ for each v such that v_0, v are T-adjacent, then we may set $y_{v_0} = y$; so we assume there is no such y. Consequently there is a T-neighbour v of v_0 such that for at least $|B_{v_0}|/t \geq nt^{-x-2}$ vertices $y \in B_{v_0}$, y has fewer than p_1 neighbours in $B_v \setminus X_v$. Choose a set X of exactly $\lceil nt^{-x-2} \rceil$ such vertices y. Since $|X| \leq nt^{-x-2} + 1 \leq 2nt^{-x-2}$, and $|B_{v_0}| \geq 8nt^{-x-2}$ (because $t \geq 4$), it follows that $|X| \leq |B_{v_0}|/4$. Since $|X_v| \leq |B_v|/2$, it follows that $|B_v \setminus X_v| \geq 2|X|$, and so at least $|X|$ vertices in $B_v \setminus X_v$ have at most p_1 neighbours in X. Since $p_1 = dt^{-Kk_1} = dt^{-4K}$, the maximality of x implies that $|X| < nt^{-x-4}$, a contradiction. So we can satisfy the three bullets above when $s = 1 \leq r - 1$.

Suppose that $2 \leq s \leq r$, and we have chosen $P_v \subseteq B_v \setminus X_v$ for each $v \in L_{s-1}$ and $y_v \in P_v$ for each $v \in V(T_{s-2})$. We must define $P_v \subseteq B_v \setminus X_v$ for each $v \in L_s$, and $y_v \in P_v$ for each $v \in L_{s-1}$, satisfying the bullets above. From the symmetry we may assume that $L_s \subseteq V_1(T)$.

Let C be the set of vertices in A_2 that are equal or adjacent to y_v for some $v \in V(T_{s-2})$. Let $y_u \in P_u$ for each $u \in L_{s-1}$; we call $(y_u : u \in L_{s-1})$ a *transversal*. A transversal $(y_u : u \in L_{s-1})$ is *valid* if for each edge uv of T with $u \in L_{s-1}$ and $v \in L_s$, there are at least p_s vertices in $B_v \setminus C$ that are adjacent to y_u and that have no other neighbour in $\{y_{u'} :\ u' \in L_{s-1}\}$.

(4) *There is a valid transversal.*

Suppose not. Let E be the set of ordered pairs (u, v) such that uv is an edge of T with $u \in L_{s-1}$ and $v \in L_s$. Then for every transversal $(y_u : u \in L_{s-1})$, there exists $(u, v) \in E$ such that there are fewer than p_s vertices in $B_v \setminus C$ that are adjacent to y_u and that have no other neighbour in $\{y_{u'} :\ u' \in L_{s-1}\}$. Call (u, v) a *problem* for the transversal $(y_u : u \in L_{s-1})$. Since $|E| = |L_s|$, there are only $|L_s|$ possible problems, and so there exists $(u, v) \in E$ that is a problem for at least a fraction $1/|L_s|$ of all transversals. Hence there exist a subset $X \subseteq P_u$ with $|X| \geq |P_u|/|L_s| \geq p_{s-1}/t$, and a choice of $y_{u'} \in P_{u'}$ for each $u' \in L_{s-1} \setminus \{u\}$, such that for all $y_u \in X$, (u, v) is a problem for the transversal $(y_{u'} : u' \in L_{s-1})$. Let C' be the set of vertices in A_2 that are adjacent to a vertex in $(y_{u'} : u' \in L_{s-1} \setminus \{u\})$. Since every vertex in $C \cup C'$ has a neighbour y_w for some $w \in V(T_{s-1}) \setminus \{u\}$, it follows that $|C \cup C'| \leq dt$. Every vertex in X has fewer than p_s neighbours in $B_v \setminus (C \cup C')$. If $(C \cup C') \cap B_v \not\subseteq Y_{uv}$ let $Y = (C \cup C') \cap B_v$, and otherwise let Y be a singleton subset of $B_v \setminus Y_{uv}$. Thus $|Y| \leq dt \leq f_s|X|$, since $dt \geq 1$ by (1). Consequently $|Y \cup Y_{uv}| \leq f_s|X \cup X_{uv}|$, since $X \cap X_{uv} = \emptyset$; and since $|C \cup C' \cup Y_{uv}| \leq |B_v|/2$ (because $|C \cup C'| \leq dt$ and by (2)), and every vertex in $X \cup X_{uv}$ has fewer than p_s neighbours in $B_v \setminus (Y \cup Y_{uv})$, this contradicts the maximality of Y_{uv}. This proves (4).

From (4), the inductive definition of y_v ($v \in V(T_{r-1})$) and P_v ($v \in V(T)$) is complete. For each $v \in L_s$, choose $y_v \in P_v$. Then the map sending each $v \in V(T)$ to y_v is an ordered parity-preserving isomorphism of T to an induced subgraph of G, a contradiction. This proves 3.1. ∎

4 Parades

Now we begin the proof of 1.10, the second main result mentioned in the introduction. This proof was derived from, and still has some ingredients in common with, the proof of the main theorem of [6], but it has needed some serious modification, in order to persuade one of the two sets Z_1, Z_2 to be linear.

Let G be a bigraph (not necessarily ordered), and let I be a set of nonzero integers. We denote $I^+ = \{i \in I : i > 0\}$ and $I^- = I \setminus I^+$. Let the sets B_i $(i \in I)$ be nonempty, pairwise disjoint subsets of $V(G)$, such that $B_i \subseteq V_1(G)$ if $i < 0$ and $B_i \subseteq V_2(G)$ if $i > 0$. We call $\mathcal{P} = (B_i : i \in I)$ a *parade* in G. Its *length* is the pair $(|I^-|, |I^+|)$, and its *width* is the pair (w_1, w_2) where $w_1 = \min(|B_i| : i \in I^-)$ and $w_2 = \min(|B_i| : i \in I^+)$, taking $w_k = |V_k(G)|$ if the corresponding set I^- or I^+ is empty. We call the sets B_i the *blocks* of the parade. What matters is that the blocks are not too small. (We used the same word in [5] for a similar but slightly different object.)

If $I' \subseteq I$, then $(B_i : i \in I')$ is a parade, called a *sub-parade* of $\mathcal{P}$. If $B'_i \subseteq B_i$ is nonempty for each $i \in I$, then $(B'_i : i \in I)$ is a parade, called a *contraction* of $\mathcal{P}$.

