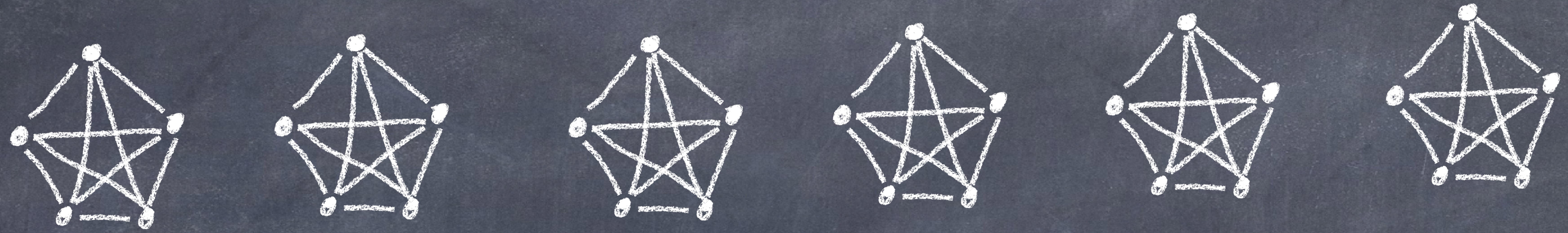
The threshold bias of the Clique-factor game



Anita Liebenau & Rajko Nenadov

Oxford Discrete Mathematics & Probability Seminar

20 October 20

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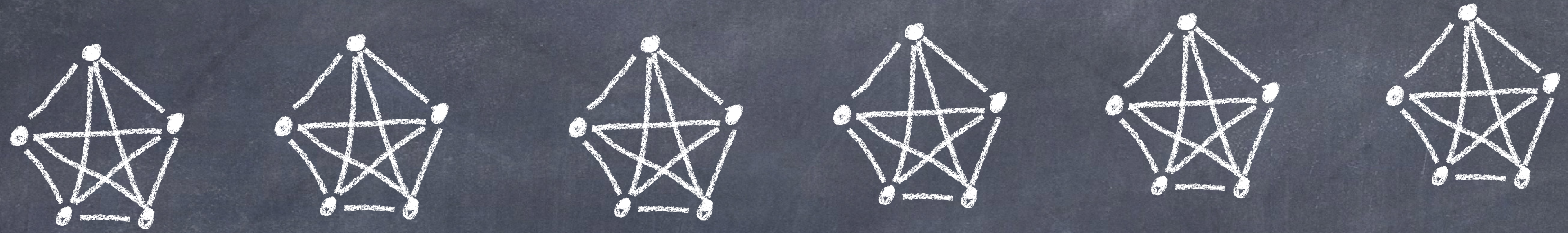
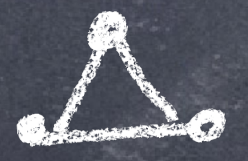


$\frac{n}{r}$ vertex-disjoint copies of K_r

- spanning
- every vertex contained in 
- r constant, n very large

b - biased Maker-Breaker games — the rules

- $(X, \mathcal{F})$ board & family of winning sets
- in round i: Maker claims 1 element of X, Breaker claims b elements
- winner: Maker if Maker occupies all elements of some $F \in \mathcal{F}$

  Otherwise Breaker wins

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Examples: $X = E(K_n)$

$\mathcal{F} = \{$all copies of $\triangle\}$

$\mathcal{F} = \{$all PM's$\}$

$\mathcal{F} = \{$all spanning trees$\}$

$\mathcal{F} = \{$all HAM cycles$\}$

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Threshold bias $b^* = b^*(\mathcal{F}, n)$

smallest b s.t. Breaker wins

$(\mathcal{F} \neq \emptyset$ & $|A| \geq 2 \ \forall A \in \mathcal{F})$

Threshold biases for some natural games on K_n

Connectivity game $(1+o(1))\frac{n}{\ln n}$ [Chvátal & Erdős '78, Beck '82, Gebauer & Szabó '09]

Perfect Matching game $(1+o(1))\frac{n}{\ln n}$

Hamiltonicity game $(1+o(1))\frac{n}{\ln n}$

[C&E '78, Beck '85
Bollobás & Papaioannou '82
Krivelevich & Szabó '08
Krivelevich '11]

$\triangle$-game $\Theta(\sqrt{n})$ [C&E '78, Balogh & Samotij '11
Glazik & Srivastav '18+]

Threshold biases for some natural games on K_n

	Clever Game		Random Game
Connectivity game	$(1+o(1))\frac{n}{\ln n}$	$=$	$(1+o(1))\frac{n}{\ln n}$
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$\triangle$- game	$\Theta(\sqrt{n})$	$\neq$	$\Theta(n)$

$$\boxed{\begin{array}{c} M \sim G(n,m) \\ m = \frac{1}{b+1}\binom{n}{2} \end{array}}$$

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Random Graph Intuition — Erdős paradism

