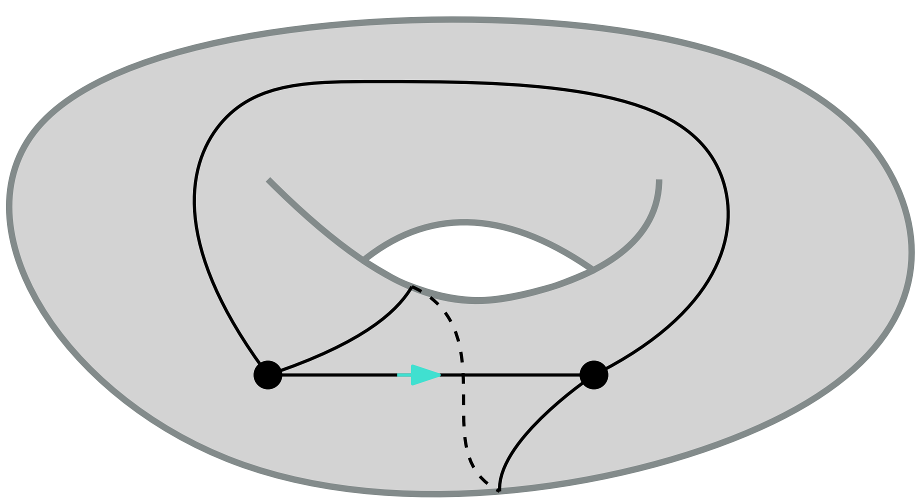
Combinatorial maps vs hyperbolic surfaces in large genus

Baptiste Louf (Uppsala Universitet)
joint work with Svante Janson



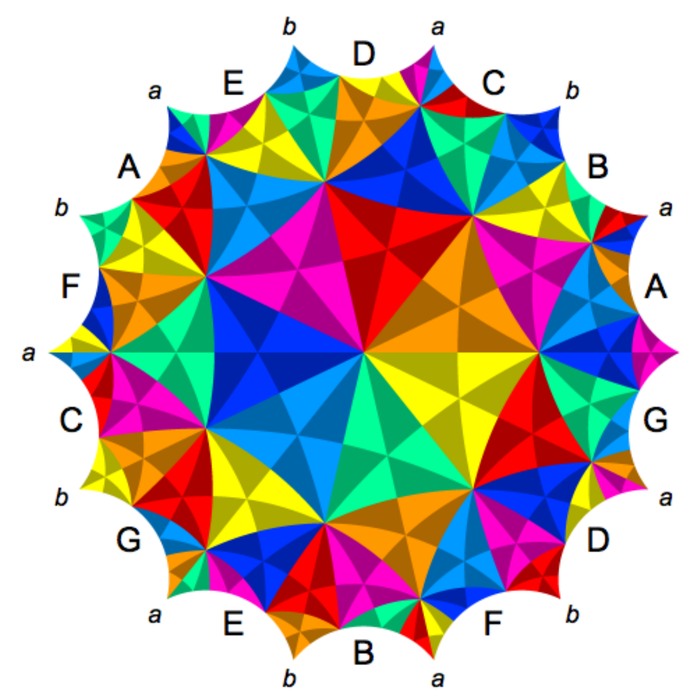
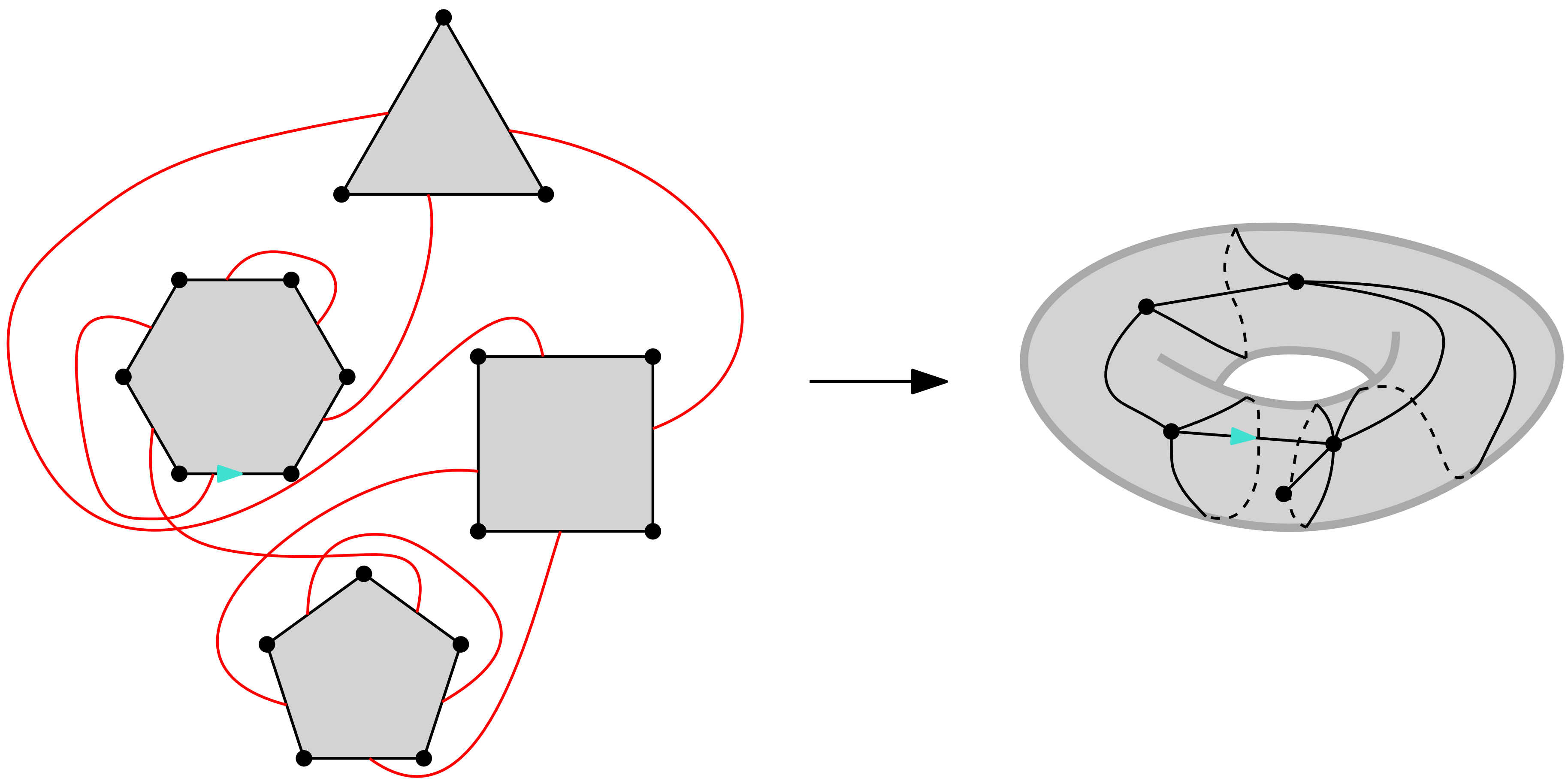


