# Critical Core Percolation on Random Graphs

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Oxford Discrete Maths and Probability Seminar

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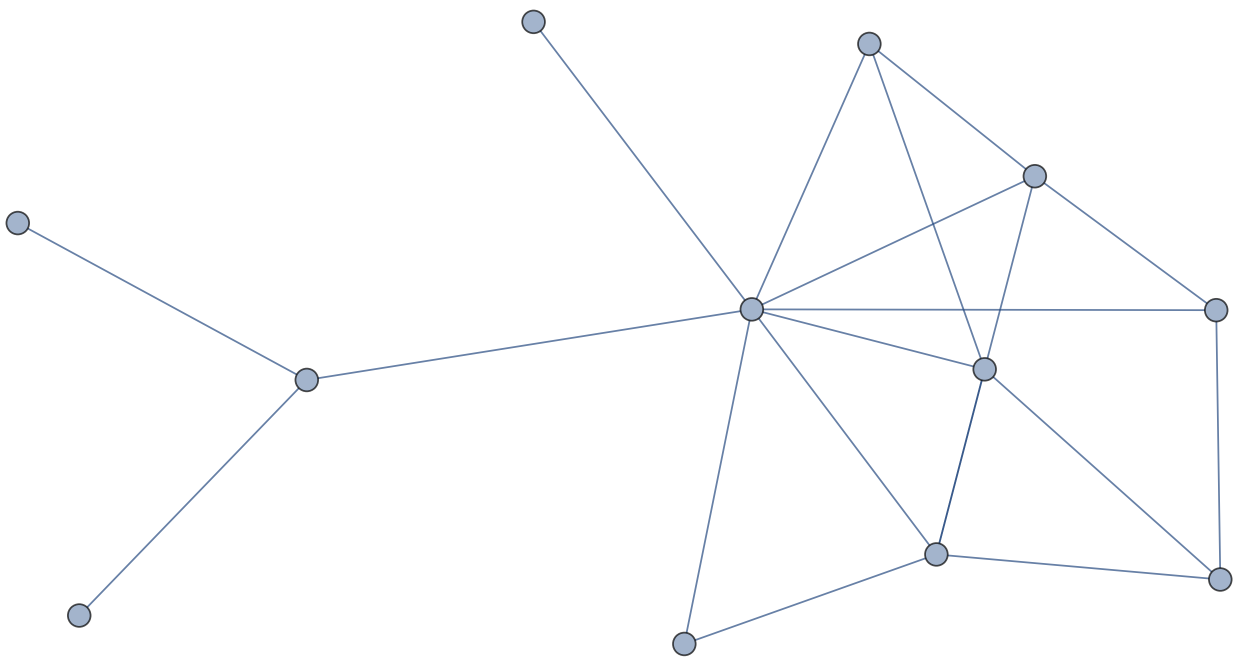
joint with Thomas Budzinski and Nicolas Curien

# *k*-core

**Definition**

The *k-core* of a graph *G* is the (unique) maximal subgraph of G in which all vertices have *induced* degree at least k.



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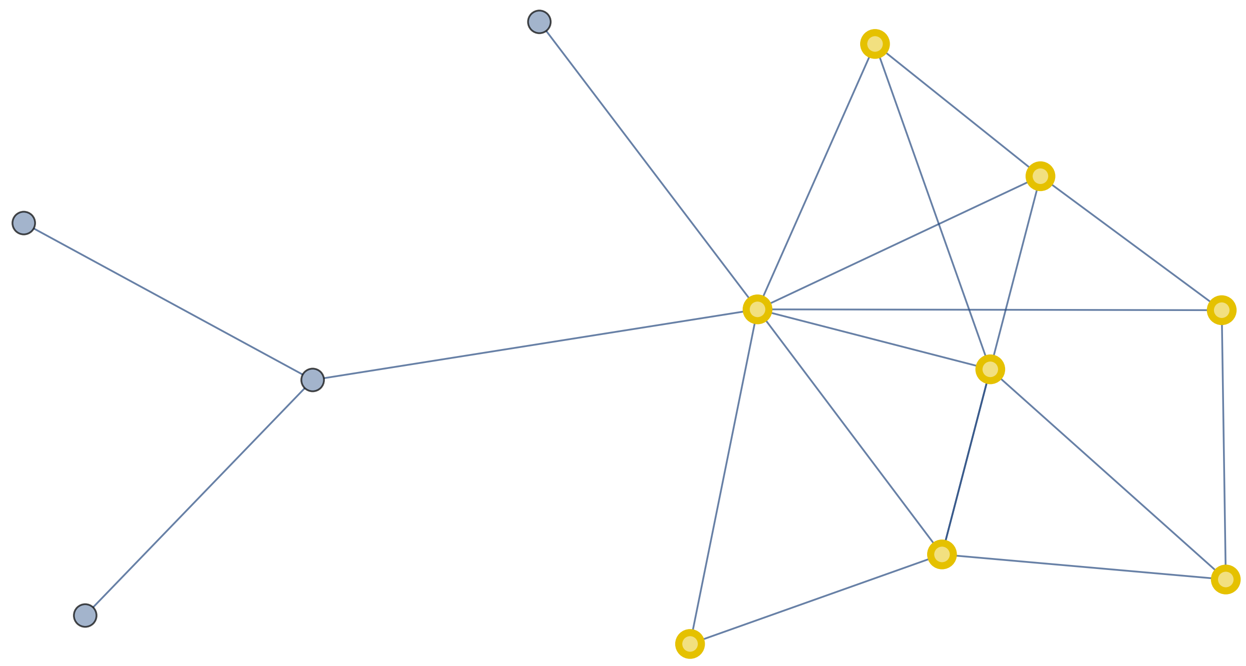

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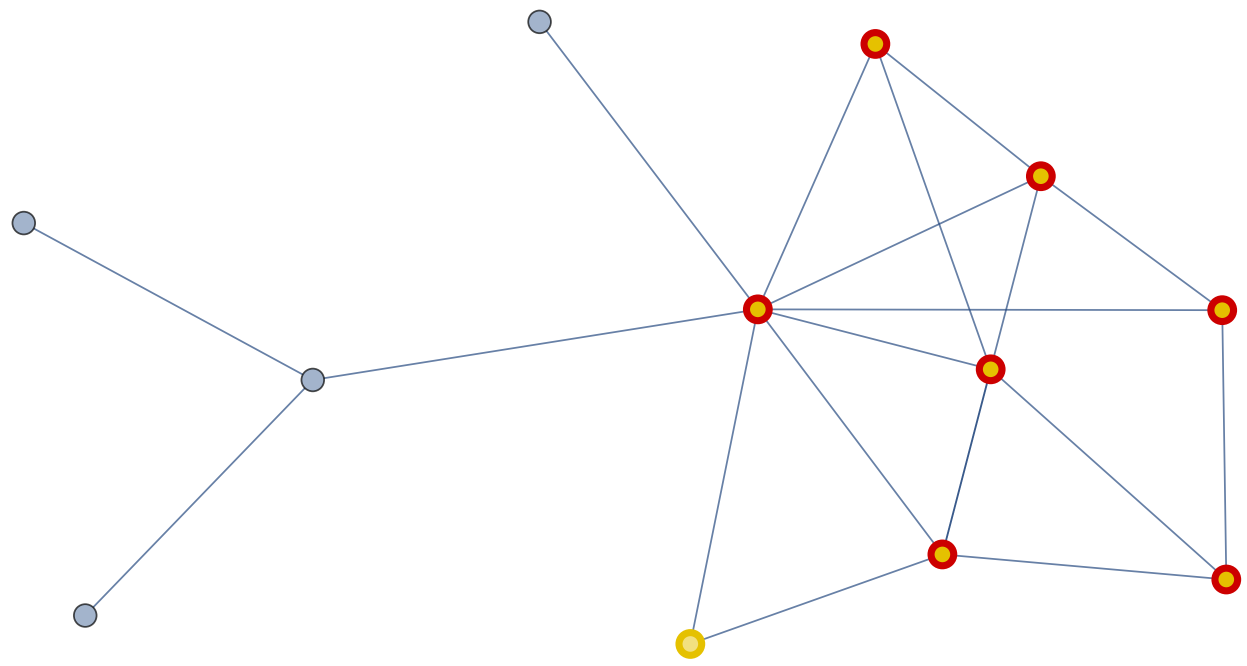
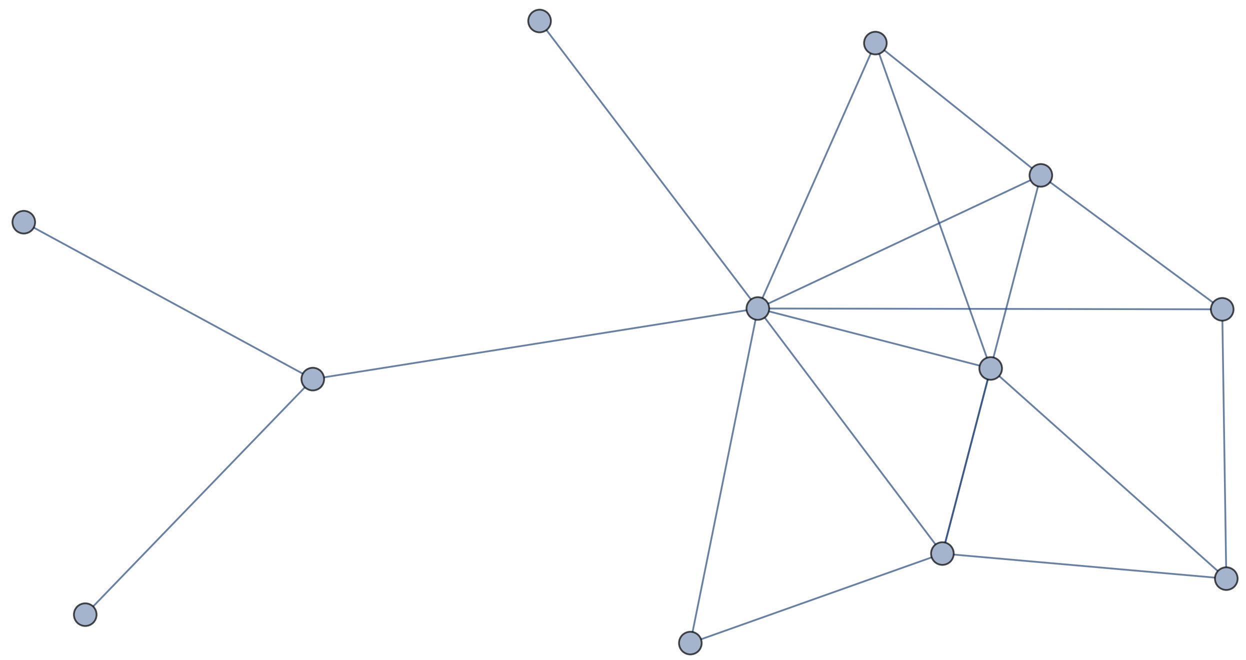


# $k$-core

**Lemma**

*We can obtain the $k$-core of $G$ by recursively removing the vertices of degree less than $k$.*