Bednarska & Łuczak 2000

$\underline{\text{Theorem : H-game } (v(H) \geq 3)}$

$\forall$ such H $\exists\, c, C > 0$ s.t. for all n :

$$c\, n^{1/m_2(H)} \leq b^*(H\text{-game}, n) \leq C \cdot n^{1/m_2(H)}$$

- $m_2(H) = \max \left\{ \frac{e(H')-1}{v(H')-2} : H' \subseteq H, v(H') \geq 3 \right\}$

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Theorem : H-game ($v(H) \geq 3$)

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winning strategy for Breaker

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- winning strategy for Maker $\longrightarrow$ random strategy
- $G(n,m)$ robustly contains H :

$$\forall F \subseteq G(n,m) \text{ s.t. } |F| \leq \varepsilon \cdot m : H \subseteq G(n,m) \setminus F$$

Bednarska & Łuczak 2000

$\boxed{\begin{array}{c}\underline{\text{Theorem : } H\text{-game} \quad (v(H) \geq 3)} \\ \forall \text{ such } H \ \exists\, c, C > 0 \text{ s.t. for all } n: \\ c\, n^{1/m_2(H)} \leq b^*(H\text{-game}, n) \leq C \cdot n^{1/m_2(H)}\end{array}}$

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- $G(n,m)$ robustly contains H:

  $\forall F \subseteq G(n,m)$ s.t. $|F| \leq \varepsilon \cdot m$: $H \subseteq G(n,m) \setminus F$
- Problem: if winning structure is spanning

Ferber, Krivelevich, Naves 2015

"Local Resilience"

- For every fixed strategy of Breaker Maker draws a random graph $\Gamma \sim G(n,p)$.

$\boxed{\text{If } C \cdot \frac{\ln n}{n} \leqslant p \leq c \cdot \frac{1}{b} \text{ then w.h.p. } \Gamma \text{ is such that Maker can claim a subgraph } M \subseteq \Gamma \text{ s.t. } \delta(M) \geq (1-\varepsilon)\, p \cdot n}$

$\longrightarrow M = G(n,p) \setminus B$

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- Example: PM, HAM, Connectivity for $b = c \cdot \frac{n}{\ln n}$

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- Example: PM, HAM, Connectivity for $b = c \cdot \frac{n}{\ln n}$
- Problem: If every vertex is in $\triangle$ in winning structure

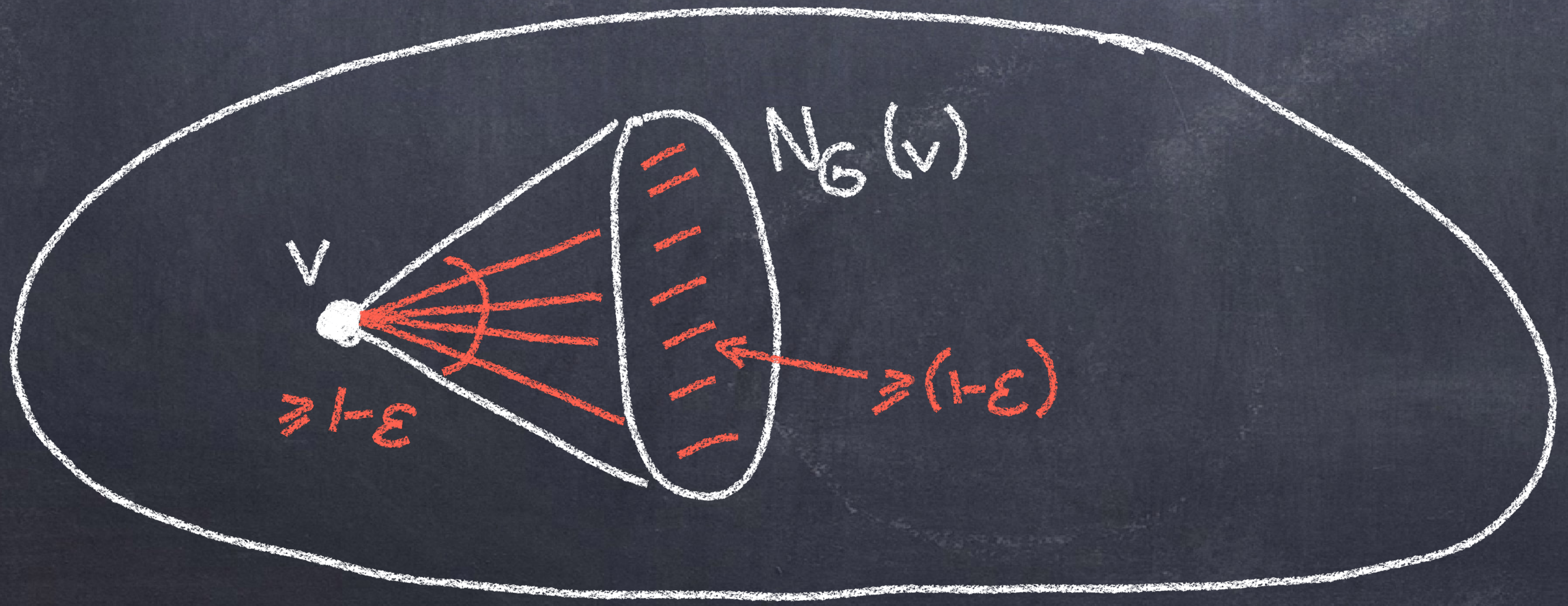
Allen, Böttcher, Kohayakawa, Naves, Person 2017+

Theorem

Let n, ε be given. Let $p \geq 10^8 \varepsilon^{-2} n^{-1/2}$ and $p \leq 10^{-24} \varepsilon^6 b^{-1}$.

