Sharp phase transition for interlacements percolation

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Based on a joint work with Duminil-Copin, Goswami, Rodriguez and Severo

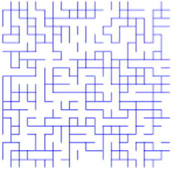
Overview of the talk

1. **Sharpness in Percolation**
2. **Random walks and Random Interlacements**
3. **Main results**
4. **Interpolation technique**

Introduction to Percolation

Bernoulli percolation

- Introduced by Broadbent and Hammerley in 1957.
- Very simple model.
- Extensively studied.
- Physics: understood. Math: deep open questions.

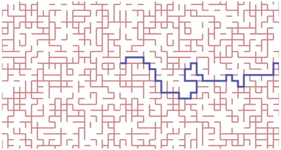


- ($d \geq 2$) Consider $\mathbb{Z}^d$ with edges between nearest neighbors.
- Fix $p \in [0, 1]$.
- Every edge is declared open with probability p and closed w.p. $(1 - p)$.
- This is done independently for every edge.

Phase transition

Consider:

$$[0 \leftrightarrow \infty] := \text{there exists an open path from 0 to infinity.} \tag{1}$$



Its probability $\theta(p)$ is weakly monotone in p:

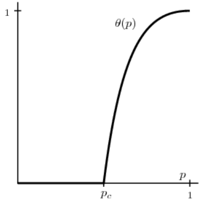
$$\theta(p) := P[0 \leftrightarrow \infty] \tag{2}$$

A beautiful path-counting argument (Peierls) shows that:

- $\theta(p) = 0$ for p small;
- $\theta(p) > 0$ for p close to one.

Phase transition!

Open questions



Define

$$p_c = \sup\{p \in [0,1]; \theta(p) = 0\}.$$

There are still many question that remain open concerning this model:

- Is $\theta(p)$ continuous for dimensions $3, 4, ..., 10$?
- How does $\theta(p)$ behave as p approaches p_c?

Sub-critical Sharpness

From definition

$$p < p_c \quad \Rightarrow \quad P\big[0 \leftrightarrow \partial B_n\big] \to 0.$$

From path counting

$$p < \frac{1}{2d} \quad \Rightarrow \quad P\big[0 \leftrightarrow \partial B_n\big] \leq Ce^{-cn}.$$

Definition

$$p_{**} := \sup\big\{p \in [0,1]; \exists c, C \text{ with } P[0 \leftrightarrow \partial B_n] \leq Ce^{-cn}\big\}.$$

Theorem (Hammersley)

One has $p_{**} = p_{***}$, where

$$p_{***} := \sup\big\{p \in [0,1]; E[\mathcal{C}_0] < \infty\big\}.$$

Theorem (Menshikov, Aizenman+Barski, Duminil-Copin+Tassion)

$$p_c = p_{***}.$$

Super-critical Sharpness

Super-critical Sharpness

- Ex(p):
$$P_p\Big[\exists \mathcal{C} \in B_n; \mathrm{diam}(\mathcal{C}) \geq n/10\Big] \geq 1 - Ce^{-cn}.$$
- Uniq(p):
$$P_p\Big[\forall \mathcal{C}_1, \mathcal{C}_2 \in B_n; \mathrm{diam}(\mathcal{C}_i) \geq n/10, \mathcal{C}_1 \underset{B_{2n}}{\longleftrightarrow} \mathcal{C}_2\Big] \geq 1 - Ce^{-cn}.$$

  Not monotone!

Counting surfaces:

$$p \sim 1 \quad \Rightarrow \quad \mathrm{Ex}(p) \text{ and } \mathrm{Uniq}(p)$$

Define:

$$\bar{p} := \inf \Big\{ p \in [0,1]; \mathrm{Ex}(p) \text{ and } \mathrm{Uniq}(p) \Big\}$$

Super-critical

Theorem (Chayes, Chayes, Newman)

$$p_c = p_{slab}$$

Theorem (Grimmett, Marstrand)

$$p_{slab} = p_c$$

Random Interlacements

Introduced by Sznitman in 2007.

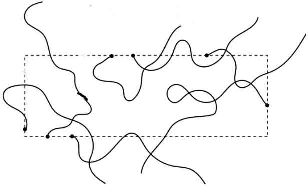
$\mathcal{I}^u \subseteq \mathbb{Z}^d$ ($d \geq 3$), related to

- trace of SRW on torus
- large deviations for SRW
- ...
- Example of strongly dependent percolation.

Construction of RI

Fix $u \geq 0$.

Construction of $\mathcal{I}^u \cap R$.



- For each $x \in \partial R$;
- Sample $N_x \overset{d}{\sim} \text{Poisson}(e_R(x))$;
- Run N_x independent SRW's.

Phase transition for RI

$$\mathrm{Cov}(x \in \mathcal{I}^u, y \in \mathcal{I}^u) \sim \frac{1}{|x-y|^{d-2}}$$

$\mathcal{I}^u$ is infinite and connected a.s. $(\forall u \geq 0)$!

$\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$.

$$u_* := \inf \left\{ u \geq 0; P\left[0 \overset{\mathcal{V}^u}{\longleftrightarrow} \infty\right] = 0 \right\}$$

Theorem (Sznitman, Sidoravicius + Sznitman)

$$(d \geq 3), 0 < u_* < \infty$$

see also Ráth.