# Phase transition

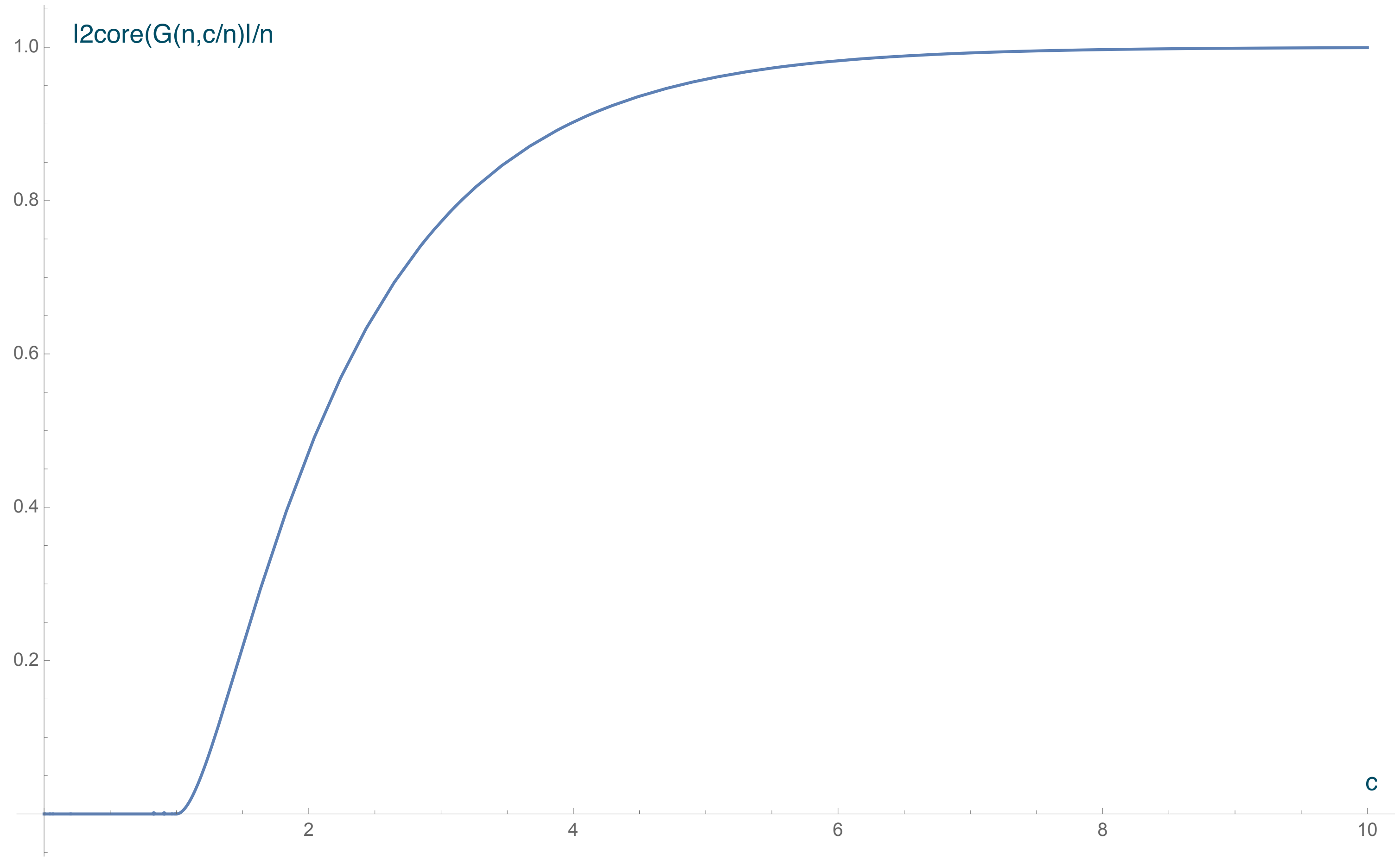
Let $G(n, p)$ be an Erdős–Rényi random graph.

**Theorem (Pittel, 90; Chvátal, 91)**

*Let $p = \frac{c}{n}$. There exists $\alpha_k > 0$ such that*

- *(subcritical case) If $c < \alpha_k$, then there is no $k$-core with positive probability (and with high probability for $k \geq 3$).*
- *(supercritical case) If $c > \alpha_k$, then the $k$-core has asymptotic size $\beta(c) \cdot n$.*

# Critical 2-core




**Theorem (Janson, Knuth, Luczak & Pittel, 93)**

*Let $p = \frac{1}{n}$. Then the 2-core of $G(n, p)$ has size of order $n^{1/3}$ as $n$ goes to infinity.*

# Discontinuity for the 3-core

For the 3-core, the phase transition is discontinuous.

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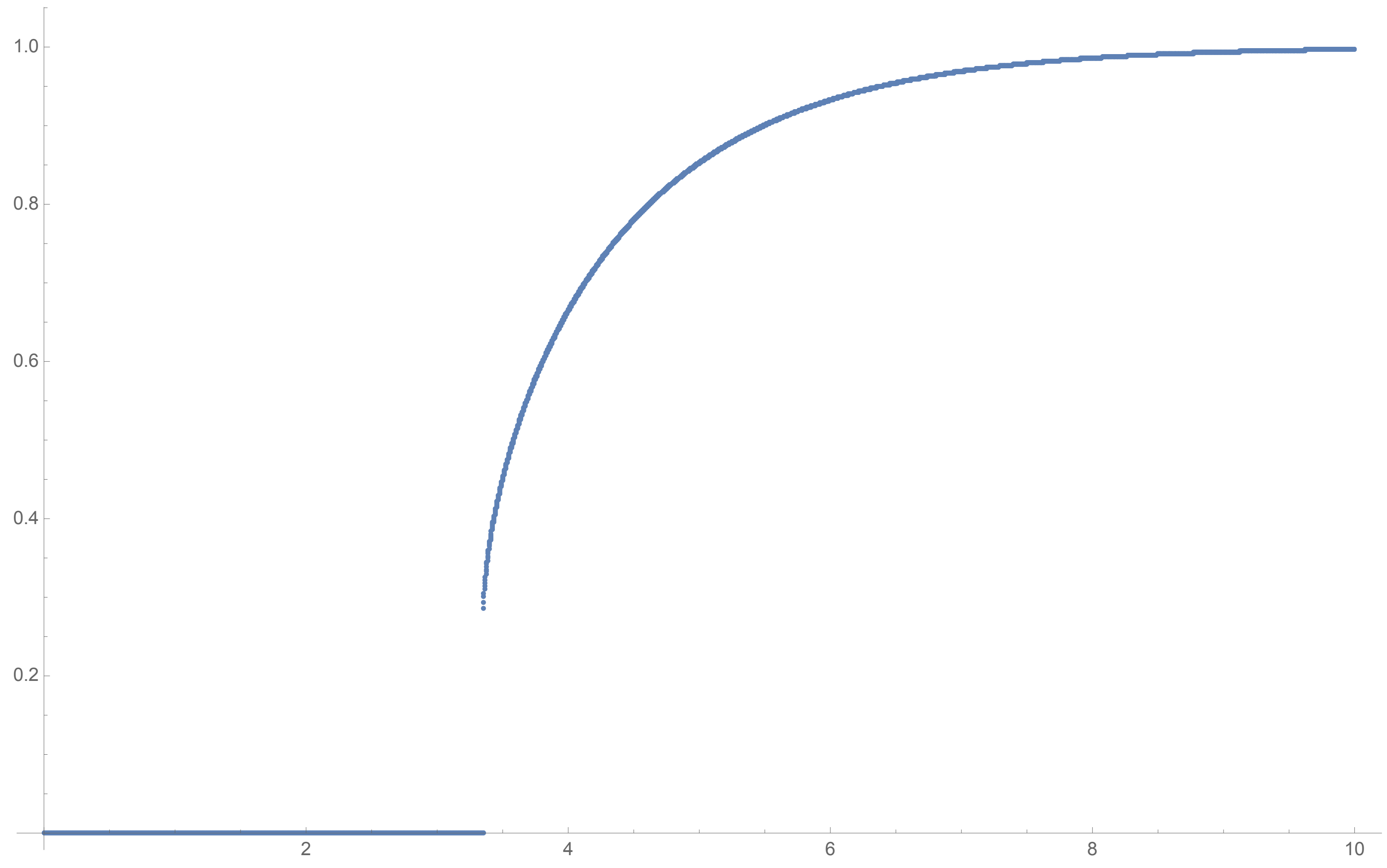

# Karp–Sipser Core

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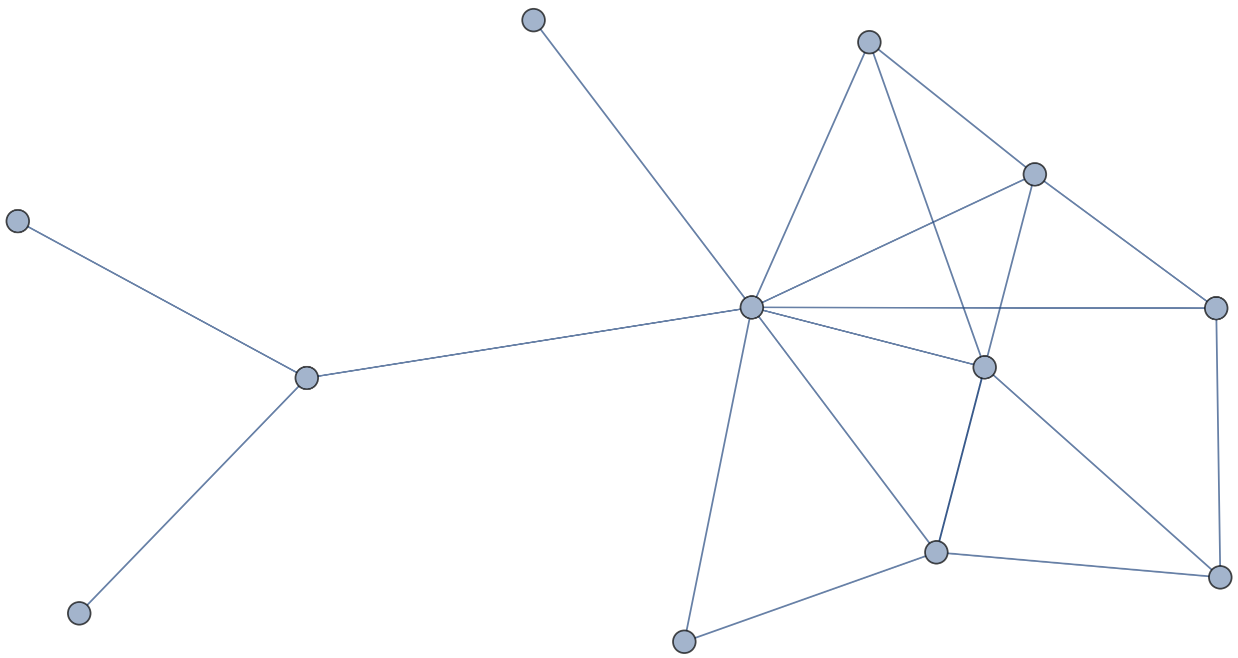
The Karp–Sipser Core of a graph $G$ is the subgraph of $G$ obtained by recursively removing the leaves of $G$ **and their neighbors**.



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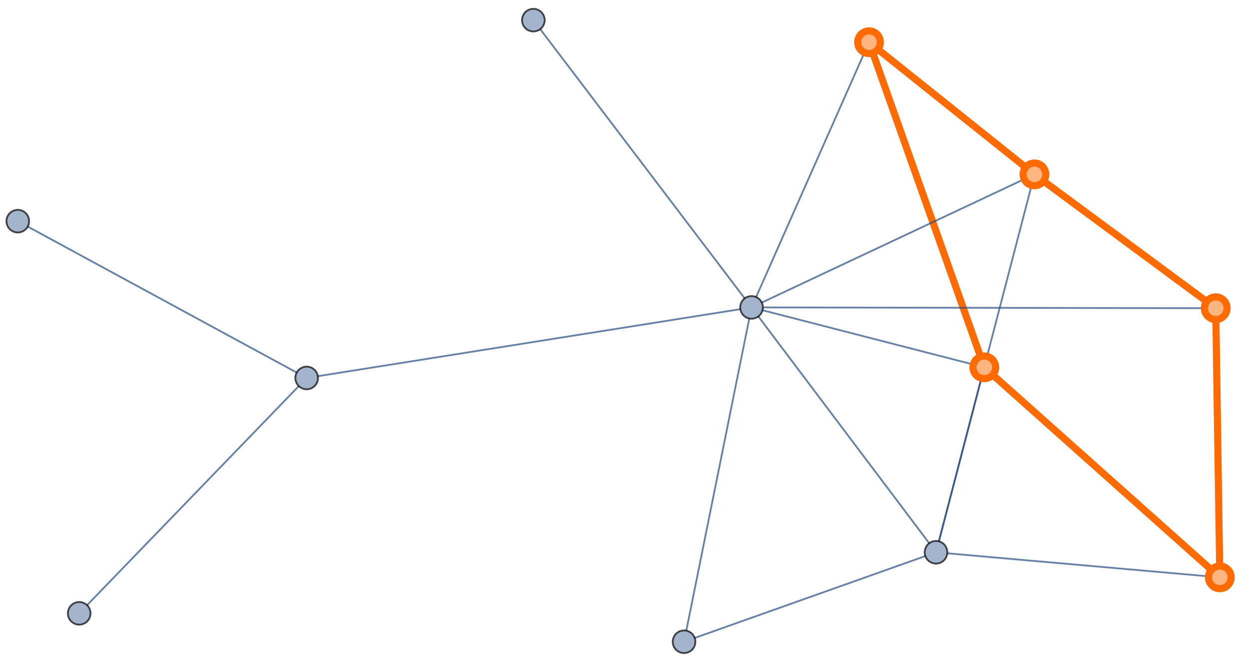


# Phase transition

**Theorem (Karp & Sipser, 81)**

- *(subcritical case) If $c < \mathrm{e}$, then as $n \to \infty$ we have*

$$\left|\mathrm{KSCore}\left(G\left(n, \frac{c}{n}\right)\right)\right| = O_{\mathbb{P}}(1).$$