image : G. Egan

Maps

Definition : maps

Map = **discrete surfaces**
i.e. gluing of polygons along their edges to create a (compact, connected, oriented) surface
Genus g of the map = genus of the surface = $\#$ of handles



Random maps

Question : what does a large random map look like ?

Classical setting : uniform maps of the sphere
Main result: **scaling limit** = the Brownian sphere [Le Gall, Miermont '11]

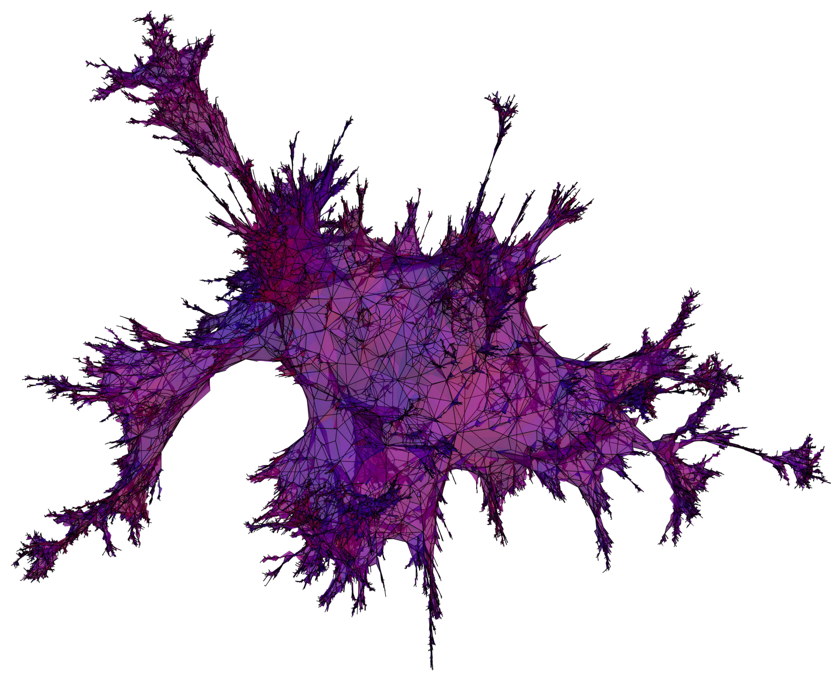


image : J. Bettinelli

High genus maps

What does a random uniform map of size n and genus g look like when $n, g \to \infty$?

Average discrete curvature is negative $\to$ (expected) **hyperbolic** properties !

High genus maps