Let X, Y be disjoint nonempty subsets of $V(G)$. The *max-degree from X to Y* is defined to be the maximum over all $v \in X$ of the number of neighbours of v in Y. Let $(B_i : i \in I)$ be a parade in a bigraph G, and for all $i, j \in I$ of opposite sign, let $d_{i,j}$ be the max-degree from B_i to B_j. (For all other pairs i, j we define $d_{i,j} = 0$.) We call $d_{i,j}$ $(i, j \in I)$ the *max-degree function* of the parade. The product of the numbers $d_{j,h}$ for all pairs h, j where $h \in I^-$ and $j \in I^+$ is called the *max-degree product* of $\mathcal{B}$. We just need this "product" definition for the next theorem.

Let $0 < \phi, \mu$. We say that $\mathcal{B}$ is (ϕ, μ)-*shrink-resistant* if for all $h \in I^-$ and $j \in I^+$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $d_{j,h}|V_1(G)|^{-\phi}$. We begin with:

4.1 *Let $\mathcal{B} = (B_i : i \in I)$ be a parade in a bigraph G, and let $0 < \phi, \mu$ with $\mu \leq 1$. Let $\beta = \mu^{1+|I|^2/\phi}$. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_h$ and $Y \subseteq B_j$ with $\frac{|X|}{|B_h|}, \frac{|Y|}{|B_j|} \geq \beta$, such that X, Y are anticomplete; or*
- *there is a (ϕ, μ)-shrink-resistant contraction $(B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta|B_i|$ for each $i \in I$.*

Let $S = \lfloor |I|^2/\phi \rfloor$. Choose an integer s with $0 \leq s \leq S + 1$ and with s maximum such that there is a contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$ with

- $|B'_i| \geq \mu^s|B_i|$ for each $i \in I^-$; and
- max-degree product at most $|V_1(G)|^{|I|^2 - \phi s}$.

(This is possible since we may take $s = 0$ and $\mathcal{B}' = \mathcal{B}$.) Let $d_{h,j}$ $(h, j \in I)$ be the max-degree function of $\mathcal{B}'$.

(1) *We may assume that $d_{j,h} \geq 1$ for all $h \in I^-$ and $j \in I^+$, and so $s \leq S$.*

If $d_{j,h} < 1$, then $d_{j,h} = 0$, since it is an integer. Thus B'_h, B'_j are anticomplete. Since $s \leq S + 1$ and hence $\mu^s \geq \mu^{S+1} \geq \beta$, it follows that $|B'_h|/|B_h|, |B'_j|/|B_j| \geq \beta$, and the first outcome of the theorem holds. Thus we may assume that $d_{j,h} \geq 1$. Hence the max-degree product of $\mathcal{B}'$ is at least one, and since it is at most $|V_1(G)|^{|I|^2-\phi s}$, it follows that $|I|^2 - \phi s \geq 0$. Hence $s \leq S$. This proves (1).

(2) *$(B'_i : i \in I)$ is (ϕ, μ)-shrink-resistant.*

Let $h \in I^-$ and $j \in I^+$, and let $C_h \subseteq B'_h$ and $C_j \subseteq B'_j$, with $|C_h| \geq \mu|B'_h|$ and $|C_j| \geq \mu|B'_j|$. For all $i \in I$ with $i \neq h, j$ let $C_i = B'_i$, and let d be the max-degree from C_j to C_h. From the maximality of s, and since $s \leq S$, it follows that the max-degree product of $(C_i : i \in I)$ is more than $|V_1(G)|^{|I|^2-\phi(s+1)}$. Since the first is at most $d/d_{h,j}$ times the max-degree product of $(B'_i : i \in I)$, which is at most $|V_1(G)|^{|I|^2-\phi s}$, it follows that $d/d_{h,j} > |V_1(G)|^{-\phi}$. This proves (2).

Since $|B'_i| \geq \mu^S|B_i| \geq \beta|B_i|$ for each $i \in I$, the second outcome of the theorem holds. This proves 4.1. ∎

Let $(B_i : i \in I)$ be a parade in a bigraph G, and let $0 < \tau, \phi, \mu$. We say that τ is a *(ϕ, μ)-band* for $(B_i : i \in I)$ if for all $h \in I^-$ and $j \in I^+$:

- the max-degree from B_j to B_h is at most $\tau|B_h|$; and
- for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $\tau|V_1(G)|^{-\phi}|B_h|$.

4.2 *Let $k \geq 0$ be an integer, and let $0 < \phi, \mu$. Then there exists an integer $K \geq k$ with the following property. Let G be a bigraph, and let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant parade in G, of length at least (K, K). Then there exists $J \subseteq I$ with $|J^-| = |J^+| = k$ such that $(B_i : i \in J)$ has a $(2\phi, \mu)$-band.*

Proof. Let $K \geq 1$ be an integer such that for every complete bipartite graph with bipartition (H, J) where $|H|, |J| \geq K$, and every colouring of its edges with $\lfloor 1/\phi + 1 \rfloor$ colours, there exist $H' \subseteq H$ and $J' \subseteq J$ with $|H'|, |J'| = k$ such that all edges between H', J' have the same colour.

Now let $(B_i : i \in I)$ be a (ϕ, μ)-shrink-resistant parade in G, with max-degree function $d_{i,j}$ $(i, j \in I)$.

For all $h \in I^-$ and $j \in I^+$, there is an integer s such that

$$|V_1(G)|^{-(s+1)\phi} < d_{j,h}/|B_h| \leq |V_1(G)|^{-s\phi}.$$

We call s the *type* of the pair (h, j). Since $|V_1(G)|^{-(s+1)\phi} < d_{j,h}/|B_h| \leq 1$, it follows that $-(s+1)\phi < 0$, and so $s \geq 0$; and since $1/|V_1(G)| \leq d_{j,h}/|B_h| \leq |V_1(G)|^{-s\phi}$ (because $d_{j,h} > 0$ from the definition of (ϕ, μ)-shrink-resistant), it follows that $1 \leq |V_1(G)|^{1-s\phi}$, and so $s \leq 1/\phi$. Hence s is one of the integers $0, 1, \ldots, \lfloor 1/\phi \rfloor$. From the choice of K, there exists $J \subseteq I$ with $|J^-| = |J^+| = k$, such that every pair (h, j) with $h \in J^-$ and $j \in J^+$ has the same type, s say. Let $\tau = |V_1(G)|^{-s\phi}$; then for all $h \in J^-$ and $j \in J^+$,