- *(supercritical case) If $c > \mathrm{e}$, then*

$$n^{-1} \cdot \left|\mathrm{KSCore}\left(G\left(n, \frac{c}{n}\right)\right)\right| \xrightarrow[n\to\infty]{(\mathbb{P})} \beta(c) > 0.$$

# Critical KS

**Conjecture (Bauer & Golinelli, 2001, Table 1 line c)**

In the critical case, we have

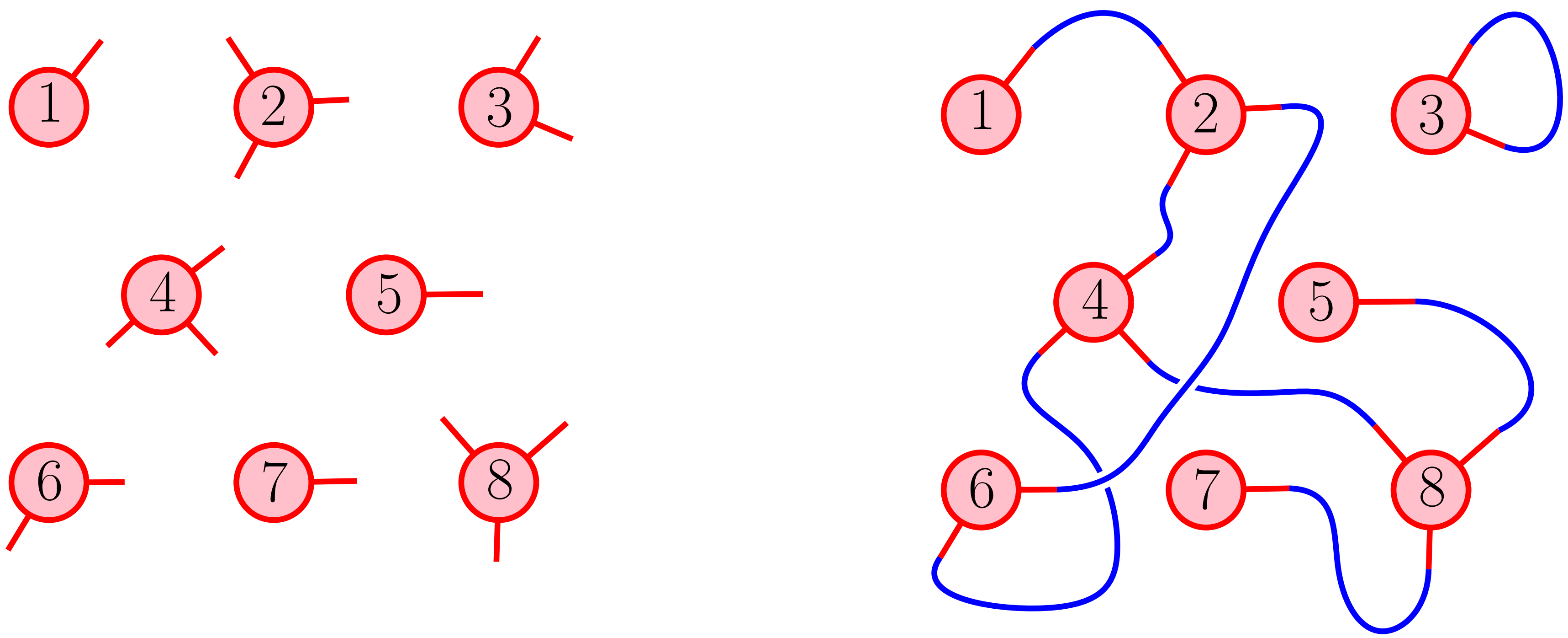
$$\left|\mathrm{KSCore}\left(G\left(n, \frac{\mathrm{e}}{n}\right)\right)\right| \approx n^{3/5}$$

# Our model

Fix $\mathbf{d}^n = (d_1^n, d_2^n, d_3^n)_{n\geq 1}$ (number of vertices) such that

$$n = d_1^n + 2d_2^n + 3d_3^n \text{ is even.}$$

Consider a random multi-graph $\mathrm{CM}(\mathbf{d}^n)$ sampled by pairing the edges emanating for the $d_1^n + d_2^n + d_3^n$ vertices uniformly at random.



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Assume that

$$\frac{d_1^n}{n} \xrightarrow[n\to\infty]{} p_1, \quad \frac{2d_2^n}{n} \xrightarrow[n\to\infty]{} p_2, \quad \text{and} \quad \frac{3d_3^n}{n} \xrightarrow[n\to\infty]{} p_3.$$

# Phase transition revisited

## Theorem (Budzinski, C. & Curien, 2022)

*Let*

$$\Theta = (p_3 - p_1)^2 - 4p_1.$$

# Phase transition revisited

**Theorem (Budzinski, C. & Curien, 2022)**

*Let*

$$\Theta = (p_3 - p_1)^2 - 4p_1.$$

- *(subcritical case) If* $\Theta < 0$*, then as* $n \to \infty$ *we have*

$$|\mathrm{KSCore}\,(\mathrm{CM}(\mathbf{d}^n))| = O_{\mathbb{P}}(\log(n)^2).$$

- *(supercritical case) If* $\Theta > 0$*, then*

$$n^{-1} \cdot |\mathrm{KSCore}\,(\mathrm{CM}(\mathbf{d}^n))| \xrightarrow[n\to\infty]{(\mathbb{P})} \frac{4\Theta}{3+\Theta} > 0.$$

# Critical KS

**Theorem**

*Assume* $\Theta = (p_3 - p_1)^2 - 4p_1 = 0$ *(strictly critical case),*

# Critical KS

## Theorem

*Assume* $\Theta = (p_3 - p_1)^2 - 4p_1 = 0$ *(strictly critical case), and let* $D_2(n)$ *(resp.* $D_3(n)$*) be the total number of half-edges attached to a vertex of degree* 2 *(resp.* 3*) in the KS-core. Then we have*

$$\begin{pmatrix} n^{-3/5} \cdot D_2(n) \\ n^{-2/5} \cdot D_3(n) \end{pmatrix} \xrightarrow[n\to\infty]{(d)} \cdot \begin{pmatrix} C_2 \cdot \vartheta^{-2} \\ C_3 \cdot \vartheta^{-3} \end{pmatrix},$$

*where* $\vartheta = \inf\{t \geq 0 : B_t = t^{-2}\}$, *for a standard linear Brownian motion* $(B_t : t \geq 0)$ *issued from* 0.

# Markovian Exploration

Main idea: Construct the core and attach the half-edges simultaneously.

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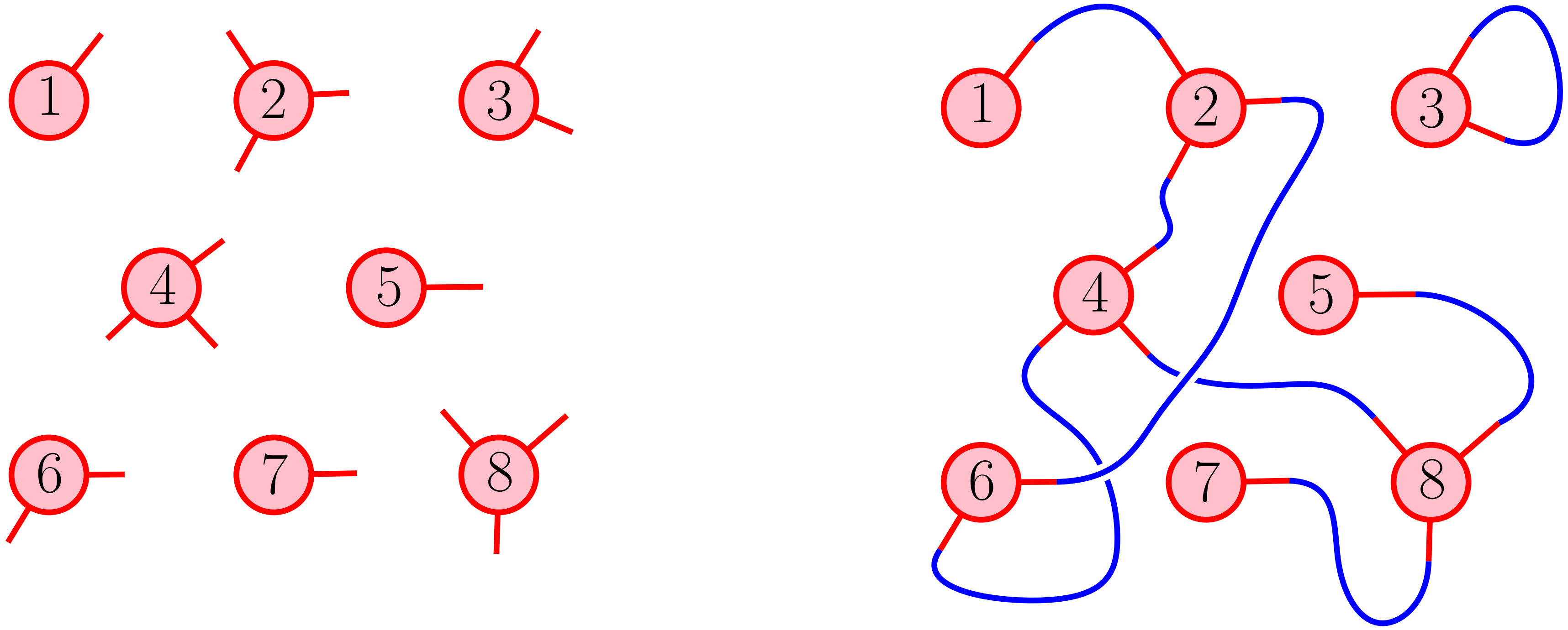

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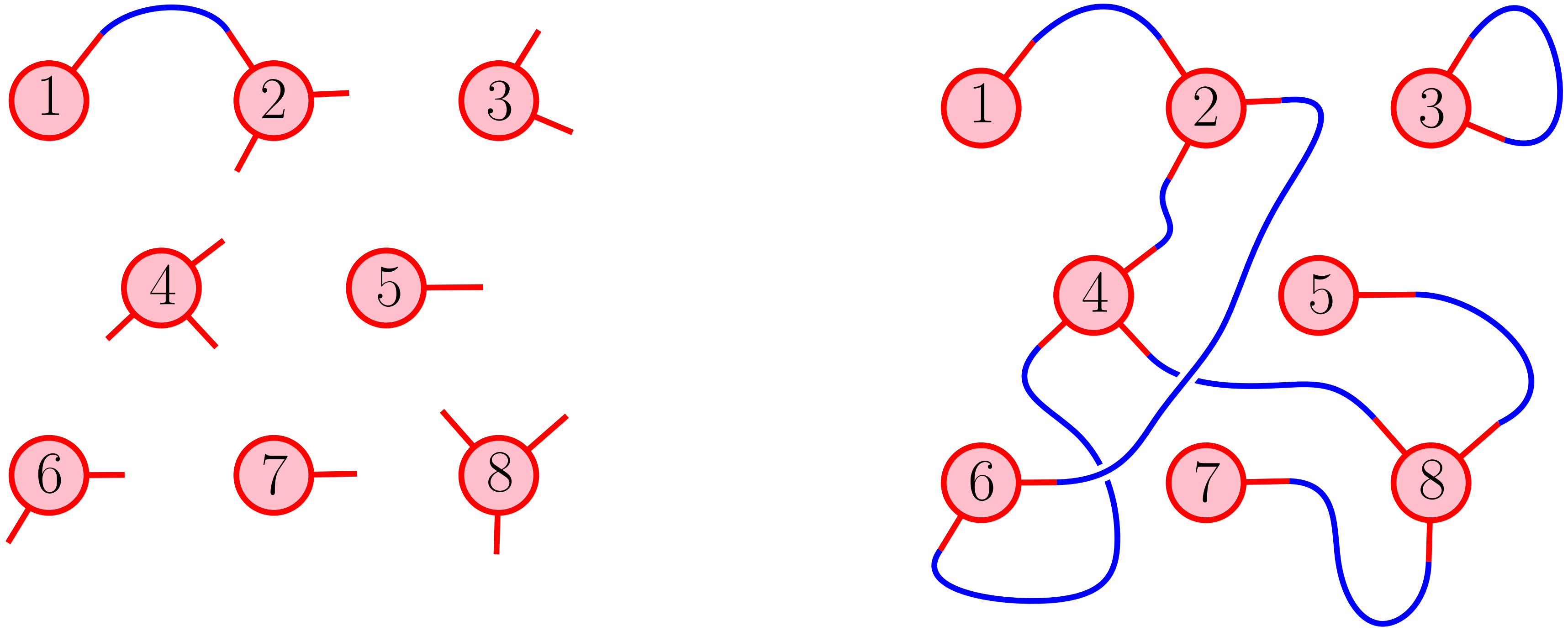