What does a random uniform map of size n and genus g look like when $n, g \to \infty$?

Average discrete curvature is negative $\to$ (expected) **hyperbolic** properties !

We already have a few results in this regime

Example : **local limit** of high genus triangulations [Budzinski,L. 19]

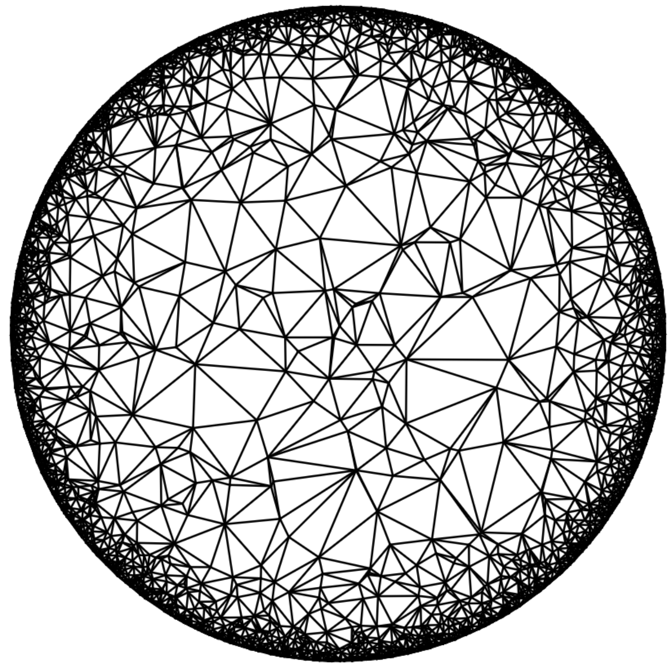


image : N. Curien

High genus maps

What does a random uniform map of size n and genus g look like when $n, g \to \infty$?

Average discrete curvature is negative $\to$ (expected) **hyperbolic** properties !

We already have a few results in this regime

Example : **local limit** of high genus triangulations [Budzinski,L. 19]

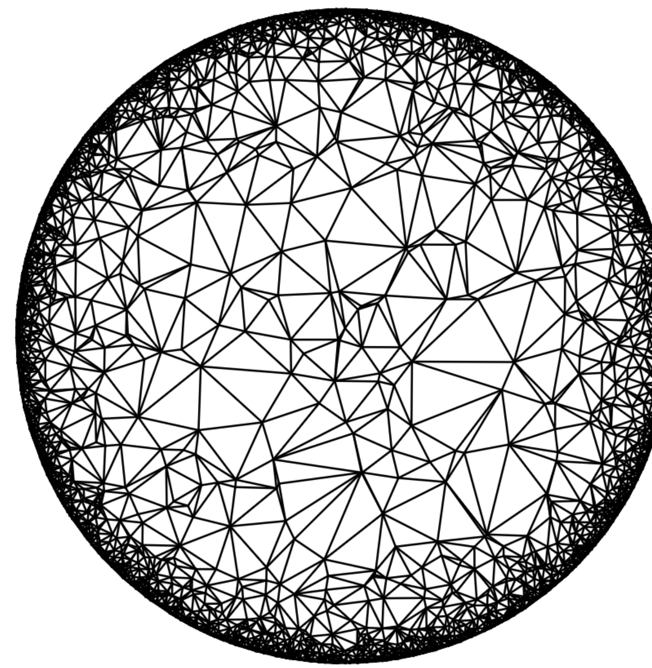
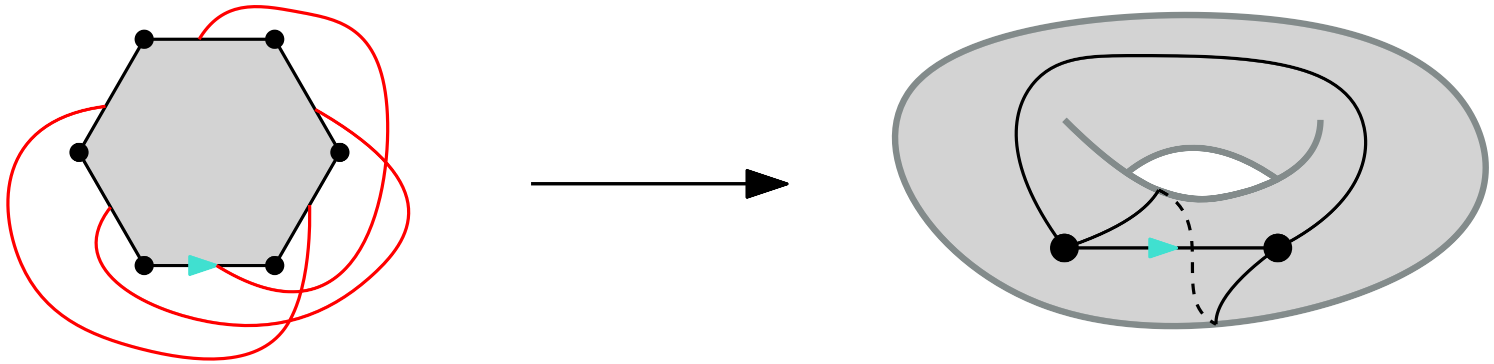