$$\tau|V_1(G)|^{-\phi} < d_{j,h}/|B_h| \leq \tau.$$

We claim that τ is a $(2\phi, \mu)$-band for $(B_i : i \in J)$. To show this, it remains to show that for all $h \in J^-$ and $j \in J^+$, and for all $X \subseteq B_h$ and $Y \subseteq B_j$ with $|X| \geq \mu|B_h|$ and $|Y| \geq \mu|B_j|$, the max-degree from Y to X is more than $\tau|V_1(G)|^{-2\phi}|B_h|$. But $\mathcal{B}$ is (ϕ, μ)-shrink-resistant, and so the max-degree from Y to X is more than $d_{j,h}|V_1(G)|^{-\phi}$; and since $d_{j,h} \geq \tau|V_1(G)|^{-\phi}|B_h|$, the claim follows. This proves 4.2. ∎

By combining 4.1 and 4.2, we deduce:

4.3 *Let $k \geq 0$ be an integer, and let $0 < \phi, \mu$ with $\mu \leq 1$. Then there exists an integer $K > 0$ with the following property. Let $\mathcal{B} = (B_i : i \in I)$ be a parade of length at least (K, K) in a bigraph G. Let $\beta = \mu^{1+2K^2/\phi}$. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_h$ and $Y \subseteq B_j$ with $\frac{|X|}{|B_h|}, \frac{|Y|}{|B_j|} \geq \beta$, such that X, Y are anticomplete; or*
- *there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and a subset $B'_i \subseteq B_i$ with $|B'_i| \geq \beta |B_i|$ for each $i \in J$, such that $(B'_i : i \in J)$ has a (ϕ, μ)-band.*

Proof. Let K satisfy 4.2 with ϕ replaced by $\phi/2$. Let G be a bigraph, and let $\mathcal{B} = (B_i : i \in I)$ be a parade in G, of length at least (K, K). By 4.1, either

- there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq B_i$ and $Y \subseteq B_j$ with $|X|/|B_h|, |Y|/|B_j| \geq \beta$, such that X, Y are anticomplete; or
- there is a $(\phi/2, \mu)$-shrink-resistant contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$, such that $|B'_i| \geq \beta |B_i|$ for each $i \in I$.

In the first case the first outcome of the theorem holds. In the second case, by 4.2 applied to $\mathcal{B}'$, the second outcome of the theorem holds. This proves 4.3. ∎

5 Covering with leaves

Again, this section concerns graphs rather than ordered graphs. If G is a graph and $A, B \subseteq V(G)$ are disjoint, we say A *covers* B if every vertex of B has a neighbour in A.

5.1 *Let $k \geq 1$ be an integer, and let $0 < \tau, \phi, \mu$, with $\mu \leq 1/(8k)$ and $\tau \leq 1/(8k^2)$. Let G be a bigraph and let let $\mathcal{A} = (A_i : i \in I)$ be a parade in G, with $|I^+|, |I^-| \leq k$, such that τ is a (ϕ, μ)-band for $\mathcal{A}$. Then for each $h \in I^-$ there exist $B_h, C_h \subseteq A_h$, and for each $h \in I^-$ and $j \in I^+$ there exists $D_{h,j} \subseteq A_j$, with the following properties:*

- *$C_h \subseteq B_h$, and $|B_h| \geq |A_h|/2$, and $|C_h| \geq |V_1(G)|^{-k\phi}|A_h|/16$, for each $h \in I^-$; and*
- *$D_{h,j}$ is anticomplete to B_i for all $i \in I^- \setminus \{h\}$, and is anticomplete to $B_h \setminus C_h$, and covers C_h, for each $h \in I^-$ and $j \in I^+$.*

Proof. For each $h \in I^-$ and $j \in I^+$, every vertex in A_j has at most $\tau|A_h|$ neighbours in A_h, and so there are at most $\tau|A_h| \cdot |A_j|$ edges between A_h, A_j. Hence at most $1/(2k)$ vertices in A_h have at least $2k\tau|A_j|$ neighbours in A_j. For each $h \in I^-$, let P_h be the set of vertices $v \in A_h$ such that for each $j \in I^+$, v has fewer than $2k\tau|A_j|$ neighbours in A_j. It follows that $|P_h| \geq |A_h|/2$ for all $h \in I^-$.

Choose $H \subseteq I^-$ maximal such that for each $h \in H$ there exists $Q_h \subseteq P_h$ with $|Q_h| \geq |V_1(G)|^{-k\phi}|A_h|/8$, and for all $h \in H$ and $j \in I^+$ there exists $D_{h,j} \subseteq A_j$, satisfying:

- $|D_{h,j}| \leq 1/(8k^2\tau)$; and

- $D_{h,j}$ covers Q_h;
- every vertex in $D_{h,j}$ has at most $4k^2\tau|Q_i|$ neighbours in Q_i for all $i \in H \setminus \{h\}$.

(This is possible since setting $H = \emptyset$ satisfies the bullets.) Suppose that there exists $g \in I^- \setminus H$.

Let $j \in I^+$. For each $h \in H$, each vertex in Q_h has at most $2k\tau|A_j|$ neighbours in A_j, and so there are at most $2k\tau|Q_h| \cdot |A_j|$ edges between Q_h and A_j; and hence at most $|A_j|/(2k)$ vertices in A_j have at least $4k^2\tau|Q_h|$ neighbours in Q_h. Let S_j be the set of vertices $v \in A_j$ such that for each $h \in H$, v has fewer than $4k^2\tau|Q_h|$ neighbours in Q_h. It follows that $|S_j| \geq |A_j|/2$.

For each $h \in H$ and $j \in I^+$, the set $D_{h,j}$ has cardinality at most $1/(8k^2\tau)$, and since each of its vertices has at most $\tau|A_g|$ neighbours in A_g, it follows that at most $|A_g|/(8k^2)$ vertices in A_g have a neighbour in $D_{h,j}$. Consequently at most $|A_g|/8$ vertices in A_g have a neighbour in some $D_{h,j}$; and since $|P_g| \geq |A_g|/2$, there is a subset $T_g \subseteq P_g$ with $|T_g| \geq |A_g|/4$ such that, for all $h \in H$ and $j \in J^+$, T_g is anticomplete to $D_{h,j}$.