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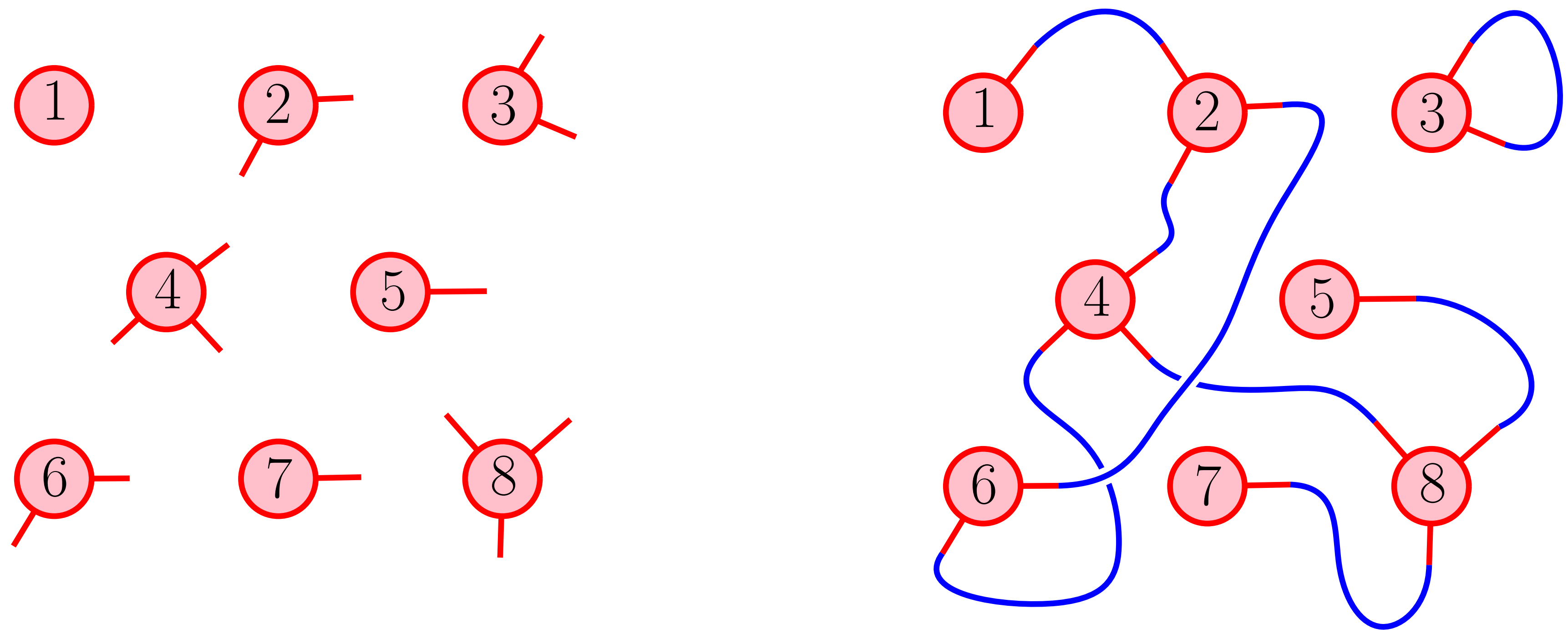

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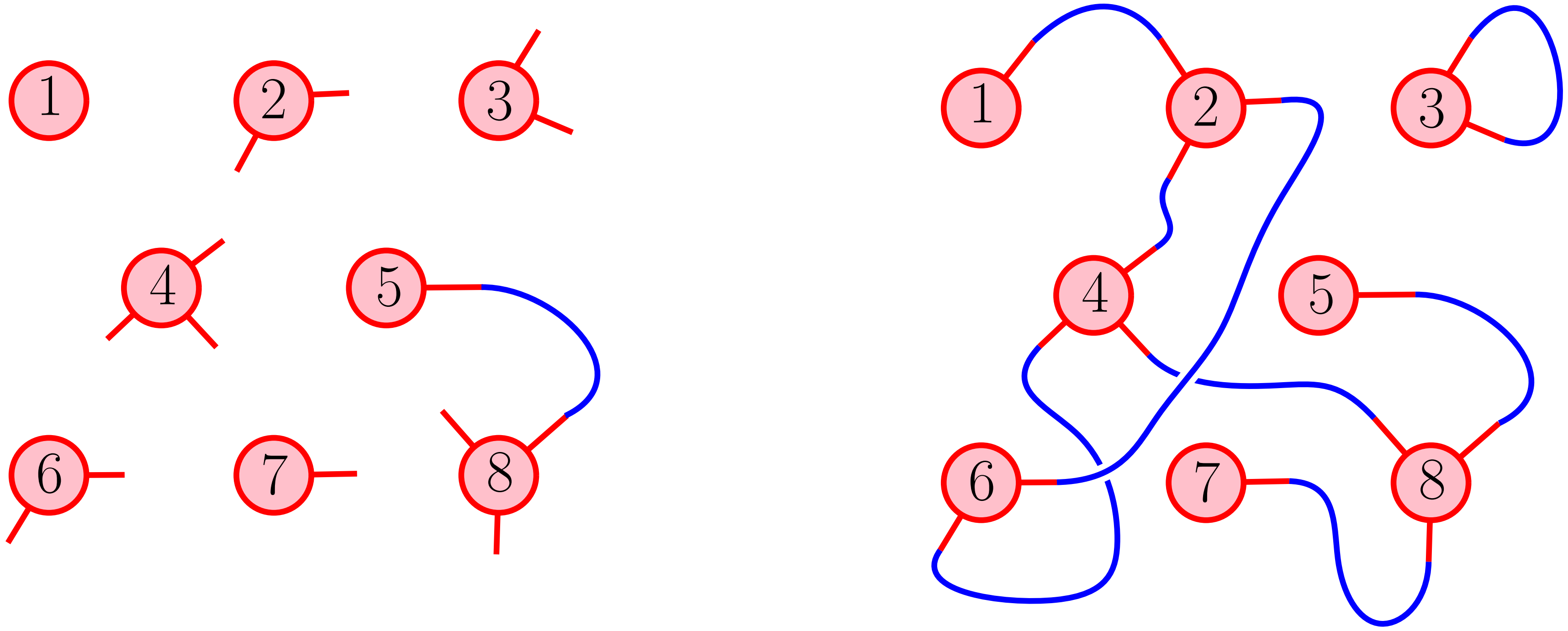

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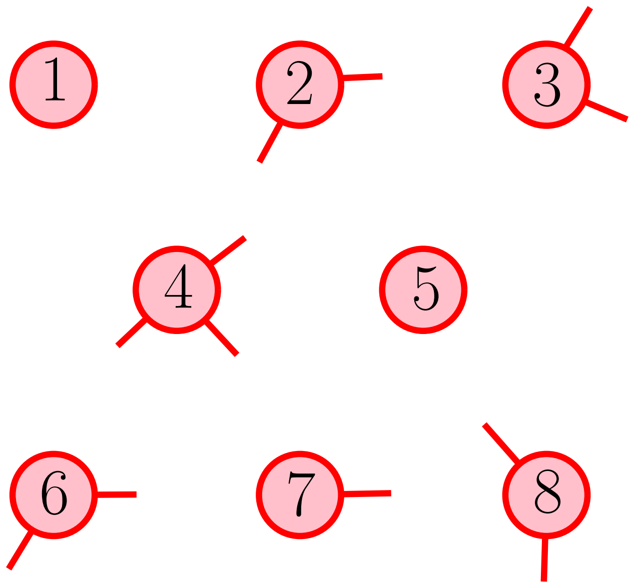
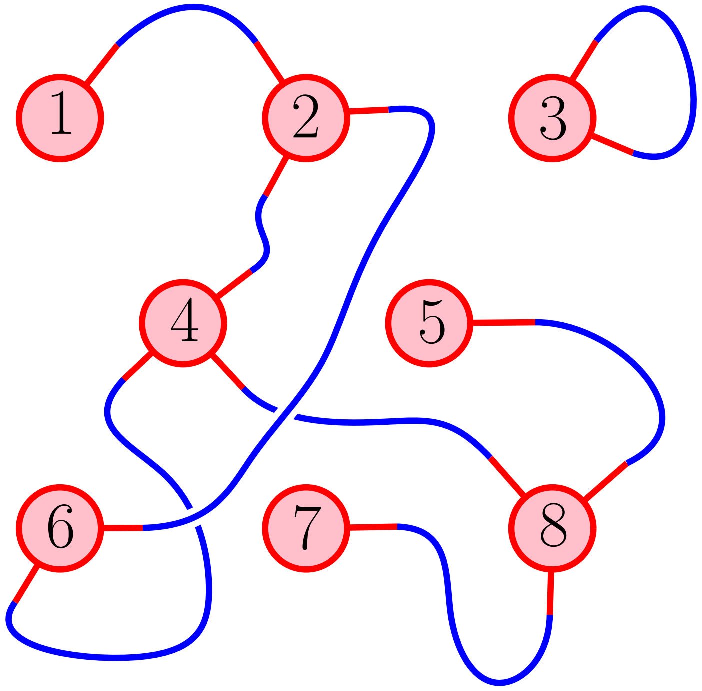


# Markovian Exploration

Main idea: Construct the core and attach the half-edges simultaneously.

We denote by

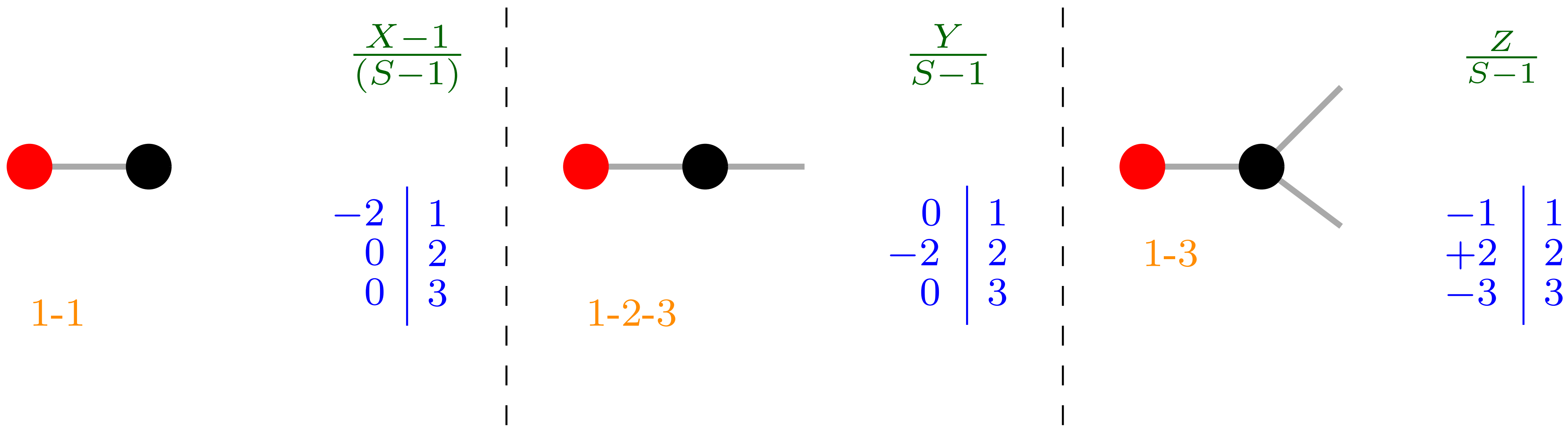
$$(X_k^n, Y_k^n, Z_k^n : k \geq 0)$$

the number of unmatched half-edges linked to vertices of *unmatched degree* $1, 2, 3$ at step $k$.

**Proposition**

$(X_k^n, Y_k^n, Z_k^n : k \geq 0)$ is a Markov chain.