In the b-biased game on $E(K_n)$, Maker can claim a subgraph M of $G \sim G(n,p)$ such that for all $v \in V$:

$$d_M(v) \geq (1-\varepsilon)np \quad \text{and} \quad e(M[N_G(v)]) \geq (1-\varepsilon)\frac{p^3n^2}{2}.$$



$G \sim G(n,p)$

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

<u>Corollary</u>

For all D there exists c such that for all n &
for all H on n vertices with $\Delta(H) \leq D$:
Maker wins the b-biased H-game on K_n if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/D}$$

Proof:

- $M \subseteq G(n,p)$ s.t. $\delta(M) \geq (1-\varepsilon)np$ & $e(M[N_G(v)]) \geq (1-\varepsilon)\frac{n^2p^3}{2}$
- "Sparse Blow-up lemma" for $G(n,p)$

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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Remarks: (1) universal result

(2) more general version for "bounded degree & bounded degeneracy"

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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For all D there exists c such that for all n & for all H on n vertices with $\Delta(H) \leq D$:

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Example 1: If $b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/2}$ then Maker wins Δ-factor game.

Remember: Breaker wins Δ-game for $b \geq 2\sqrt{n}$.

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

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Example 2: If $b \leq c \cdot \left(\frac{n}{\log n} \right)^{1/3}$ then Maker wins K_4-factor game.

They prove: Breaker wins K_4-factor game for $b \geq Cn^{1/3}$.

Allen, Böttcher, Kohayakawa, Naves, Person 2017+

Corollary

For all D there exists c such that for all n & for all H on n vertices with $\Delta(H) \leq D$:
Maker wins the b-biased H-game on K_n if

$$b \leq c \cdot \left(\frac{n}{\log n}\right)^{1/D}$$

- $b^*(K_{D+1}\text{-factor}, n) = n^{1/D + o(1)}$ for $D = 2, 3$
- "$\frac{1}{D}$ probably not correct for larger D"

L. & Nenadov 2020+

Theorem

For all $D \geq 3$ $\exists c$ such that for every $n \in (D+1)\mathbb{Z}$

in the b-biased MB-game on K_n

Maker wins the K_{D+1}-factor game if $b \leq c \cdot n^{\frac{2}{D+3}}$

L. & Nenadov 2020+

Theorem

For all $D \geq 3$ $\exists\, c, C$ such that for every $n \in (D+1)\mathbb{Z}$ in the b-biased MB-game on K_n

(a) Maker wins the K_{D+1}-factor game if $b \leq c \cdot n^{\frac{2}{D+3}}$

(b) Breaker wins the K_{D+1}-factor game if $b \geq C \cdot n^{\frac{2}{D+3}}$

Threshold bias $b^*(\mathcal{F}, n)$

	Clever Game		Random Game
Connectivity, PM, HAM	$(1+o(1))\frac{n}{\ln n}$	$=$	$(1+o(1))\frac{n}{\ln n}$
$\triangle$- game	$\Theta(\sqrt{n})$	$\neq$	$\Theta(n)$
$\triangle$- factor game	$n^{\frac{1}{2}+o(1)}$		$n^{\frac{2}{3}+o(1)}$
K_{D+1}- factor game	$n^{\frac{2}{D+3}}$		$n^{\frac{2}{D+1}+o(1)}$
	our result		Johansson, Kahn, Vu 2008

Why $n^{\frac{2}{D+3}}$?