image : N. Curien

Many open questions remain (global properties, asymptotic enumeration, . . .) !

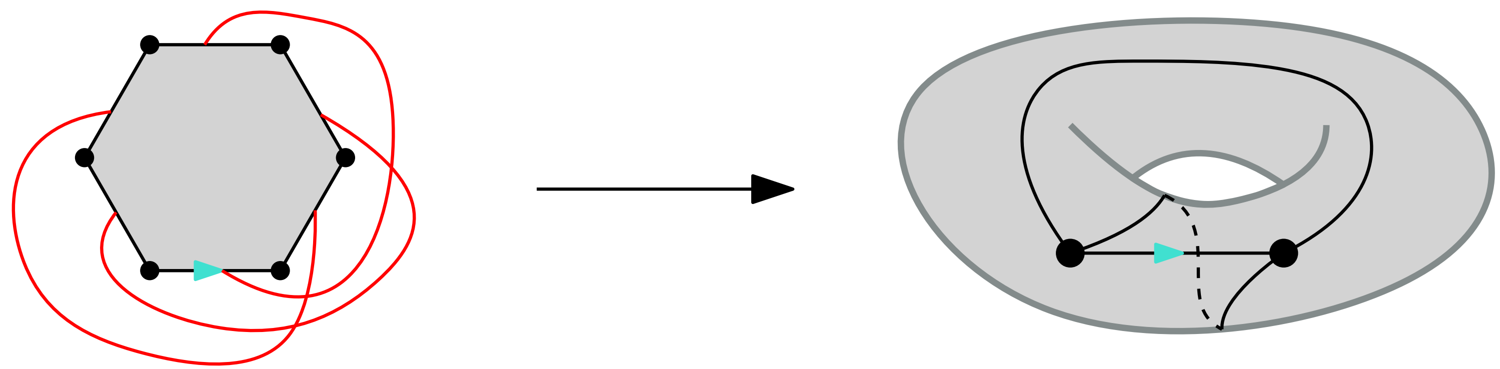
Our model today

Unicellular maps: maps with only one face/gluing of a single polygon



Our model today

Unicellular maps: maps with only one face/gluing of a single polygon



$\mathbf{U}_{n,g}$: random uniform unicellular map of genus g and n edges
metric on $\mathbf{U}_{n,g}$:

$$d := d_{\mathsf{graph}} \times \sqrt{\frac{12g}{n}}$$

Goal: study $\mathbf{U}_{n,g}$ as $n, g \to \infty$ with $g = o(n)$

Hyperbolic surfaces

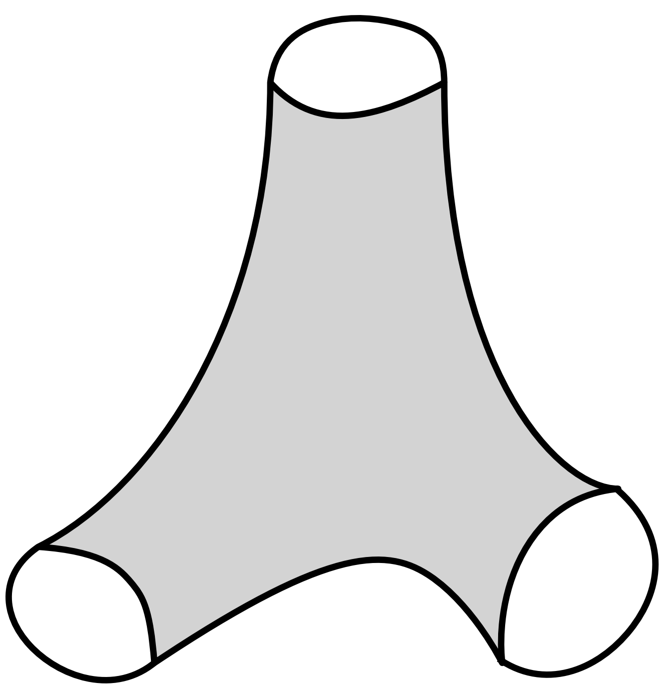
Hyperbolic surfaces

Hyperbolic surface $=$ surface with a metric w/ constant **curvature** -1 $=$ looks locally like the **hyperbolic plane** $\mathbb{H}$.

Hyperbolic surfaces

Hyperbolic surface $=$ surface with a metric w/ constant **curvature** -1 $=$ looks locally like the **hyperbolic plane** $\mathbb{H}$.

$\forall a, b, c > 0$ there exists a unique hyperbolic **pair of pants** with boundary lengths a, b, c, i.e. a genus 0 surface with 3 geodesic boundaries

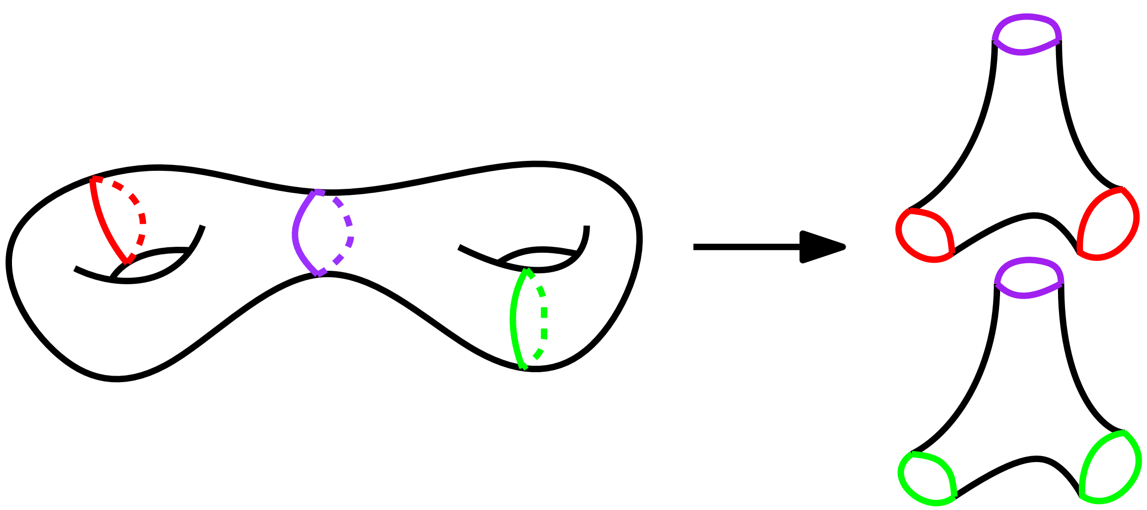


geodesic $=$ curve that is "locally shortest path"

Cutting up a surface

Take a hyperbolic surface S of genus $g \geq 2$.

Pair of pants decomposition: there exists $3g-3$ simple closed geodesics dividing S into $2g-2$ pairs of pants.