# Example: Transitions for the 2-core

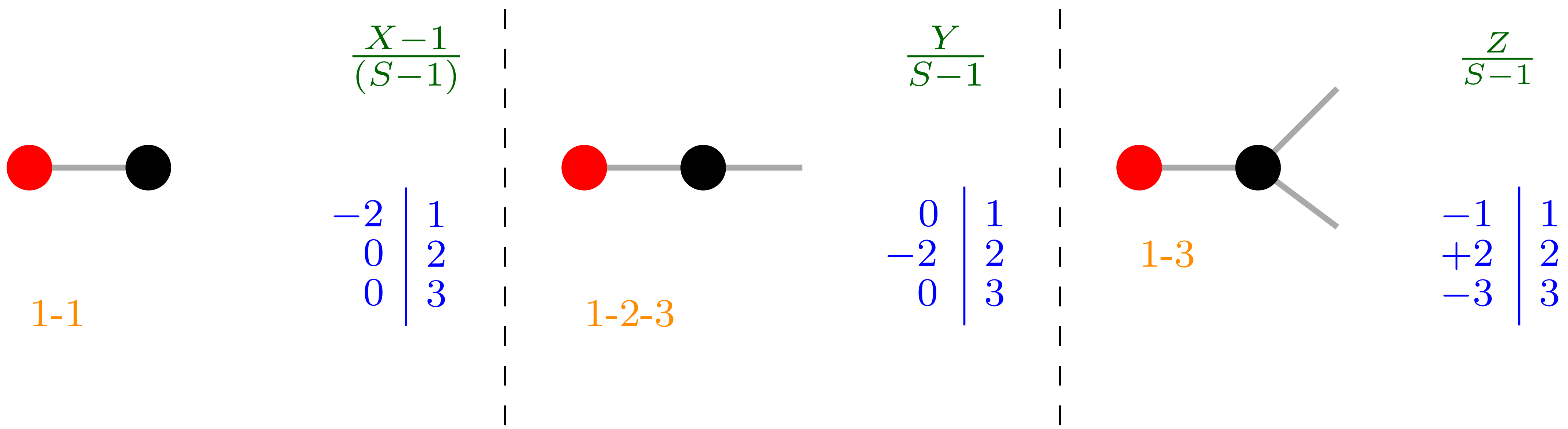


# Fluid limit approximation

**Proposition**

$$\left(\frac{X^n_{\lfloor tn\rfloor}}{n}, \frac{Y^n_{\lfloor tn\rfloor}}{n}, \frac{Z^n_{\lfloor tn\rfloor}}{n}\right)_{0\le t\le \theta^n/n} \xrightarrow[n\to\infty]{(\mathbb{P})} (\mathscr{X}, \mathscr{Y}, \mathscr{Z})_{0\le t\le t_{\mathrm{ext}}}.$$

# Example: Fluid limit for the 2-core




$$x' = \frac{-2x - z}{x + y + z}$$

$$y' = \frac{-2y + 2z}{x + y + z}$$

$$z' = \frac{-3z}{x + y + z}$$

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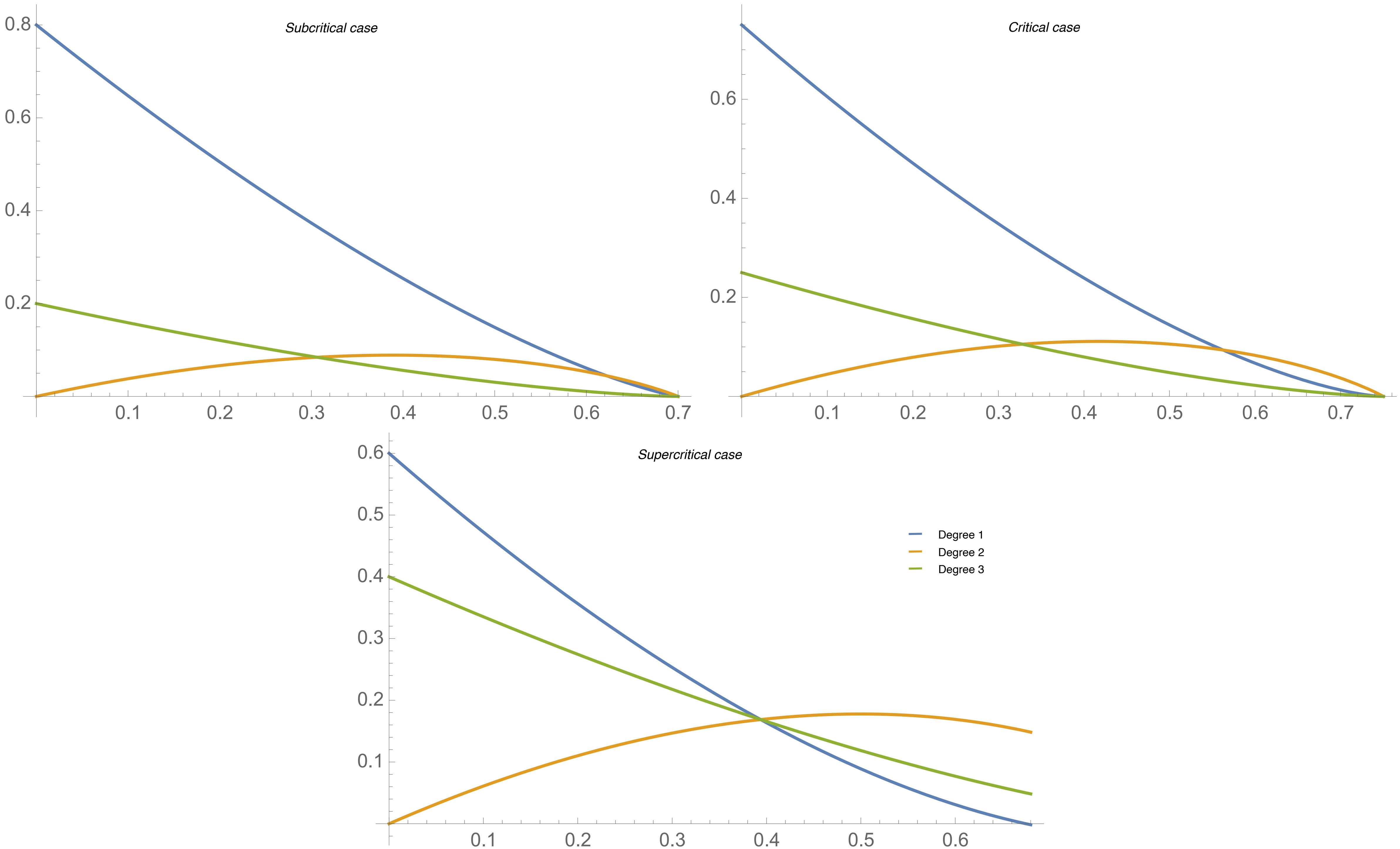
$$x' = \frac{-2x - z}{x + y + z}, \qquad y' = \frac{-2y + 2z}{x + y + z}, \qquad z' = \frac{-3z}{x + y + z}.$$

- We have $(x + y + z)' = -2$.
- We assume $y(0) = 0$ and obtain

$$\begin{cases} x(t) = (1 - 2z_0)(1 - 2t) + z_0(1 - 2t)^{3/2}, \\ y(t) = 2z_0((1 - 2t) - (1 - 2t)^{3/2}), \\ z(t) = z_0(1 - 2t)^{3/2}, \end{cases}$$

# Fluid limit approximation of the 2-core



# KS-core : transitions

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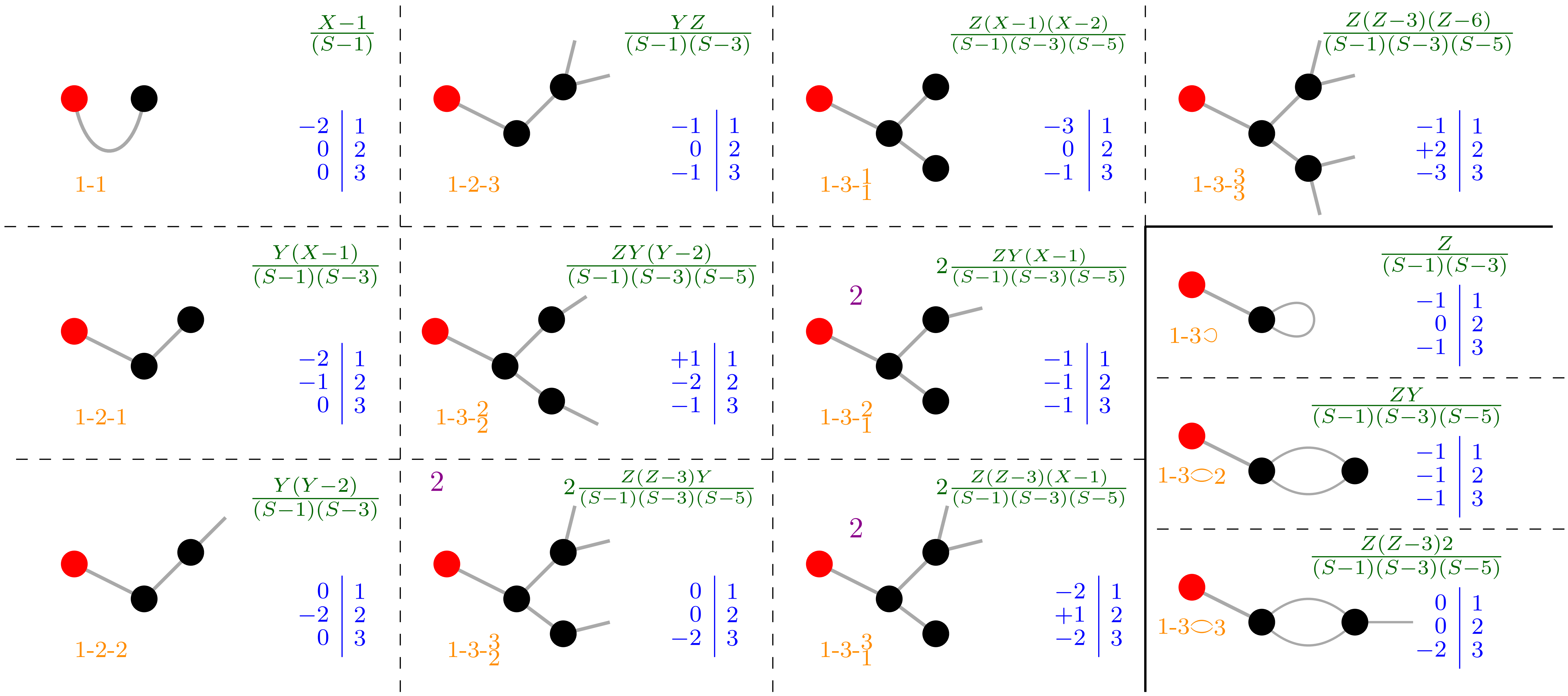

...and its fluid limit approximation:

$$\left( \frac{X^n_{\lfloor tn \rfloor}}{n}, \frac{Y^n_{\lfloor tn \rfloor}}{n}, \frac{Z^n_{\lfloor tn \rfloor}}{n} \right)_{0 \le t \le \theta^n / n} \xrightarrow[n \to \infty]{(\mathbb{P})} (\mathscr{X}, \mathscr{Y}, \mathscr{Z})_{0 \le t \le t_{\mathrm{ext}}},$$

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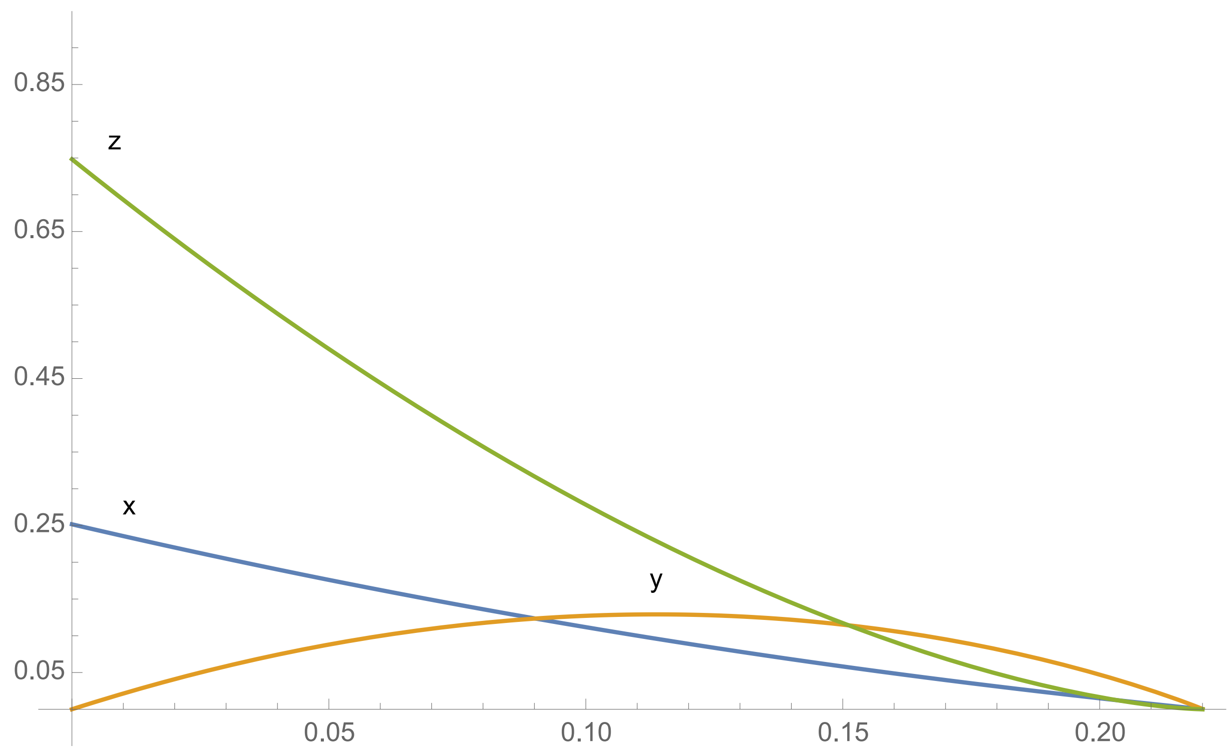
where $(\mathscr{X}, \mathscr{Y}, \mathscr{Z})$ is the unique solution to the differential equation