(1) *Let $j \in I^+$. Then there exists $Y_j \subseteq T_g$ with $|T_g \setminus Y_j| < \mu|A_g|$, and $X \subseteq S_j$, such that X covers Y_j, and $|X| \leq 2|V_1(G)|^\phi/\tau$.*

Choose $X \subseteq S_j$ maximal such that

- $|X| \leq 2|V_1(G)|^\phi/\tau$; and
- $|Y| \geq \tau|V_1(G)|^{-\phi}|X| \cdot |A_g|$, where Y is the set of vertices in T_g that have a neighbour in X.

Suppose that $|T_g \setminus Y| \geq \mu|A_g|$. Since $|S_j| \geq \mu|A_j|$ and τ is a (ϕ, μ)-band for $(A_i : i \in I)$, it follows that the max-degree from S_j to $T_g \setminus Y$ is more than $\tau|V_1(G)|^{-\phi}|A_g|$. Choose $v \in S_j$ with more than $\tau|V_1(G)|^{-\phi}|A_g|$ neighbours in $T_g \setminus Y$. Since v has a neighbour in $T_g \setminus Y$, it follows that $v \notin X$, and from the maximality of X, adding v to X contradicts one of the two bullets in the definition of X. The second bullet is satisfied, and so the first is violated; and hence $|X| + 1 > 2|V_1(G)|^\phi/\tau$. Since $2|V_1(G)|^\phi/\tau \geq 1$, it follows that $X \neq \emptyset$, and so $2|X| \geq |X| + 1 > 2|V_1(G)|^\phi/\tau$, and therefore $|X| > |V_1(G)|^\phi/\tau$. So $|Y| > \tau|V_1(G)|^{-\phi}(|V_1(G)|^\phi/\tau)|A_g| = |A_g|$, a contradiction. This proves that $|T_g \setminus Y| < \mu|A_g|$, and so proves (1).

(2) *There exists $Q_g \subseteq T_g$ with $|Q_g| \geq (64k^2|V_1(G)|^\phi)^{-k}|A_g|/8$, and for each $j \in I^+$ there exists a subset $D_{g,j} \subseteq S_j$ with $|D_{g,j}| \leq 1/(8k^2\tau)$, such that $D_{g,j}$ covers Q_g.*

For each $j \in I^+$, let Y_j be as in (1), and choose $X_j \subseteq D_j$, such that X_j covers Y_j, and $|X_j| \leq 2|V_1(G)|^\phi/\tau$. Let Y be the intersection of the sets Y_j ($j \in I^+$). Since each Y_j satisfies $|T_g \setminus Y_j| < \mu|A_g|$, it follows that $|Y| \geq |T_g| - k\mu|A_g| \geq |A_g|/8$ (since $k\mu \leq 1/8$ and $|T_g| \geq |A_g|/4$). Let $j \in I^+$. Since $\lfloor 1/(8k^2\tau) \rfloor \geq 1/(16k^2\tau)$ (because $8k^2\tau \leq 1$), there is a partition of X_j into at most $\lceil 16k^2\tau|X_j| \rceil$ sets each of cardinality at most $1/(8k^2\tau)$. But $|X_j| \leq 2|V_1(G)|^\phi/\tau$, and so

$$\lceil 16k^2\tau|X_j| \rceil \leq \lceil 32k^2|V_1(G)|^\phi \rceil \leq 64k^2|V_1(G)|^\phi$$

since $32k^2|V_1(G)|^\phi \geq 1$. Thus X_j admits a partition $\mathcal{R}_j$ into at most $64k^2|V_1(G)|^\phi$ sets each of cardinality at most $1/(8k^2\tau)$. For each $v \in Y$, there exists $u \in X_j$ adjacent to Y; choose some such u, choose $R \in \mathcal{R}_j$ containing u, and say R is the *j-type* of v. Each vertex of v has a j-type, for each

$j \in I^+$; and since there are only at most $64k^2|V_1(G)|^\phi$ j-types for each j, and $|I^+| \le k$, it follows that there exists $Q_g \subseteq Y$ with

$$|Q_g| \ge (64k^2|V_1(G)|^\phi)^{-k}|Y| \ge (64k^2|V_1(G)|^\phi)^{-k}|A_g|/8,$$

such that for all $j \in I^+$, all members of Q_g have the same j-type, say $D_{g,j} \in \mathcal{R}_j$, and each $D_{g,j}$ covers B_g. This proves (2).

From (2), this contradicts the maximality of H. (Note that for $j \in I^+$ and $h \in H$, every vertex in $D_{h,j}$ has no neighbours in Q_g, since $Q_g \subseteq T_g$; and every vertex of $D_{g,j}$ has at most $4k^2\tau|Q_h|$ neighbours in Q_h, because $D_{g,j} \subseteq S_j$.) This proves that $H = I^-$. Thus we have shown that for all $h \in I^-$ there exists $Q_h \subseteq P_h$ with $|Q_h| \ge (64k^2|V_1(G)|^\phi)^{-k}|A_h|/8$, and for all $h \in I^-$ and $j \in I^+$ there exists $D_{h,j} \subseteq A_j$, satisfying:

- $|D_{h,j}| \le 1/(8k^2\tau)$;
- $D_{h,j}$ covers Q_h; and
- every vertex in $D_{h,j}$ has at most $4k^2\tau|Q_i|$ neighbours in Q_i for all $i \in H \setminus \{h\}$.