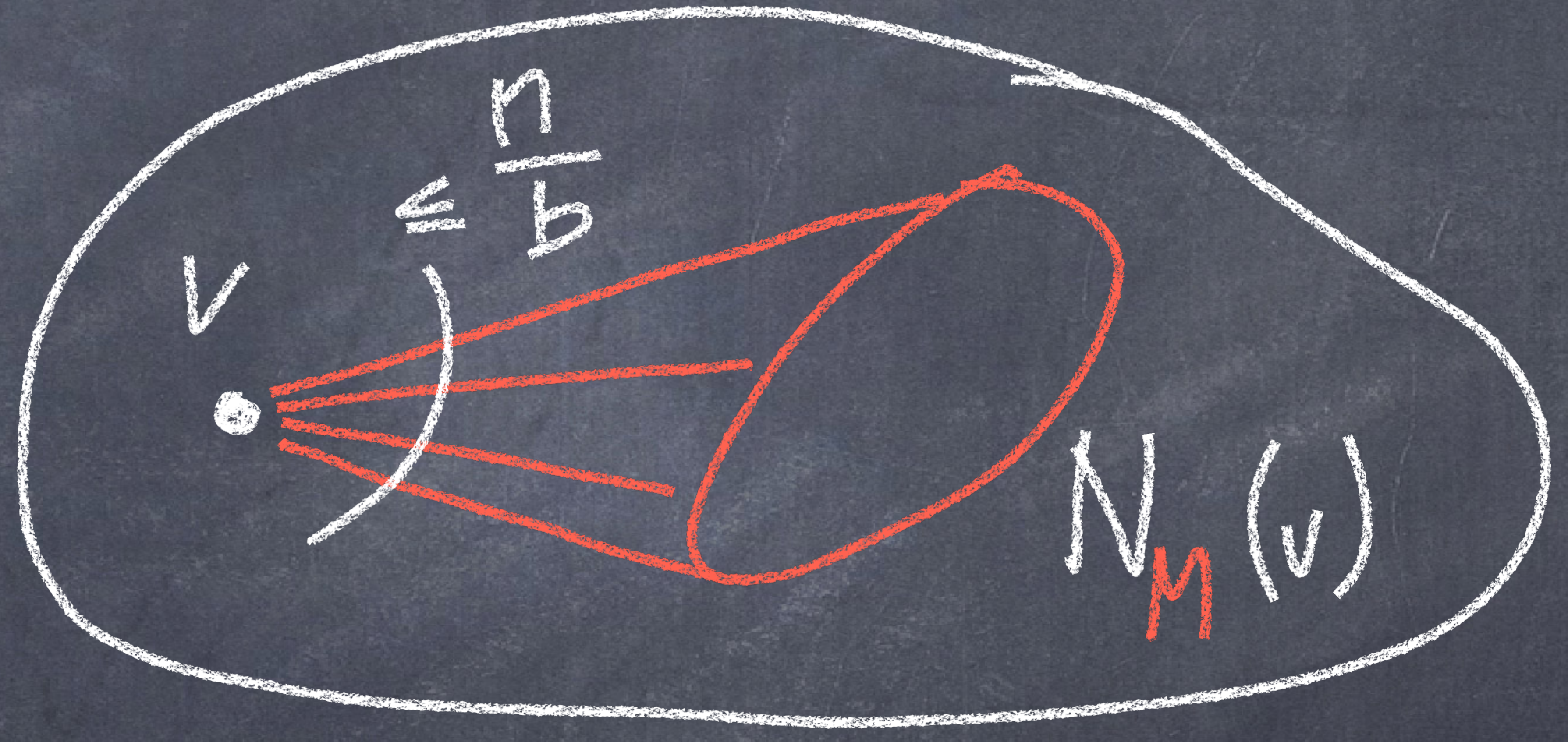
- every v is in a copy of K_{D+1}
- For which b can Breaker achieve "$\exists\, v$ that is not in a copy of K_{D+1}" ?

Why $n^{\frac{2}{D+3}}$?

▷ For which b can Breaker achieve

"$\exists\, v$ that is not in a copy of K_{D+1}" ?

- Fix v
- always play b edges at v
- prevent K_D in $N_M(v)$



→ use Bednarska-Łuczak

→ works if $b \geq C \cdot \left(\frac{n}{b}\right)^{\frac{1}{m_2(K_D)}} \iff b \geq C' n^{\frac{1}{m_2(K_D)+1}} = C' n^{\frac{2}{D+3}}$