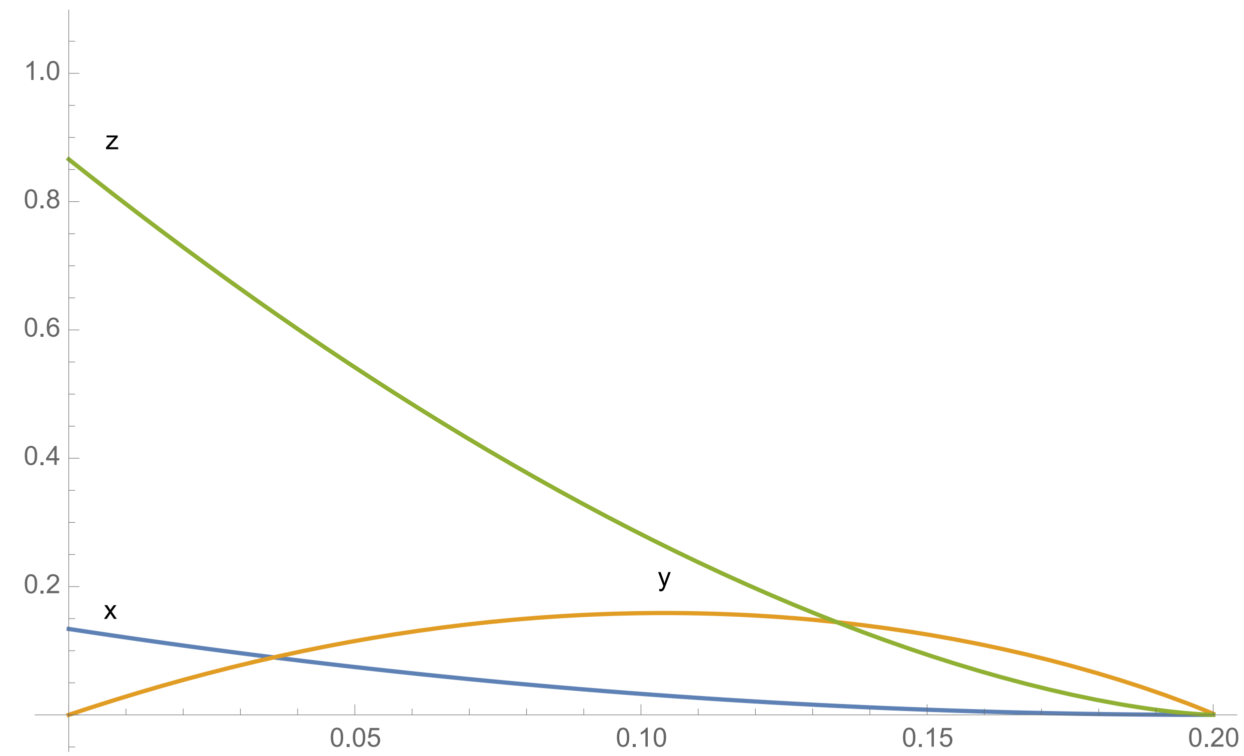
$$\begin{pmatrix} \mathscr{X}' \\ \mathscr{Y}' \\ \mathscr{Z}' \end{pmatrix} = \begin{pmatrix} -2\mathbf{x} - \mathbf{yz} - 3\mathbf{x}^2\mathbf{z} - 2\mathbf{yx} + \mathbf{zy}^2 - 2\mathbf{zxy} - \mathbf{z}^3 - 4\mathbf{z}^2\mathbf{x} \\ 4\mathbf{z}^3 - 2\mathbf{xy} - 4\mathbf{zy}^2 - 4\mathbf{xyz} - 4\mathbf{y}^2 + 4\mathbf{z}^2\mathbf{x} \\ -3\mathbf{yz} - 3\mathbf{zy}^2 - 12\mathbf{z}^2\mathbf{y} - 3\mathbf{zx}^2 - 6\mathbf{xyz} - 12\mathbf{z}^2\mathbf{x} - 9\mathbf{z}^3 \end{pmatrix},$$

$$\text{where } (\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{\mathscr{X} + \mathscr{Y} + \mathscr{Z}}(\mathscr{X}, \mathscr{Y}, \mathscr{Z}) \text{ is the proportion vector,}$$

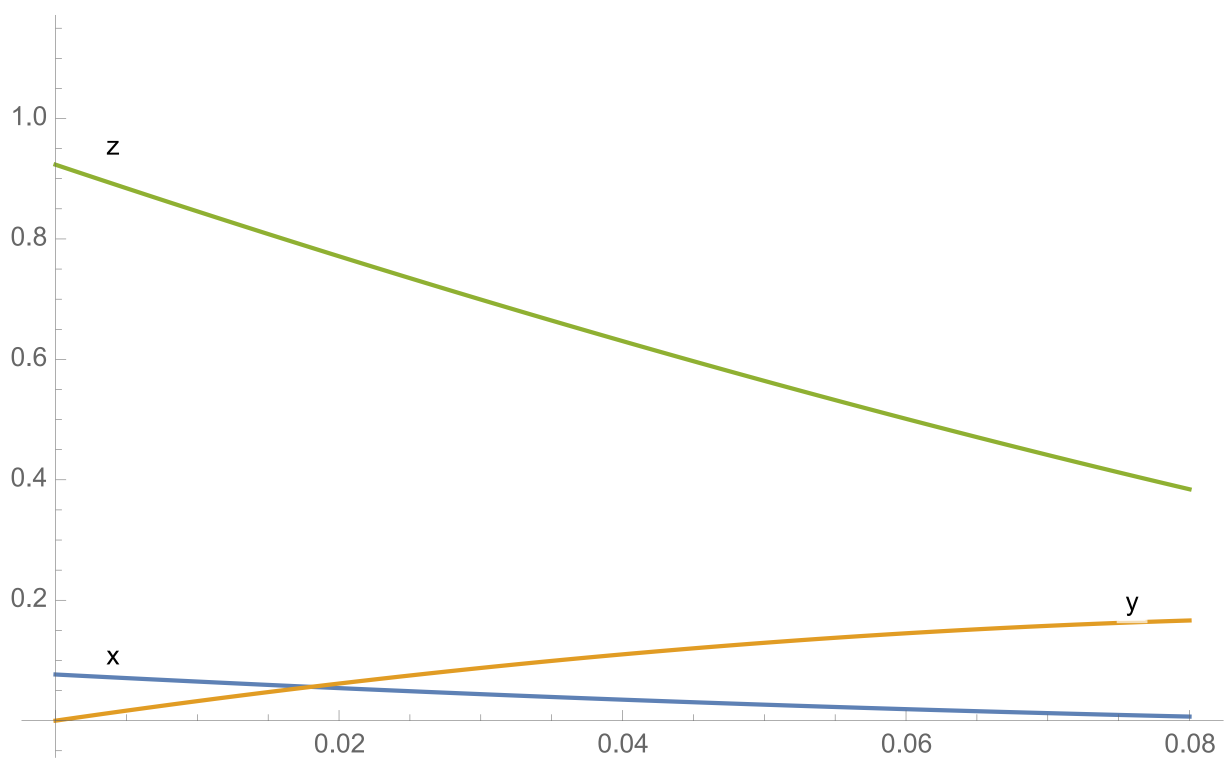
with initial conditions $(p_1, p_2, p_3)$ and where $t_{\mathrm{ext}}$ is the first hitting time of 0 by the continuous process $\mathscr{X}$.



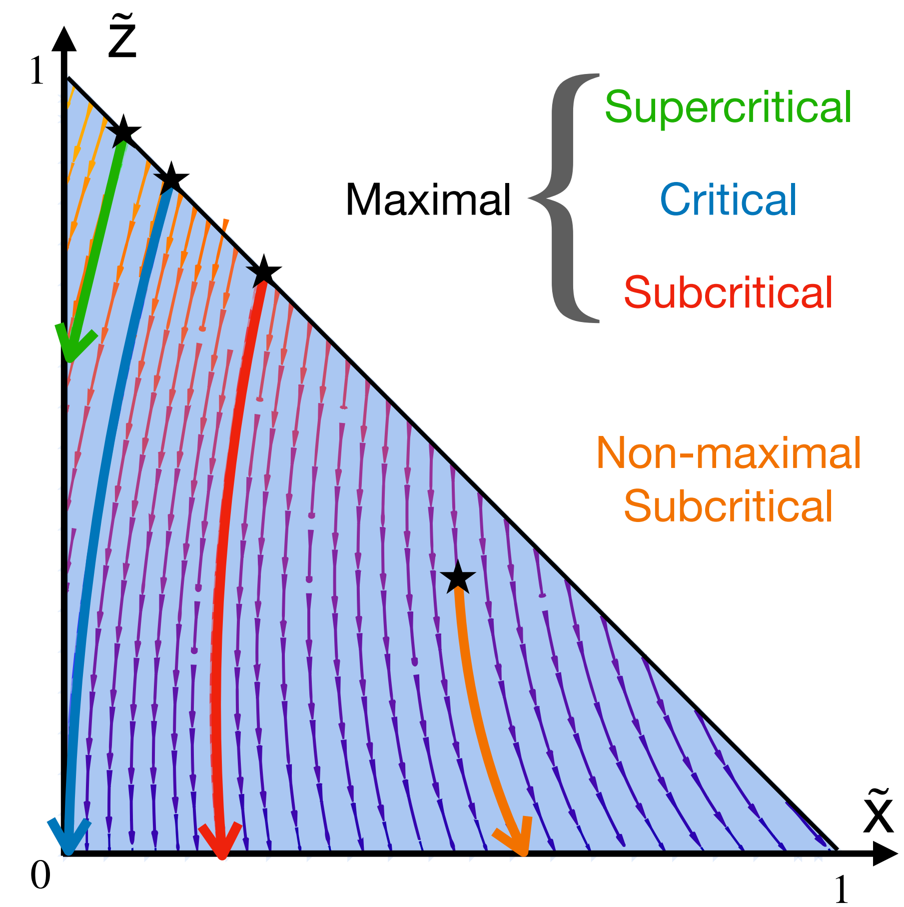
0.85
0.65
0.45
0.25
0.05
z
x
y
0.05
0.10
0.15
0.20

1.0
0.8
0.6
0.4
0.2
z
x
y
0.05
0.10
0.15
0.20

1.0
0.8
0.6
0.4
0.2
z
x
y
0.02
0.04
0.06
0.08

$\tilde{z}$
1
Maximal
Supercritical
Critical
Subcritical
Non-maximal
Subcritical
$\tilde{x}$
0
1

# The fluid limit is not sufficient : Three examples

Two tribes, initially $n$ individuals in each tribe

At each step :

$((X_k, Y_k) : k \geq 0)$ number of individuals in the tribes at step $k$

| Pick an individual uniformly at random and it dies | Pick a tribe uniformly at random and an individual of this tribe dies | Pick an individual uniformly at random and it kills someone in the other tribe |
|---|---|---|
| $\Delta(X_k, Y_k)$ $(X_k, Y_k) =$ | | |
| $\begin{cases} (-1,0) \text{ with proba } \dfrac{X_k}{X_k + Y_k} \\ (0,-1) \text{ with proba } \dfrac{Y_k}{X_k + Y_k} \end{cases}$ | $\begin{cases} (-1,0) \text{ with proba } \dfrac{1}{2} \\ (0,-1) \text{ with proba } \dfrac{1}{2} \end{cases}$ | $\begin{cases} (-1,0) \text{ with proba } \dfrac{Y_k}{X_k + Y_k} \\ (0,-1) \text{ with proba } \dfrac{X_k}{X_k + Y_k} \end{cases}$ |

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$$\begin{cases} \text{(-1,0) with proba } \dfrac{1}{2} \\ \text{(0,-1) with proba } \dfrac{1}{2} \end{cases}$$