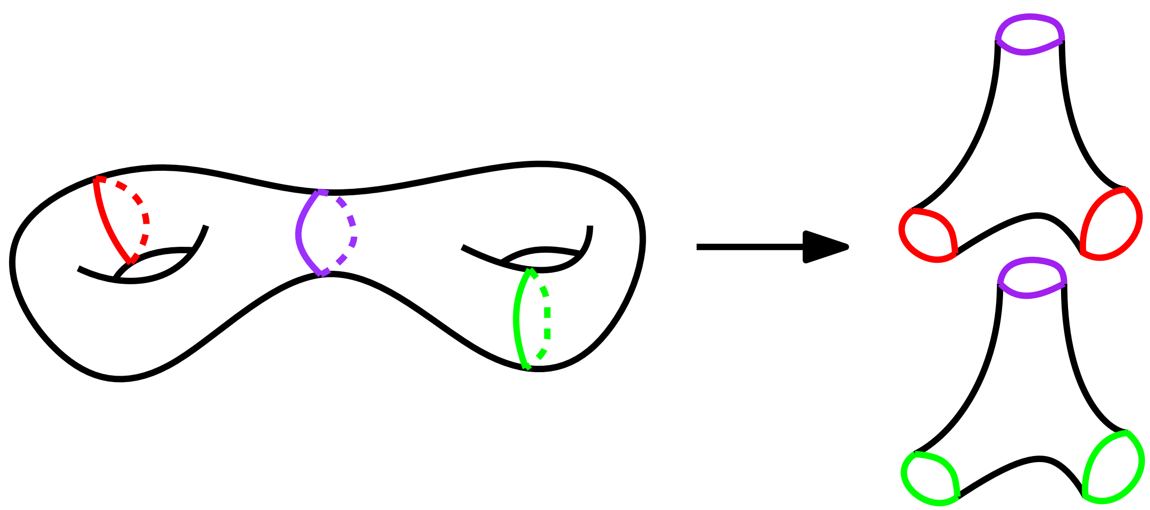


the choice of curves is not unique !

Cutting up a surface

Take a hyperbolic surface S of genus $g \geq 2$.

Pair of pants decomposition: there exists $3g-3$ simple closed geodesics dividing S into $2g-2$ pairs of pants.



the choice of curves is not unique !

Fenchel–Nielsen coordinates:
Each geodesic has a length $\ell_i \in \mathbb{R}^+$, and a “twist factor” $\tau_i \in \mathbb{R}$.
it determines the surface uniquely !

Teichmüller space: $\mathcal{T}_g =$ space of hyperbolic surfaces of genus $g \cong (\mathbb{R}^+ \times \mathbb{R})^{3g-3}$

Parametrizing hyperbolic surfaces

Moduli space: $\mathcal{M}_g = \mathcal{T}_g/\text{isometries}$ ⟵ not easy to understand !

Parametrizing hyperbolic surfaces

Moduli space: $\mathcal{M}_g = \mathcal{T}_g/$isometries ← not easy to understand !

Weil–Petersson volume form: [Wolpert '85]

$$\mathrm{d}vol_{WP} = \prod_{i=1}^{3g-3} \mathrm{d}\ell_i \mathrm{d}\tau_i$$

"Magic formula":

- Doesn't depend on the choice of curves !
- Defined on $\mathcal{T}_g$ but "descends" on $\mathcal{M}_g$!
- $vol_{WP}(\mathcal{M}_g)$ is finite → probability measure !

Parametrizing hyperbolic surfaces

Moduli space: $\mathcal{M}_g = \mathcal{T}_g/$isometries ← not easy to understand !

Weil–Petersson volume form: [Wolpert '85]

$$\mathrm{d}vol_{WP} = \prod_{i=1}^{3g-3} \mathrm{d}\ell_i \mathrm{d}\tau_i$$

"Magic formula":

- Doesn't depend on the choice of curves !
- Defined on $\mathcal{T}_g$ but "descends" on $\mathcal{M}_g$!
- $vol_{WP}(\mathcal{M}_g)$ is finite → probability measure !

$\mathbf{S}_g$ = random hyperbolic surface under WP measure.
Properties of $\mathbf{S}_g$ as $g \to \infty$ were first studied 10 years ago [Mirzakhani, Guth–Parlier–Young]

A coincidence and a conjecture

A surprising coincidence

Theorem [Mirzakhani–Petri '17]
For all $y > x > 0$, the number of **simple closed geodesics** in $\mathbf{S}_g$ of length $\in [x, y]$ converges in distribution to a Poisson law of parameter

$$\int_x^y \frac{\cosh t - 1}{t} dt$$

as $g \to \infty$.

Theorem [Janson–L. '21]
For all $y > x > 0$, the number of **simple cycles** in $\mathbf{U}_{n,g}$ of length $\in [x, y]$ converges in distribution to a Poisson law of parameter

$$\int_x^y \frac{\cosh t - 1}{t} dt$$

as $g \to \infty$.