Now let $i \in I^-$. Since there are only at most k^2 sets $D_{h,j}$, and each has cardinality at most $1/(8k^2\tau)$, and every vertex in a set $D_{h,j}$ for $h \ne i$ has at most $4k^2\tau|Q_i|$ neighbours in Q_i, it follows that at most $|Q_i|/2$ vertices in Q_i have a neighbour in some set $D_{h,j}$ with $h \ne i$; and so there exists $C_i \subseteq Q_i$ with $|C_i| \ge |Q_i|/2$ such that C_i is anticomplete to $D_{h,j}$ for all $h \in I^- \setminus \{i\}$ and $j \in I^+$. Hence $|C_i| \ge (64k^2)^{-k}|V_1(G)|^{-k\phi}|A_h|/16$ for each $i \in I^-$. Moreover, since the union of all the sets $D_{h,j}$ has cardinality at most $1/(8\tau)$, and each vertex of this union has at most $\tau|A_i|$ neighbours in A_i, it follows that at most $|A_i|/8$ vertices in A_i have a neighbour that belongs to some $D_{h,j}$; and consequently there exists $B_i \subseteq A_i$ with $C_i \subseteq B_i$, and with $|B_i| \ge |A_i|/2$, such that $B_i \setminus C_i$ is anticomplete to all the sets $D_{h,j}$. This proves 5.1. ∎

By combining 4.3 and 5.1, and fixing values for ϕ and μ, we obtain:

5.2 *Let $k \ge 0$ be an integer, and let $c > 0$. Then there exists an integer $K > 0$ with the following property. Let $\mathcal{A} = (A_i : i \in I)$ be a parade of length at least (K, K) in a bigraph G. Let $\beta = (8k)^{-1-2K^2k/c}$. Then either*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq A_h$ and $Y \subseteq A_j$ with $\frac{|X|}{|A_h|}, \frac{|Y|}{|A_j|} \ge \beta$, such that X, Y are anticomplete; or*
- *there exist $h \in I^-$ and $j \in I^+$ such that some $v \in A_j$ has at least $\frac{\beta}{8k^2}|A_h|$ neighbours in A_h; or*
- *there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and for each $h \in J^-$ there exists $B_h \subseteq A_h$ with $|B_h| \ge \beta|A_h|/2$, and there exists $C_h \subseteq B_h$ with $|C_h| \ge \beta|V_1(G)|^{-c}|A_h|/16$; and for each $h \in J^-$ and $j \in J^+$ there exists $D_{h,j} \subseteq A_j$ covering C_h, such that $D_{h,j}$ is anticomplete to $B_i \setminus C_i$ for all $i \in J^-$, and is anticomplete to C_i for all $i \in J^- \setminus \{h\}$.*

Proof. By 4.3, taking $\mu = 1/(8k)$ and $\phi = c/k$, we may assume that there exist $J \subseteq I$ with $|J^-| = |J^+| = k$, and a subset $F_i \subseteq A_i$ with $|F_i| \ge \beta|A_i|$ for each $i \in J$, such that $(F_i : i \in J)$

has a (ϕ, μ)-band τ. We may assume that for all $j \in J^+$ and $v \in A_j$ and $h \in I^-$, v has fewer than $(\beta/(8k^2))|A_h|$ neighbours in A_h, and hence has fewer than $|F_h|/(8k^2)$ neighbours in F_h. Consequently we may assume that $\tau \le 1/(8k^2)$. By 5.1 applied to $\mathcal{F} = (F_i : i \in J)$, for each $h \in J^-$ there exists $B_h \subseteq F_h$ with $|B_h| \ge |F_h|/2 \ge \beta|A_h|/2$, and there exists $C_h \subseteq B_h$ with

$$|C_h| \ge |V_1(G)|^{-k\phi}|F_h|/16 \ge \beta|V_1(G)|^{-k\phi}|A_h|/16;$$

and for each $h \in J^-$ and $j \in J^+$ there exists $D_{h,j} \subseteq F_j$ covering C_h, such that $D_{h,j}$ is anticomplete to $B_i \setminus C_i$ for all $i \in I^-$, and is anticomplete to C_i for all $i \in J^- \setminus \{h\}$. Then the theorem is satisfied. This proves 5.2. ∎

6 The proof of 1.10

If T is a tree and $w \in V(T)$, we say that the *w-radius* of T is the maximum integer r such that some path of T with one end w has r edges. For $v \in V(T) \setminus \{w\}$, the *$w$-parent* of v is the neighbour of v in the path of T between v, w. We define $d_T(u, v)$ to be the distance in T between u, v.

If G is a bigraph and T is an induced sub-bigraph that is a tree bigraph, we say that T is a *induced subtree* of G. If $\mathcal{B}$ is a parade in a bigraph G, an induced subtree T of G is *$\mathcal{B}$-rainbow* if every vertex of H belongs to some block of $\mathcal{B}$, and every block of $\mathcal{B}$ contains at most one vertex of H.

If I is a set of nonzero integers, a *shape* in I is a tree S with $V(S) \subseteq I$, such that for every edge ij of S, i and j have opposite sign. Let $\mathcal{A} = (A_i : i \in I)$ be a parade in a bigraph G, let T be an $\mathcal{A}$-rainbow induced subtree of G, and let S be a shape in I. We say that S is the *shape of* T if

- for each $i \in I$, $i \in V(S)$ if and only if some vertex of A_i belongs to $V(T)$; and
- for all $i, j \in I$, i is adjacent to j in S if and only if there exists $u \in A_i$ and $v \in A_j$ such that uv is an edge of T.

Thus every induced subtree of G has a unique shape.

Let $\mathcal{A}^0 = (A_i^0 : i \in I)$ be a parade in a bigraph G, and let $r \ge 0$ be an integer. For each $i \in I$, and for $1 \le q \le r$ let $A_i^q \subseteq A_i^{q-1}$ be nonempty. For $0 \le q \le r$ let $\mathcal{A}^q$ be the parade $(A_i^q : i \in I)$. We call the sequence $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ a *nested parade sequence.* Let $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ be a nested parade sequence with notation as above, let T be an $\mathcal{A}^0$-rainbow induced subtree of G, and let $w \in V(T)$, such that T has w-radius at most r. We say that T is *w-isolated in* $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ if:

- for each $v \in V(T)$, $v \in A_i^{r-d_T(v,w)}$ for some $i \in I$; and
- for each $v \in V(T) \setminus \{w\}$, let $q = r + 1 - d_T(v, w)$; for $i \in I$, v has a neighbour in A_i^q only if the w-parent of v in T also belongs to A_i^q.