Pick an individual uniformly at random and it kill someone in the other tribe

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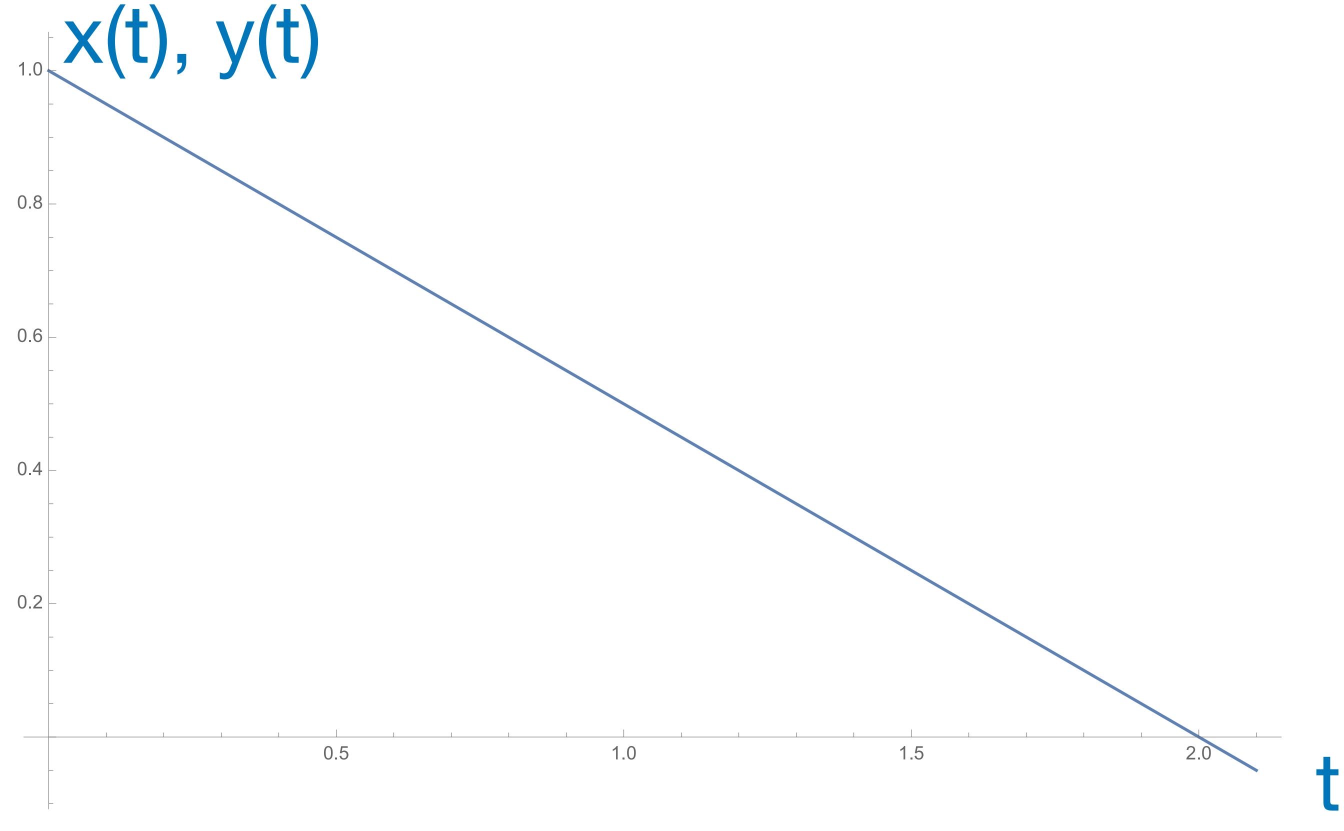

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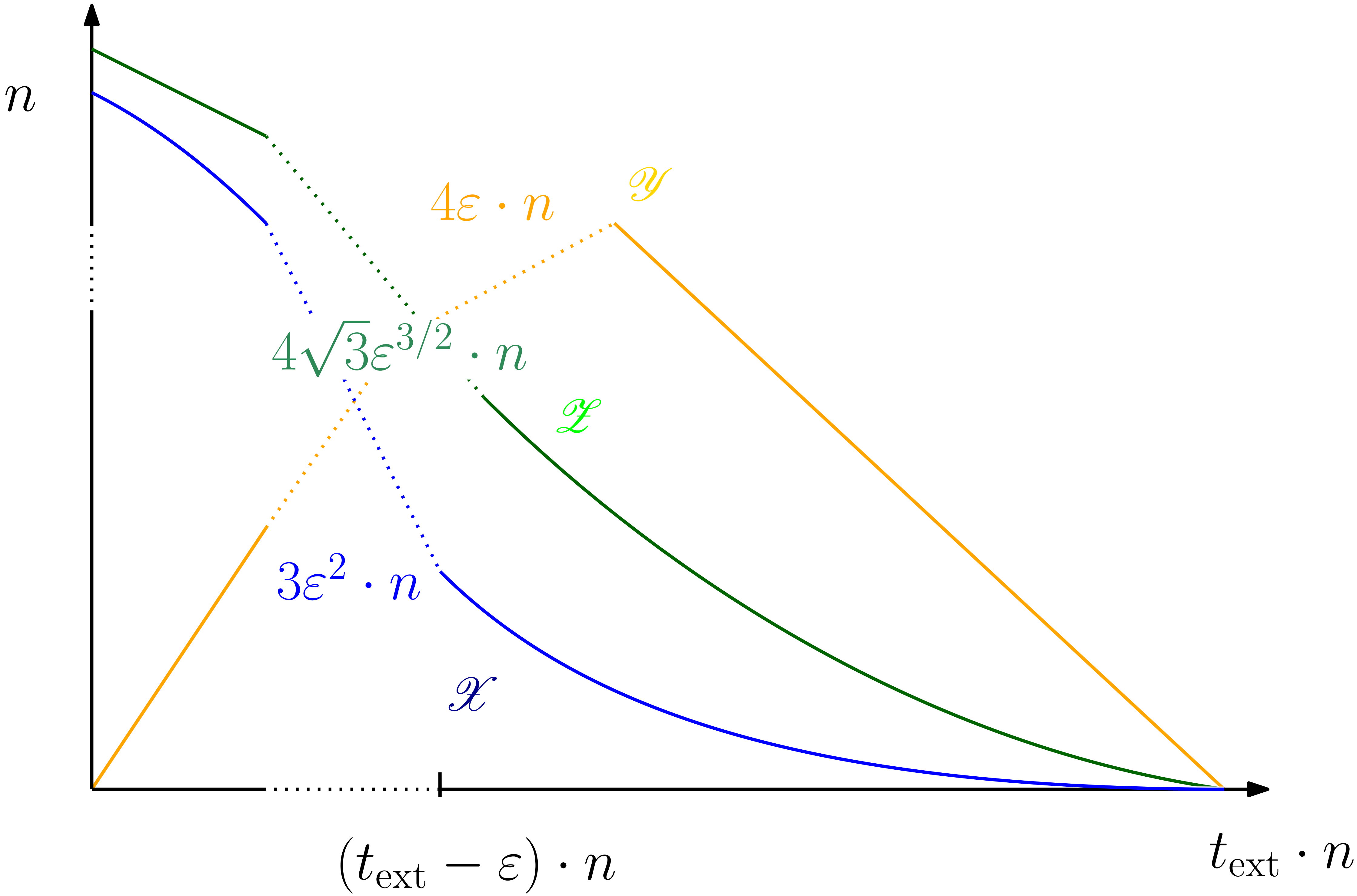
Number of individuals remaining when one tribe dies out

$O_{\mathbb{P}}(1)$

$O_{\mathbb{P}}(\sqrt{n})$

$O_{\mathbb{P}}(n^{3/4})$

# Back to Karp—Sipser



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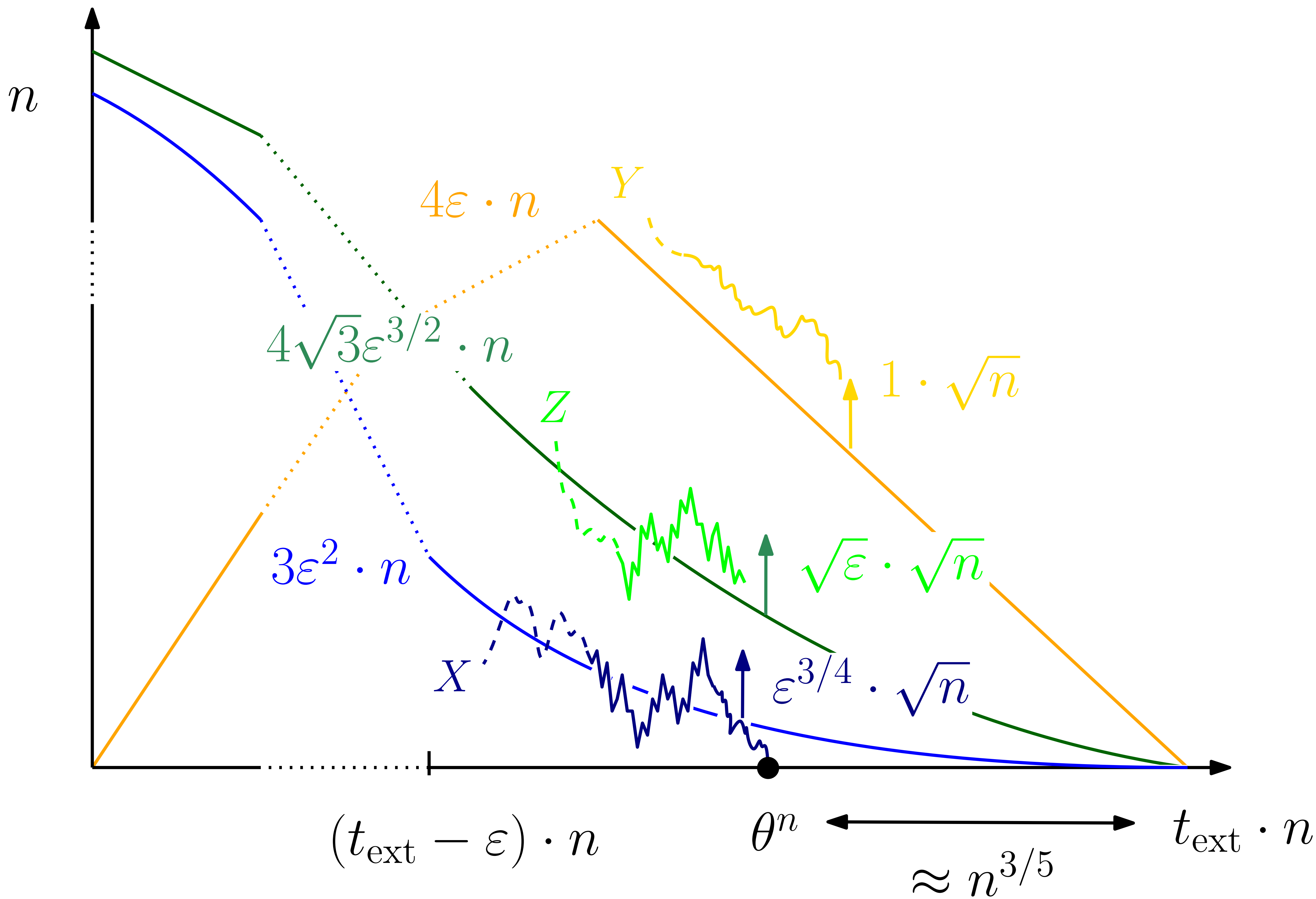
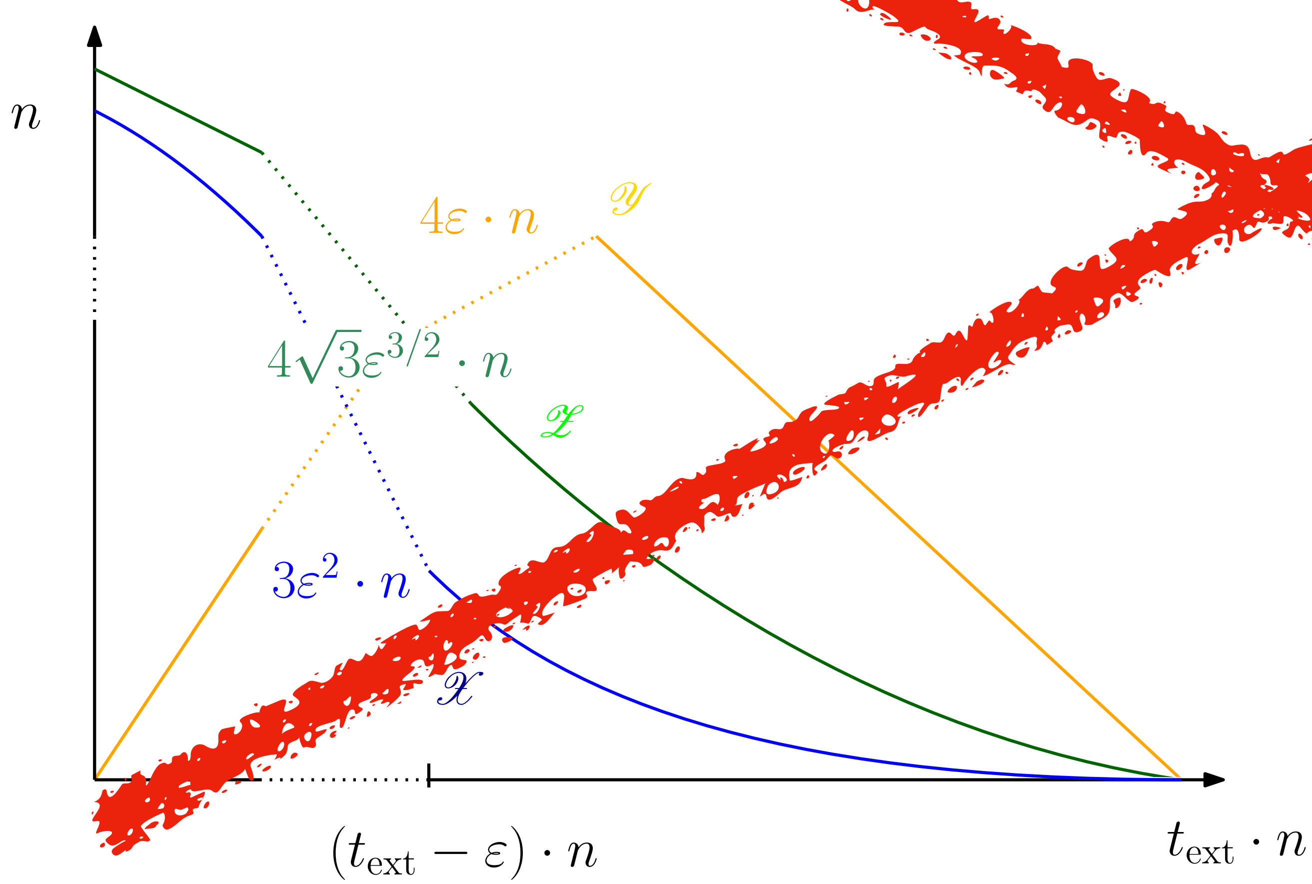