Quantitative thresholds

Sub-critical: $u_{**} := \inf\left\{u \geq 0; \forall n \;\; P\left[0 \overset{\mathcal{V}^u}{\leftrightarrow} \partial B_n\right] \leq Ce^{-cn^{\varepsilon}}\right\}$

Theorem (Sidoravicius + Sznitman)

$$u_{**} < \infty$$

and

$$u_{**} = u_{***} := \inf\left\{u \geq 0; P\left[B_n \overset{\mathcal{V}^u}{\leftrightarrow} \partial B_{2n}\right] \underset{n}{\to} 0\right\}$$

Super-critical: $\bar{u} := \sup\left\{u \geq 0; \mathrm{Ex}'(u), \mathrm{Uniq}'(u)\right\}$

Theorem (T. $d \geq 5$, Drewitz+Rath+Sapozhnikov $d \geq 3$)

$$\bar{u} > 0$$

What about u_{slab}?

Main Result

Main result

- Ex(u):

$$P_p\Big[\exists \mathcal{C} \in \mathcal{V}^u \cap B_n; \mathrm{diam}(\mathcal{C}) \geq n/10\Big] \geq 1 - Ce^{-cn^{\varepsilon}}.$$

- Uniq(u, v):

$$P_p\Big[\forall \mathcal{C}_1, \mathcal{C}_2 \in \mathcal{V}^{u} \cap B_n; \mathrm{diam}(\mathcal{C}_i) \geq n/10, \mathcal{C}_1 \overset{\mathcal{V}^v \cap B_{2n}}{\leftrightarrow} \mathcal{C}_2\Big] \geq 1 - Ce^{-cn}.$$

Note the sprinkling!

$$\bar{\bar{u}} := \sup\big\{u \geq 0; \mathrm{Ex}(u) \text{ and } \forall v < u \ \mathrm{Uniq}(u, v)\big\}$$

Theorem (Duminil-Copin+Goswami+Rodriguez+Severo+T.)

$$u_{**} = u_* = \bar{\bar{u}}$$

Applications (conditional statements):

- RW on torus/cylinder
- Large deviations for SRW . . .

Ideas of proofs

- Reuse Bernoulli sharpness!!!
- Weaken $\bar{\bar{u}}$
- General sharpness for finite range models
- Couple with finite range model
- Interpolate the coupling

Weaken $\bar{\bar{u}}$

Let

$$M(R) := \exp\left\{\log^{\gamma}(R)\right\}, \text{ with } \gamma \text{ large}$$

then

$$\tilde{u} := \sup\left\{u \geq 0; P\left[B_r \leftrightarrow \partial B_{M(R)}\right]\left(\frac{M(R)}{R}\right)^d \to 0\right\}$$

Monotone! Similar to sub-critical!

Theorem (Duminil-Copin+Goswami+Rodriguez+Severo+T.)

$$\tilde{u} = \bar{\bar{u}}$$

Inspired by Benjamini+Tassion

Sharpness for finite range models

Theorem (Duminil-Copin+Goswami+Rodriguez+Severo)

$u_{**} = u_* = \bar{\bar{u}}$ *for all models satisfying:*

a) *lattice symmetries*

b) *FKG*

c) *Finite energy*

d) *Finitary factor of i.i.d.*

e) *Sprinkling:* $\omega^u \preceq \omega^{u-\varepsilon} \cup Ber(\delta)$

Finitary RI

Define $\mathcal{I}^{u,L}$:

- PPP on $\mathbb{Z}^d$, intensity $\left(\frac{u}{L}\right)$
- For each point, take a SWR with length L

Called finitary R.I. (Bowen, Cai, Zhang, Ráth+Rokob, ...)

$$\mathcal{I}^{u,L} \to \mathcal{I}^u$$

Also $I^{u,L}$ satisfies a), b), ..., e), thus

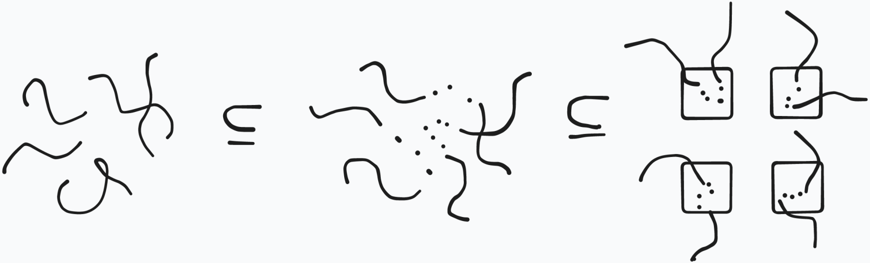
$$u_{**}^L = \tilde{u}^L$$

Interpolation

Coupling L and $2L$

Theorem (Duminil-Copin+Goswami+Rodriguez+Severo+T.)

“ $$P\left[\mathcal{I}^{u-\varepsilon,L} \subseteq \mathcal{I}^{u,2L} \subseteq \mathcal{I}^{u+\varepsilon,L} \text{ in } B_{L+K}\right] \geq 1-(L+K)^d e^{-cL^{1/4}}$$ ”



Interpolation

1) From infinite to finite:

$$P\Big[B_r \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_R\Big] \geq P\Big[B_r \overset{\mathcal{V}^{u+\varepsilon_L, L}}{\longleftrightarrow} \partial B_R\Big] - \exp\big\{ - \log^{\gamma'}(R)\big\}$$

2) From L to $2L$:

$$P\Big[B_r \overset{\mathcal{V}^{u,2L}}{\longleftrightarrow} \partial B_R\Big] \geq P\Big[B_r \overset{\mathcal{V}^{u+\varepsilon_L, L}}{\longleftrightarrow} \partial B_R\Big] - \exp\big\{ - \log^{\gamma'}(R)\big\} 1_{R<M(L)}$$

3) Box by box

$$P\Big[B_r \overset{\mathcal{V}^k}{\longleftrightarrow} \partial B_R\Big] \geq P\Big[B_r \overset{\mathcal{V}^{k+1}}{\longleftrightarrow} \partial B_R\Big]$$