Again, let $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ be a nested parade sequence with notation as before. Let $i \in I$; we say that a vertex $w \in A_i^0$ is *r-panarboreal in* $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ if for every shape S in I with $i \in V(S)$ and with i-radius at most r, there is an $\mathcal{A}^0$-rainbow induced subtree T of G with shape S that is w-isolated in $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$. We will prove:

6.1 *Let $0 < c \leq 1$. For all integers $r, k \geq 0$, there exist an integer $K > 0$, and $\gamma > 0$ with the following property. Let G be a bigraph and let $\mathcal{A} = (A_i : i \in I)$ be a parade in G with length at least (K, K). Then either:*

- *there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq A_h$ and $Y \subseteq A_j$, either with $|X| \geq \gamma|A_h|$ and $|Y| \geq \gamma|V_2(G)|^{-c}|A_j|$, or with $|X| \geq \gamma|V_1(G)|^{-c}|A_h|$ and $|Y| \geq \gamma|A_j|$, such that X, Y are anticomplete; or*

- *there exist $h, j \in I$ with opposite sign, and $v \in A_h$, such that v has at least $\gamma|A_j|$ neighbours in A_j; or*

- *there exist $J \subseteq I$ with $|J^-|, |J^+| \geq k$, and a nested parade sequence $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$ with $\mathcal{A}^q = (A^q_i : j \in J)$ for $0 \leq q \leq r$, and with the following properties. For each $j \in J$, $A^0_j \subseteq A_j$ and $|A^r_j| \geq \gamma|A_j|$; and for each $h \in J^-$ there exists $C_h \subseteq A^r_h$ with $|C_h| \geq \gamma|V_1(G)|^{-c}|A_h|$, such that every vertex in C_h is r-panarboreal in $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$.*

Proof. We will prove, by induction on r, that for each value of r the statement holds for all k. If $r = 0$ then the third outcome of the theorem is true, setting $K = k$, and $\gamma = 1$, and taking $J = I$, and $A^0_h = A_h$ for all $h \in I$, and $C_h = A_h$ for all $h \in J^-$. Thus we may assume that $r \geq 1$, and the claim holds for $r - 1$ and all k.

Choose k' such that setting $K = k'$ satisfies 5.2. From the inductive hypothesis, there exist an integer $K > 0$ and $\gamma' > 0$ such that the assertion of 6.1 holds with r, k, γ replaced by $r - 1, k', \gamma'$ respectively. Let $\beta = (8k)^{-1-2k'^2k/c}$, and $\gamma = \frac{\beta}{8k^2}\gamma'$. We claim that the theorem is satisfied. To see this, let G be a bigraph and let $\mathcal{A} = (A_i : i \in I)$ be a parade in G with length at least (K, K).

(1) *We may assume that there exist $L \subseteq I$ with $|L^-|, |L^+| \geq k'$, and a nested parade sequence $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, with $\mathcal{B}^q = (B^q_i : j \in L)$ for $0 \leq q \leq r - 1$, and with the following properties. For each $j \in L$, $B^0_j \subseteq A_j$ and $|B^{r-1}_j| \geq \gamma|A_j|$; and for each $h \in L^-$ there exists $C_h \subseteq B^{r-1}_h$ with $|C_h| \geq \gamma|V_1(G)|^{-c}|A_h|$, such that every vertex in C_h is $(r-1)$-panarboreal in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$.*

Let G^T be the bigraph obtained from G by setting $V_1(G^T) = V_2(G)$ and $V_2(G^T) = V_1(G)$, and let I^T be the set of all integers $-i$ where $i \in I$. Thus $\mathcal{A}^T = (A_i : i \in I^T)$ is a parade in G^T. From the choice of K, applied to G^T and $\mathcal{A}^T$, either

- there exist $h \in I^-$ and $j \in I^+$, and $X \subseteq A_h$ and $Y \subseteq A_j$, either with $|X| \geq \gamma'|A_h|$ and $|Y| \geq \gamma'|V_2(G)|^{-c}|A_j|$, or with $|X| \geq \gamma'|V_1(G)|^{-c}|A_h|$ and $|Y| \geq \gamma'|A_j|$, such that X, Y are anticomplete; or

- there exist $h, j \in I$ with opposite sign, and $v \in A_h$, such that v has at least $\gamma'|A_j|$ neighbours in A_j; or

- the statement of (1) holds.

In the first case, since $\gamma \leq \gamma'$, the first outcome of the theorem holds, and similarly in the second case, the second outcome of the theorem holds. So we may assume that the third case holds. This proves (1).

(2) *We may assume that there exists* $J \subseteq L$ *with* $|J^-| = |J^+| = k$*, and for each* $h \in J^-$ *there exists* $B_h^r \subseteq B_h^{r-1}$ *with* $|B_h^r| \geq \beta|B_h^{r-1}|/2$*, and there exists* $C_h \subseteq B_h^r$ *with* $|C_h| \geq \beta|V_1(G)|^{-c}|B_h^{r-1}|/16$*; and for each* $h \in J^-$ *and* $j \in J^+$ *there exists* $D_{h,j} \subseteq C_j$ *covering* C_h*, such that* $D_{h,j}$ *is anticomplete to* B_i^r *for all* $i \in J^- \setminus \{h\}$*.*

Let $A'_i = B_i^{r-1}$ for $i \in L^-$, and $A'_i = C_i$ for $i \in L^+$. Then $(A'_i : i \in L)$ is a parade of length at least (k', k'), and so from the choice of k', either

- there exist $h \in L^-$ and $j \in L^+$, and $X \subseteq A'_h$ and $Y \subseteq A'_j$ with $\frac{|X|}{|A'_h|}, \frac{|Y|}{|A'_j|} \geq \beta$, such that X, Y are anticomplete; or
- there exist $h \in L^-$ and $j \in L^+$ such that some $v \in A_j$ has at least $\frac{\beta}{8k^2}|A'_h|$ neighbours in A'_h; or
- the statement of (2) holds.

In the first case, the first outcome of the theorem holds, since

$$|X| \geq \beta|A'_h| = \beta|B_h^{r-1}| \geq \beta\gamma'|A_h| \geq \gamma|A_h|$$

(because $\beta\gamma' \geq \gamma$), and

$$|Y| \geq \beta|A'_j| = \beta|C_j| \geq \beta\gamma'|V_2(G)|^{-c}|A_j| \geq \gamma|V_2(G)|^{-c}|A_j|.$$

In the second case, the second outcome of the theorem holds, since

$$\frac{\beta}{8k^2}|A'_h| = \frac{\beta}{8k^2}|B_h^{r-1}| \geq \frac{\beta}{8k^2}\gamma'|A_h| \geq \gamma|A_h|.$$

Thus we may assume that the third case holds. This proves (2).