Let

$$A_k^n = X_k^n - n\mathscr{X}\left(\frac{k}{n}\right), \quad B_k^n = Y_k^n - n\mathscr{Y}\left(\frac{k}{n}\right), \quad C_k^n = Z_k^n - n\mathscr{Z}\left(\frac{k}{n}\right),$$

Naive guess : $A_k \approx \sqrt{n}$, so vertices of degree $1$ extinct when

$$A_k \approx n\mathscr{X}\left(\frac{k}{n}\right) \Leftrightarrow \sqrt{n} \approx \varepsilon^2 n \Leftrightarrow \varepsilon \approx n^{-1/4}$$




At that time, there are $\varepsilon n \approx n^{3/4}$ vertices of degree $2$ and $\varepsilon^{3/2} n \approx n^{5/8}$ vertices of degree $3$.

# Control of the fluctuations : the drift

- The fluctuations are smaller !
- The drift brings the $X$ "closer" to its fluid limit $\mathscr{X}$. More precisely:

$$\mathbb{E}[A_{k+1} - A_k \quad A_k, B_k, C_k] \approx -\frac{1}{nt_{\text{ext}} - k} A_k$$

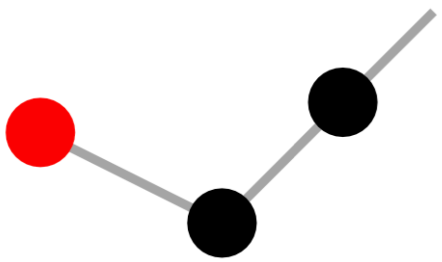
- Between $\frac{t_{\text{ext}} n}{2}$ and $k = (t_{\text{ext}} - \varepsilon)n$, we have,

$$\mathbb{E}[A_k \quad A_{t_{\text{ext}} n/2}] \approx A_{t_{\text{ext}} n/2} \cdot \prod_{i = t_{\text{ext}} n/2}^{k} \left(1 - \frac{1}{t_{\text{ext}} n - i}\right)$$

$$\approx \sqrt{n} \frac{t_{\text{ext}} n - k}{t_{\text{ext}} n} \approx \varepsilon \sqrt{n}$$

# Control of the fluctuations : the variance

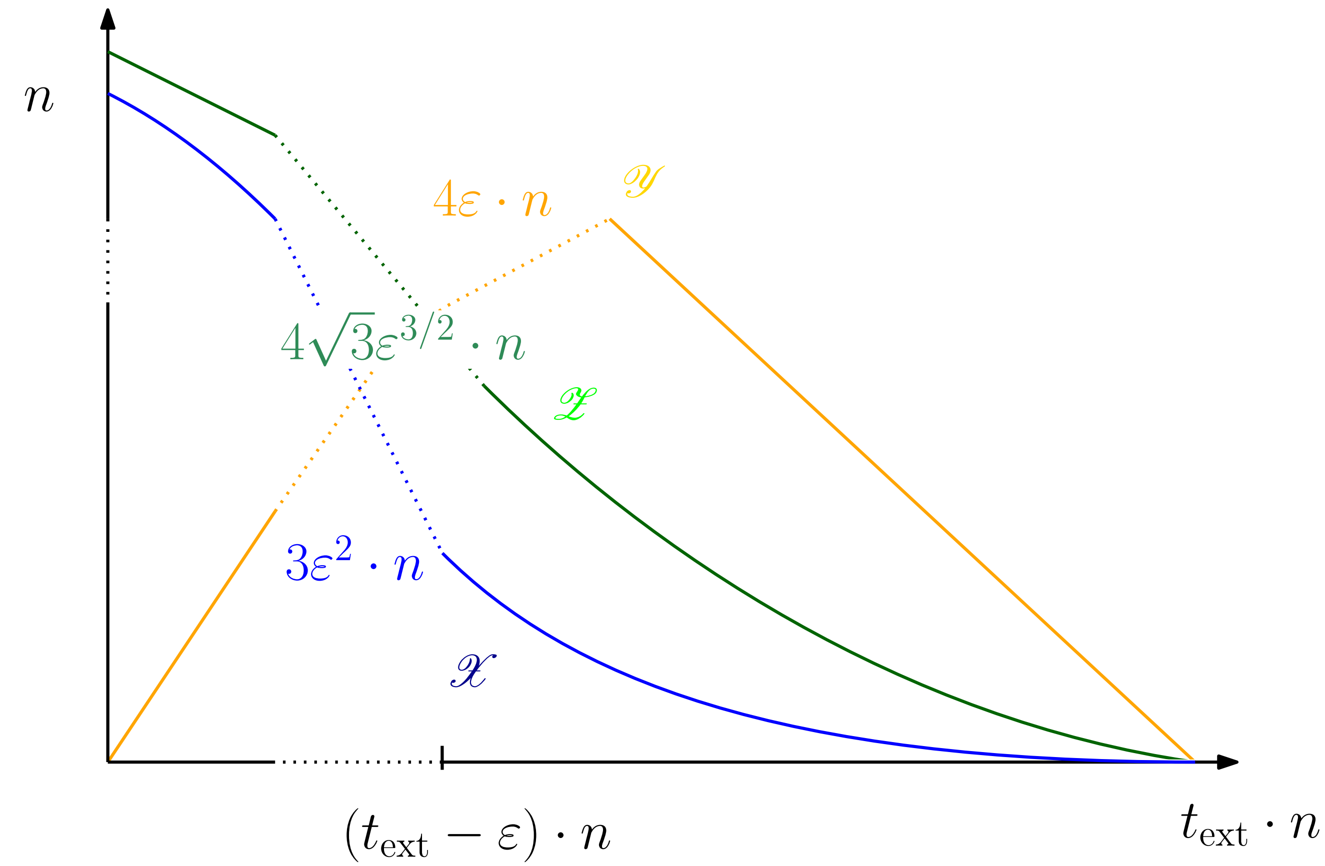
- Dominant case:



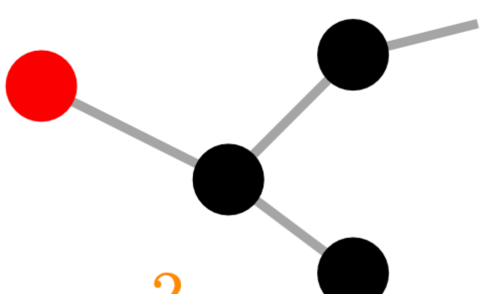
1-2-2

$$\frac{Y(Y-2)}{(S-1)(S-3)}$$

| | |
|---|---|
| 0 | 1 |
| −4 | 2 |
| 0 | 3 |



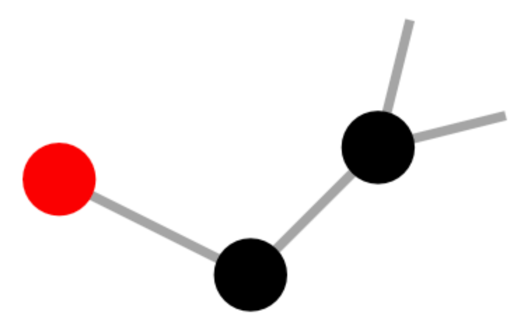

- Next order (probability $\approx \varepsilon^{1/2}$):



1-3-$^2_1$

$$\frac{ZY(X-1)}{(S-1)(S-3)(S-5)}$$

| | |
|---|---|
| −1 | 1 |
| −2 | 2 |
| −3 | 3 |



1-2-3

$$\frac{YZ}{(S-1)(S-3)}$$

| | |
|---|---|
| −1 | 1 |
| 0 | 2 |
| −3 | 3 |

# Control of the fluctuations : the variance

- So X increases or decreases by $1$ with probability $\approx \varepsilon^{1/2}$. Thus, for $k = (t_{\text{ext}} - \varepsilon)n$,

$$\text{Var}[A_{k+1} - A_k \;\; A_k] \approx \varepsilon^{1/2}.$$

- Adding all steps from $k = (t_{\text{ext}} - \varepsilon)n$ to $k' = (t_{\text{ext}} - \dfrac{\varepsilon}{2})n$, we get,

$$\text{Var}[A_{k'} - A_k \;\; A_k] \approx \varepsilon^{1/2} \cdot \varepsilon n \approx \varepsilon^{3/2} n,$$

so the fluctuations coming "from the end" are of order $\varepsilon^{3/4}\sqrt{n}$

- Extinction when $\varepsilon^{3/4}\sqrt{n} \approx \varepsilon^2 n$ i.e. when $\varepsilon \approx n^{-2/5}$.

- There are $\varepsilon n \approx n^{3/5}$ vertices of degree $2$ and $\varepsilon^{3/2} n \approx n^{2/5}$ vertices of degree $3$.

- (We also need to control the fluctuations of $Y$ and $Z$ to ensure that the fluid limit approximation is still good for vertices of degree 2 and 3 before extinction).

# Bonus : final SDE

- Focus on the time scale $k = nt_{\text{ext}} - tn^{3/5}$, and look at the rescaled fluctuations:

$$\widetilde{A_k} = \frac{1}{n^{1/5}}\left(X_k - n\mathscr{X}\left(\frac{k}{n}\right)\right).$$

- Drift and variance estimates:

$$\mathbb{E}[\widetilde{A}_{k+1} - \widetilde{A}_k \quad \widetilde{A}_k] \approx -\frac{1}{tn^{3/5}}\widetilde{A}_k, ,$$

$$\mathrm{Var}[\widetilde{A}_{k+1} - \widetilde{A}_k \quad \widetilde{A}_k] \approx 2\sqrt{3}\sqrt{t}n^{-3/5}.$$

- So, $$\left(\frac{1}{n^{1/5}}A_{nt_{\text{ext}}+tn^{3/5}} : -K \le t \le 0\right) \xrightarrow[n\to\infty]{(d)} \left(F_t : -K \le t \le 0\right),$$

where $$\mathrm{d}F_t = -\frac{1}{t}F_t\mathrm{d}t + \sqrt{2\sqrt{3}\quad t^{\,1/4}}\,\mathrm{d}B_t\,.$$

# Thank you for your attention !