Maker's strategy for $b \leq cn^{\frac{2}{D+3}}$

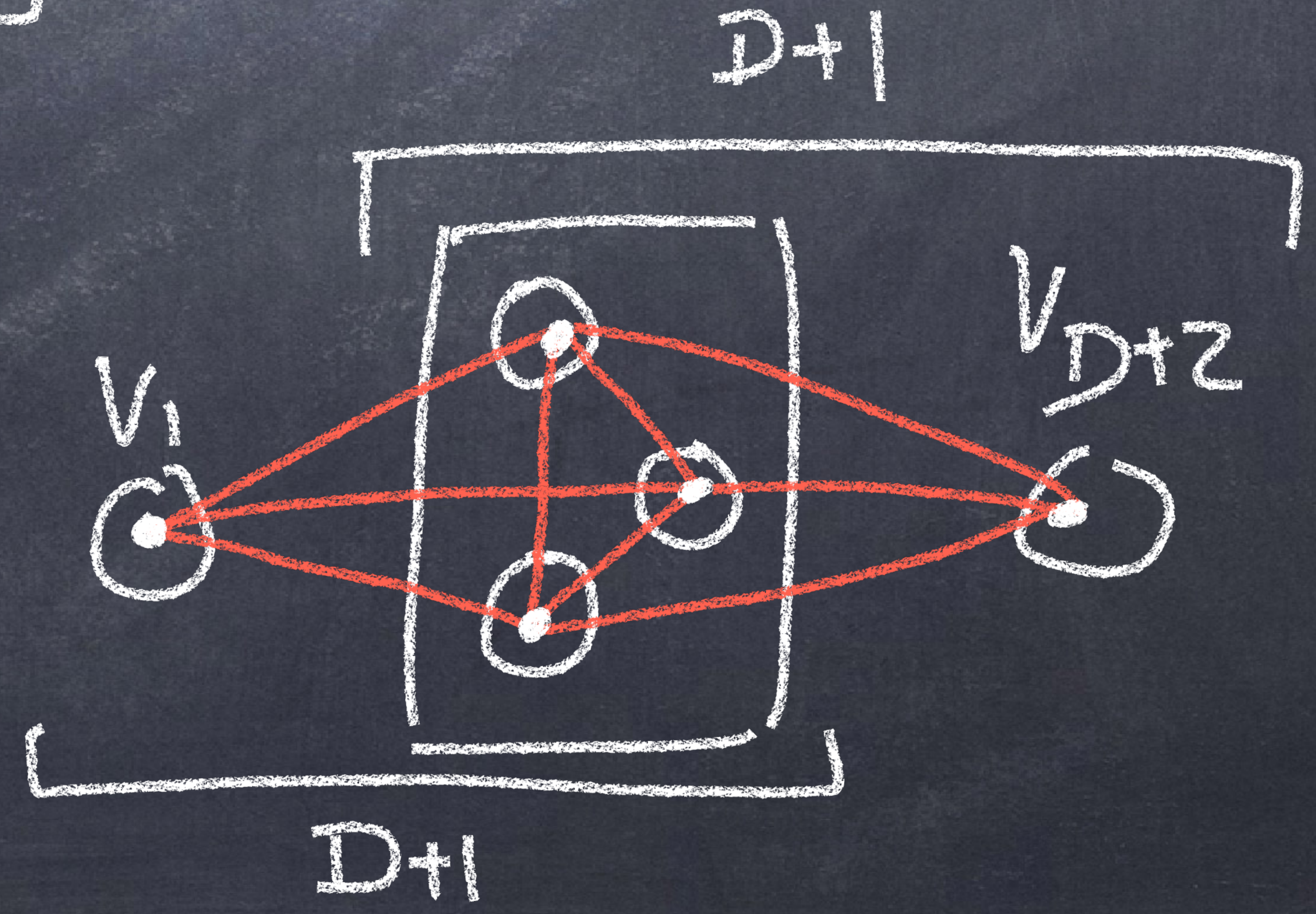
Build a "pseudo-random" graph

(P1) $\delta(M) \geq \frac{np}{2}$ & for all X, Y $|X| \geq \frac{\log n}{p}$, $|Y| \geq \alpha n$ $\exists v \in X$ s.t.

$$|N_M(v) \cap Y| \geq \frac{|Y|p}{2}$$

(P2) $\forall v \, \forall X \subseteq N(v)$ w $|X| \geq \alpha np$: $K_D \subseteq M[X]$

(P3) $\forall V_1, \dots, V_{D+2}$, $|V_i| \geq n^{1-\beta}$ $\exists K_{D+2}^-$:



Maker's strategy for $b \le cn^{\frac{2}{D+3}}$

Build a "pseudo-random" graph

Apply ABKNP 2017

▷ $M_1 \subseteq G(n,p)$ s.t. $\delta(M_1) \ge (1-\varepsilon)np$ & $e_{M_1}(N_G(v)) \ge (1-\varepsilon)p^3 \frac{n^2}{2}$

ε – small constant , $p = c_\varepsilon b^{-1} = Kn^{-\frac{2}{D+3}}$

Maker's strategy for $b \le cn^{\frac{2}{D+3}}$

Build a "pseudo-random" graph

Apply ABKNP 2017

- $M_1 \subseteq G(n,p)$ s.t. $\delta(M_1) \ge (1-\varepsilon)np$ & $e_{M_1}(N_G(v)) \ge (1-\varepsilon)\frac{p^3 n^2}{2}$

  ε – small constant, $p = c_\varepsilon b^{-1} = Kn^{-\frac{2}{D+3}}$

- $M_2 \subseteq G(n,q)$ s.t. $\delta(M_2) \ge (1-\gamma)nq$ & $e_{M_2}(N_G(v)) \ge (1-\gamma)\frac{q^3 n^2}{2}$

$$n^{-\frac{1-\beta}{m_2(H)}} \ll q \ll n^{-\frac{2}{D+3}} \gamma^6, \quad \gamma = n^{-3\beta}$$

Maker's strategy for $b \le cn^{\frac{2}{D+3}}$

Build a "pseudo-random" graph

(P1) $\delta(M) \ge \frac{np}{2}$ & for all X, Y $|X| \ge \frac{\log n}{p}$, $|Y| \ge \alpha n$ $\exists v \in X$ s.t.