Define $B_j^r = B_j^{r-1}$ for each $j \in J^+$. Then $|B_i^r| \geq \gamma|A_i|$ for each $i \in J$, since for $h \in J^-$,

$$|B_h^r| \geq \beta|B_h^{r-1}|/2 \geq (\beta/2)\gamma'|A_h| \geq \gamma|A_h|$$

and for each $j \in J^+$,

$$|B_j^r| = |B_j^{r-1}| \geq \gamma'|A_j| \geq \gamma|A_j|.$$

Moreover, for each $h \in J^-$, $C_h \subseteq B_h^r$, and

$$|C_h| \geq \beta|V_1(G)|^{-c}|B_h^{r-1}|/16 \geq \beta|V_1(G)|^{-c}\gamma'|A_h|/16 \geq \gamma|V_1(G)|^{-c}|A_h|.$$

Let $\mathcal{A}^q = (B_j^q : j \in J)$ for $0 \leq q \leq r$. To complete the proof, we will show that for each $h \in J^-$ and each $w \in C_h$, w is r-panarboreal in $(\mathcal{A}^0, \dots, \mathcal{A}^r)$.

Let S be a shape in J with $h \in V(S)$ and with h-radius at most r. Let $j_1, \dots, j_t$ be the neighbours of h in S; thus $j_1, \dots, j_t \in J^+$. For $1 \leq s \leq t$, let S_s be the component of $S \setminus \{h\}$ that contains j_s. Thus S_s has j_s-radius at most $r - 1$, and S_s is a shape in J and hence in L. For $1 \leq s \leq t$, since D_{h,j_s} covers C_h, there is a vertex $w_s \in D_{h,j_s}$ adjacent to w. Consequently w_s has no neighbours in $B_i^r \setminus C_i$ for all $i \in J^-$, and has no neighbours in C_i for all $i \in J^- \setminus \{h\}$. Since every vertex in

C_i is $(r-1)$-panarboreal in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, it follows that there is an $\mathcal{A}$-rainbow subtree T_s of G with shape S_s and with $w_s \in V(T_s)$, such that T_s is w_s-isolated in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$. Let T be the tree obtained from the union of the trees $T_1, \ldots, T_t$ by adding the vertex w and the edges $ww_1, \ldots, ww_s$.

(3) *T is an induced subtree of G.*

To see this, since each T_s is induced, it suffices to check that

- w has a unique neighbour w_s in $V(T_s)$, for $1 \le s \le t$; and
- there is no edge of G between $V(T_s)$ and $V(T_{s'})$ for distinct s, s'.

To prove the first, suppose that w is adjacent to some vertex $v \in V(T_s)$ where $v \neq w_s$. Since T_s is w_s-isolated in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, and $w \in B_h^{r-1}$, it follows that the w_s-parent of v in T_s also belongs to B_h^{r-1}; but this is not the case since $h \notin V(S_s)$. So w has a unique neighbour w_s in $V(T_s)$, for $1 \le s \le t$. This proves the first bullet. Now suppose that $1 \le s, s' \le t$ with $s \neq s'$, and some vertex $u \in V(T_s)$ is adjacent to some vertex v in $V(T_{s'})$. Since G is bipartite, one of u, v is closer in T to w than the other, say $d_T(v, w) < d_T(u, w)$, and consequently $u \neq w_s$. Since T_s is w_s-isolated in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, and $u \neq w_s$, it follows that v and the w_s-parent of u in T_s belong to the same block of $\mathcal{A}$, a contradiction since the shapes of $T_s, T_{s'}$ are vertex-disjoint. This proves the second bullet above, and so proves (3).

(4) *T is w-isolated in $(\mathcal{A}^0, \ldots, \mathcal{A}^r)$.*

We must show that:

- for each $v \in V(T)$, $v \in B_i^{r-d_T(v,w)}$ for some $i \in J$; and
- for each $v \in V(T) \setminus \{w\}$, let $q = r + 1 - d_T(v, w)$; for $i \in J$, v has a neighbour in B_i^q only if the w-parent of v in T also belongs to B_i^q.

For the first bullet, since $w \in C_h \subseteq B_h^r$, we may assume that $v \neq w$; let $v \in V(T_s)$ where $1 \le s \le t$. Then the first bullet follows, since T_s is w_s-isolated in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, and

$$r - d_T(v, w) = (r-1) - d_{T_s}(v, w_s).$$

For the second bullet, let $v \in V(T_s)$. Since, as we saw earlier, w_s has no neighbours in B_j^r for all $j \in J^- \setminus \{h\}$, we may assume that $v \neq w_s$. But then the second bullet is true, since T_s is w_s-isolated in $(\mathcal{B}^0, \ldots, \mathcal{B}^{r-1})$, and

$$r - d_T(v, w) = (r-1) - d_{T_s}(v, w_s),$$

and the w_s-parent of v in T_s is the w-parent of v in T. This proves (4).

From (3) and (4), this proves 6.1. ∎

Now we can prove 1.10, which we restate:

6.2 *Let T be an ordered tree bigraph. For all $c > 0$ there exists $\varepsilon > 0$ with the following property. Let G be an ordered bigraph not containing T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. Then there are subsets $Z_i \subseteq V_i(G)$ for $i = 1, 2$, either with $|Z_1| \geq \varepsilon|V_1(G)|$ and $|Z_2| \geq \varepsilon|V_2(G)|^{1-c}$, or with $|Z_1| \geq \varepsilon|V_1(G)|^{1-c}$ and $|Z_2| \geq \varepsilon|V_2(G)|$, such that Z_1, Z_2 are anticomplete.*

Proof. We may assume that $|V(T)| \geq 2$; choose an integer r such that T has w_1-radius at most r, for some vertex $w_1 \in V_1(T)$, and T has w_2-radius at most r, for some vertex $w_2 \in V_2(T)$. Choose an integer k such that $|V_1(T)|, |V_2(T)| \leq k$. Choose K, γ as in 6.1. Let $\varepsilon = \gamma/(2K)$. We claim that ε satisfies the theorem.