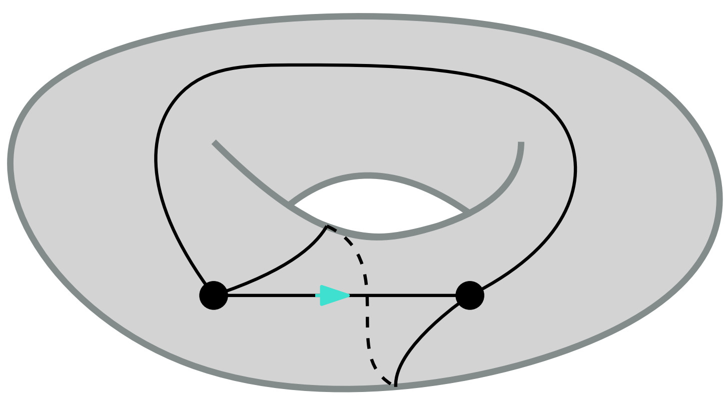
A conjecture

Is it really just a coincidence ?

A conjecture

Is it really just a coincidence ?

Conjecture: (vague version)
$\mathbf{S}_g$ and $\mathbf{U}_{n,g}$ are "the same" as $g \to \infty$ (wrt to a well chosen metric)



$$\approx$$
$$g \to \infty$$

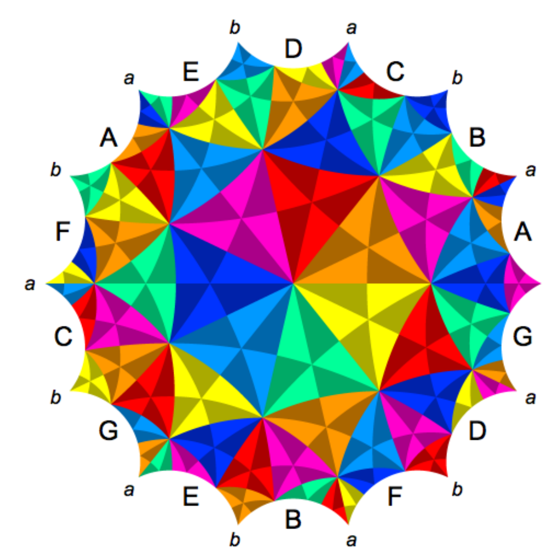


image : G. Egan

A conjecture

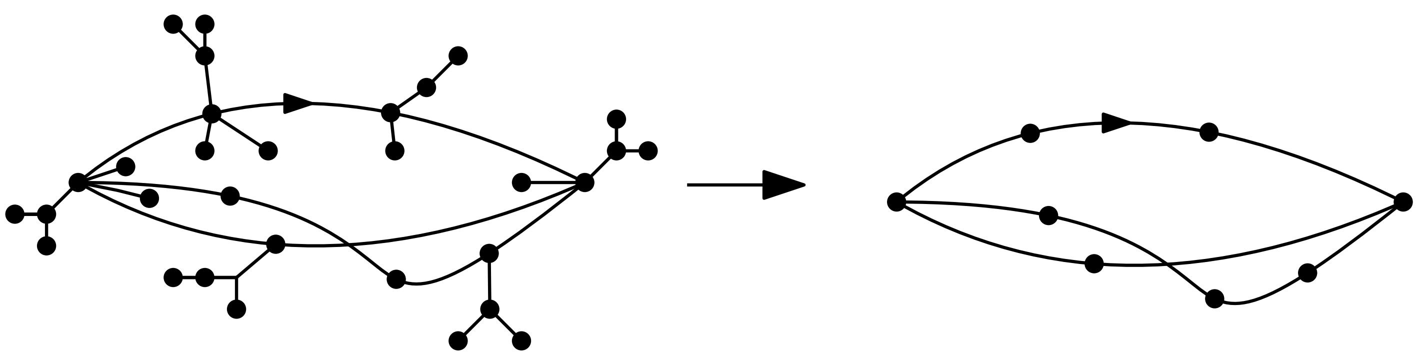
Making the conjecture really precise requires some work !

A conjecture

Making the conjecture really precise requires some work !