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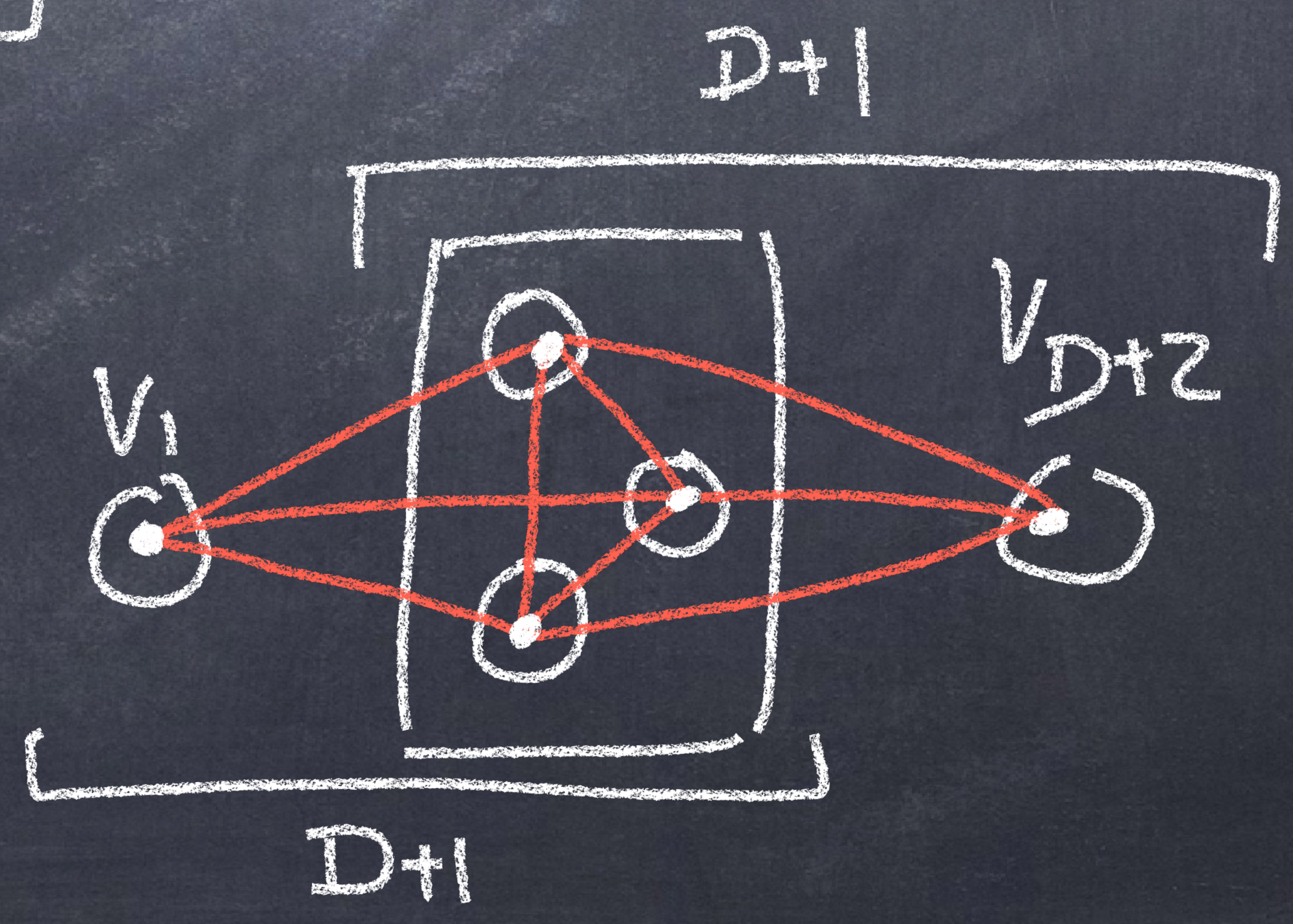
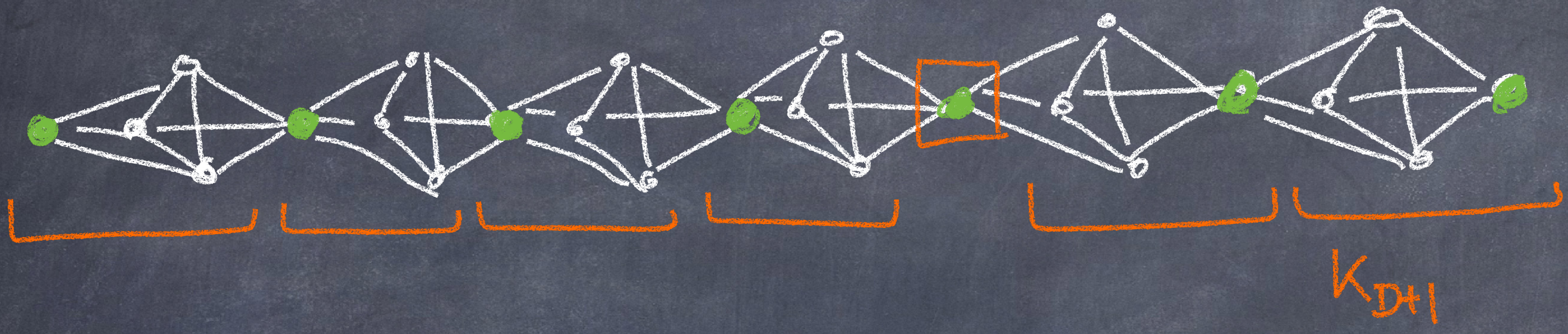