Let G be an ordered bigraph that does not contain T, such that every vertex in $V_1(G)$ has degree less than $\varepsilon|V_2(G)|$, and every vertex in $V_2(G)$ has degree less than $\varepsilon|V_1(G)|$. If G has no edges then Z_1, Z_2 exist as required, so we may assume that G has an edge; and so $\varepsilon|V_i(G)| < 1$ for $i = 1, 2$. Let $p = \lceil |V_1(G)|/(2K) \rceil$; then $p \leq |V_1(G)|/K$, since $|V_1(G)| > 1/\varepsilon \geq K$. Let the vertices of $V_1(G)$ be $u_1, \ldots, u_{n_1}$, ordered according to the linear order of $V_1(G)$ imposed by G. Let

$$A_i = \{u_{(K-i)p+1}, \ldots, u_{(K-i+1)p}\}$$

for $-K \leq i \leq -1$. Similarly, let $q = \lceil |V_2(G)|/(2K) \rceil$, and $V_2(G) = \{v_1, \ldots, v_{n_2}\}$ in order, and for $1 \leq i \leq K$ let

$$A_i = \{v_{(i-1)q+1}, \ldots, v_{iq}\}.$$

Let $I = \{-K, \ldots, -1, 1, \ldots, K\}$. Then $\mathcal{A} = (A_i : i \in I)$ is a parade in G, of length (K, K) and width (p, q), and all its blocks are intervals of the linear order, in the natural sense. Since the blocks of $(A_i : i \in J)$ are intervals and are numbered in order, it follows that:

(1) *Let R be an $\mathcal{A}$-rainbow induced subtree of G with shape S. The orders of $V_1(G)$ and $V_2(G)$ induce orders on $V_1(R), V_2(R)$, making R into an ordered tree bigraph R'. Also the orders on I^- and I^+ make S into an ordered tree bigraph S'; and R' is isomorphic to S'.*

Certainly R is isomorphic to S, but we need to check that the natural isomorphism preserves the vertex-orders. For each $v \in V(R)$, let $f(v) \in I$ such that $v \in A_{f(v)}$; then f is an isomorphism from R to S. Let $u, v \in V_1(R)$ say, where u is earlier than v in the order that R' imposes on $V_1(R')$. Hence u is earlier than v in the order that G imposes on $V_1(G)$, that is, $i < j$ where $u = u_i$ and $v = u_j$. Since the blocks of $\mathcal{A}$ are intervals, numbered in order, and $i < j$, and $f(u) \neq f(v)$ since R is $\mathcal{A}$-rainbow, it follows that $f(u) < f(v)$. Hence $f(u)$ is earlier than $f(v)$ in the order imposed on $V_1(S')$ by S'; and so f maps the order of $V_1(R')$ imposed by R' to the order of $V_1(S')$ imposed by S'. Similarly f maps the order of $V_2(R')$ imposed by R' to the order of $V_2(S')$ imposed by S'. This proves (1).

Let us apply 6.1 to $(A_i : i \in I)$, and deduce that one of the three outcomes of 6.1 holds. Suppose that the first outcome holds, that is, there exist $h \in I^-$ and $j \in I^+$, and $Z_1 \subseteq A_h$ and $Z_2 \subseteq A_j$, either with $|Z_1| \geq \gamma|A_h|$ and $|Z_2| \geq \gamma|V_2(G)|^{-c}|A_j|$, or with $|Z_1| \geq \gamma|V_1(G)|^{-c}|A_h|$ and $|Z_2| \geq \gamma|A_j|$, such that Z_1, Z_2 are anticomplete, and from the symmetry we assume the first. Since

$$\gamma|A_h| = \gamma p \geq \gamma|V_1(G)|/(2K) = \varepsilon|V_1(G)|$$

and

$$\gamma|V_2(G)|^{-c}|A_j| \geq \gamma|V_2(G)|^{-c}q \geq \gamma|V_2(G)|^{-c}|V_2(G)|/(2K) = \varepsilon|V_2(G)|^{1-c},$$

in this case the theorem holds.

Now suppose that the second outcome holds, that is, there exist $h, j \in I$ with opposite sign, and $v \in A_h$, such that v has at least $\gamma|A_j|$ neighbours in A_j. From the symmetry we may assume that $h \in I^-$. Since

$$\gamma|A_j| = \gamma q \geq \gamma|V_2(G)|/(2K) = \varepsilon|V_2(G)|$$

this is impossible.

Finally, suppose that the third outcome holds, that is, there exist $J \subseteq I$ with $|J^-|, |J^+| \geq k$, and a nested parade sequence $(\mathcal{A}^0, \dots, \mathcal{A}^r)$ with $\mathcal{A}^q = (A^q_i : j \in J)$ for $0 \leq q \leq r$, and with the following properties. For each $j \in J$, $A^0_j \subseteq A_j$ and $|A^r_j| \geq \gamma|A_j|$; and for each $h \in J^-$ there exists $C_h \subseteq A^r_h$ with $|C_h| \geq \gamma|V_1(G)|^{-c}|A_h|$, such that every vertex in C_h is r-panarboreal in $(\mathcal{A}^0, \dots, \mathcal{A}^r)$. Since $|V_1(T)|, |V_2(T)| \leq k$, it follows that there is a shape S in J, such that the ordered tree bigraph T is isomorphic to the ordered tree bigraph S' obtained from S as in (1). Let this isomorphism map w_1 to $h \in J^-$, and choose $w \in C_h$. Since w is r-panarboreal in $(\mathcal{A}^0, \dots, \mathcal{A}^r)$, it follows that there is an $(A_i : i \in I)$-rainbow induced subtree R of G, with shape S. Let R' be as in (1); then R' is isomorphic to S' by (1), and hence isomorphic to T, and therefore G contains T, a contradiction. This proves 6.2. ∎

References

[1] M. Chudnovsky, A. Scott, P. Seymour and S. Spirkl, "Pure pairs. I. Trees and linear anticomplete pairs", *Advances in Math.*, to appear, `arXiv:1809.00919`.

[2] P. Erdős, A. Hajnal and J. Pach, "A Ramsey-type theorem for bipartite graphs", *Geombinatorics* 10 (2000), 64–68.

[3] J. Fox, "A bipartite analogue of Dilworth's theorem", *Order* 23 (2006), 197–209.

[4] D. Korándi, J. Pach, and I. Tomon, "Large homogeneous submatrices", `arXiv:1903.06608`.

[5] A. Scott, P. Seymour and S. Spirkl, "Pure pairs. IV. Trees in bipartite graphs", submitted for publication, `arXiv:2009.09426`.

[6] A. Scott, P. Seymour and S. Spirkl, "Pure pairs. VI. Excluding an ordered tree", submitted for publication, `arXiv:2009.10671`.