Maker's strategy for $b \leq cn^{\frac{2}{D+3}}$

use (P3) to find chains :



length $n^{1-\beta}$

Maker's strategy for $b \leq cn^{\frac{2}{D+3}}$

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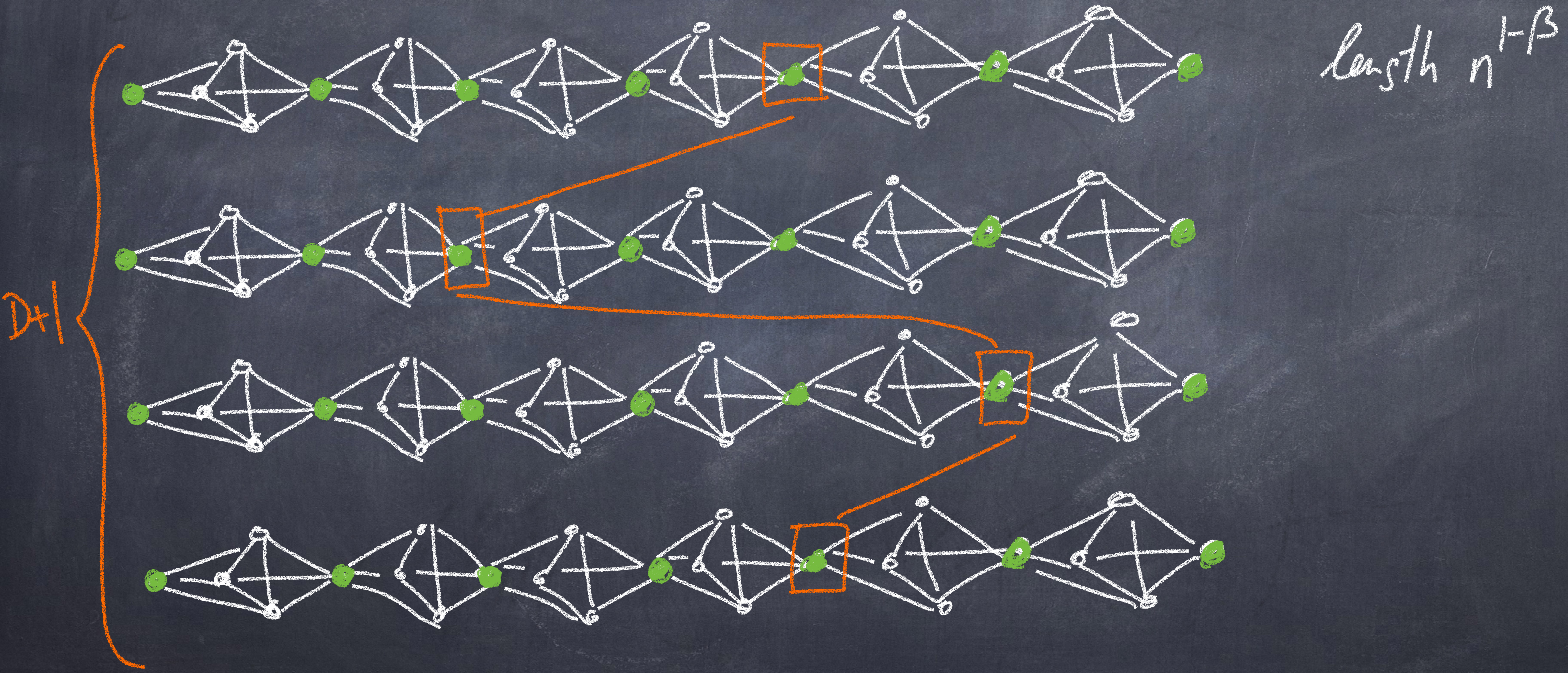


Maker's strategy for $b \le cn^{\frac{2}{D+3}}$

use (P3) to find chains:

length $n^{1-\beta}$ } n^{β} many

length $n^{1-\beta-\gamma}$ } $n^{\beta+\gamma}$ many

⋮

C_0 length 0 (vertices) } εn many

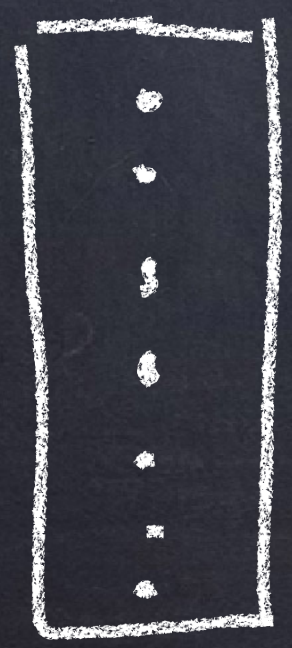
Maker's strategy for $b \le cn^{\frac{2}{D+3}}$

use (P3) to find chains:

length $n^{1-\beta}$ $\Big\}$ n^{β} many

absorbing property

for any ... there exist ... such that ... contains K_{D+1}-factor.



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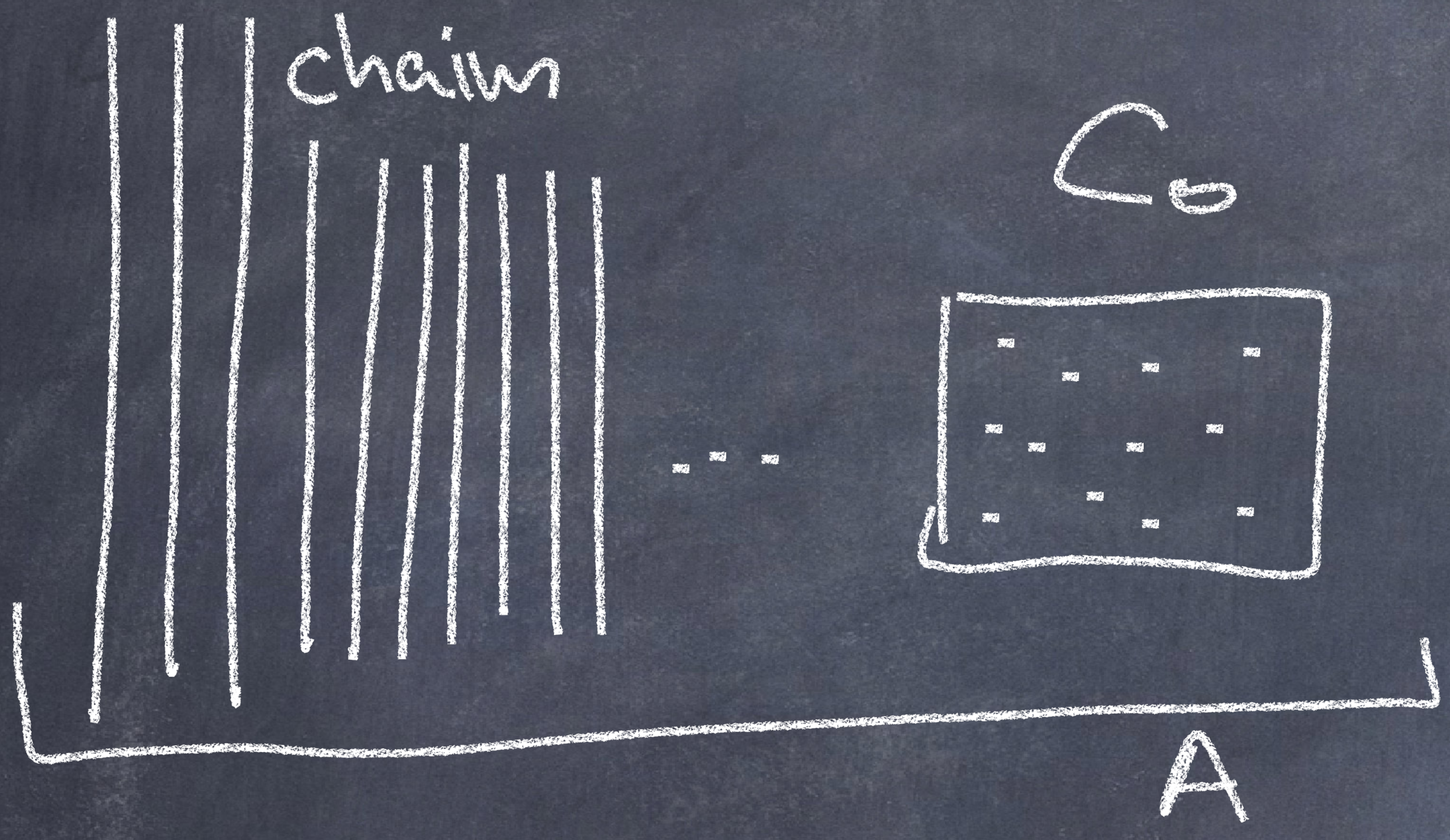

rest of graph

cover $\bar{w}$ K_{D+1}'s
(somewhat greedily)
using v's from C_0
$\rightarrow$ use (P1) & (P2)

Absorbing property

$\forall W \subseteq C_0 \quad \exists \ K_{D+1}$-factor of $A \setminus W$

- Conjecture: for every H on $\leq n$ vertices with $\Delta(H) \leq D$:

$$b^*(H\text{-game}, n) \geq c \cdot n^{\frac{2}{D+3}}$$

$\rightarrow$ K_{D+1}-factor is hardest for Maker

- Problem: Find explicit winning strategies for Maker in the H-game or the Hamiltonicity game.

- Problem: What is the threshold bias for the $(2:b)$-game?

- **Conjecture:** for every H on $\leq n$ vertices with $\Delta(H) \leq D$:

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$\longrightarrow K_{D+1}$-factor is hardest for Maker

- **Problem:** Find explicit winning strategies for Maker in the H-game or the Hamiltonicity game.

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Thank you !