Two important things seem to be necessary:

• removing the "fractal part", i.e. looking only at the **2-core**



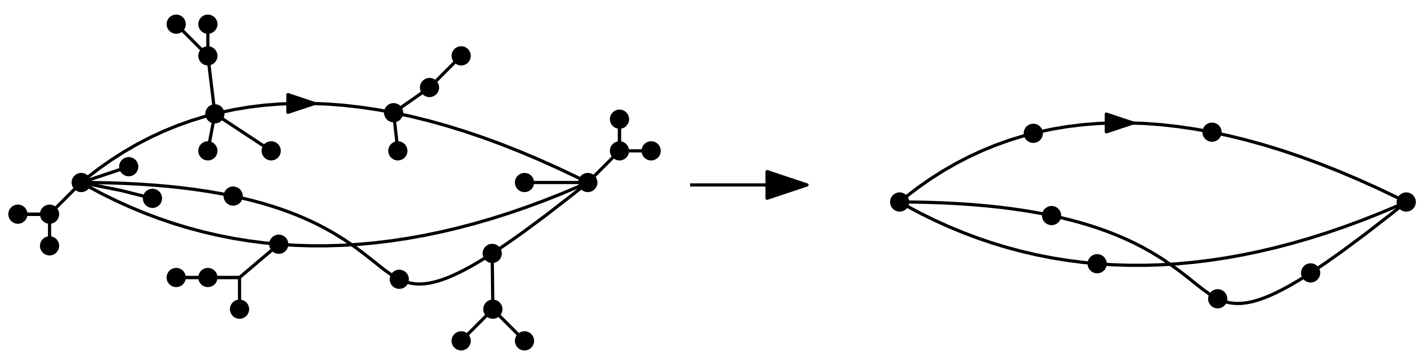
• seeing the maps as the gluing of a **hyperbolic polygon**

A conjecture

Making the conjecture really precise requires some work !

Two important things seem to be necessary:

• removing the "fractal part", i.e. looking only at the **2-core**



• seeing the maps as the gluing of a **hyperbolic polygon**

Right metric: probably **Gromov–Hausdorff** distance on metric spaces, possibly something stronger to make sense of the topology (e.g. separating curves).

Some open problems

Hope: If the conjecture is true, and we can transfer hyperbolic problems to maps problems, and thus to tree problems thanks to a magic bijection (see later).

Some open problems

Hope: If the conjecture is true, and we can transfer hyperbolic problems to maps problems, and thus to tree problems thanks to a magic bijection (see later).

We can start working on some of the open problems for hyperbolic surfaces ($\mathbf{S}_g$) directly on maps ($\mathbf{U}_{n,g}$).
For example:

- We know that

$$(1 + o(1)) \log g \leq \mathsf{diam}(\mathbf{S}_g) \leq (4 + o(1)) \log g$$

  What is the right constant ?
- spectral properties ? spectral gap, isoperimetric/Cheeger constant . . .

Ideas of proof

Morceaux choisis

The method of moments

Key principle: under reasonable assumptions, a sequence (X_n) of random variables converges in distribution towards X, if for all finite r

$$\mathbb{E}[X_n^r] \xrightarrow[n\to\infty]{} \mathbb{E}[X^r]$$

The method of moments

Key principle: under reasonable assumptions, a sequence (X_n) of random variables converges in distribution towards X, if for all finite r

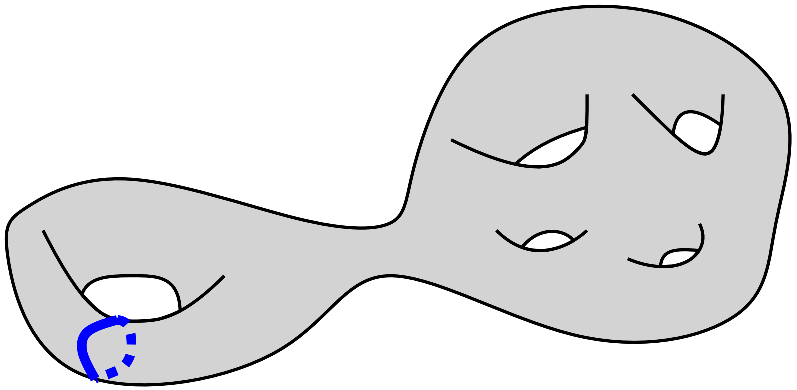
$$\mathbb{E}[X_n^r] \xrightarrow[n\to\infty]{} \mathbb{E}[X^r]$$

How to calculate moments: roughly speaking, if X counts curves (of a given length), then

$$\mathbb{E}[X^r] = \frac{\#\text{surfaces with } r \text{ marked curves}}{\#\text{surfaces}}$$

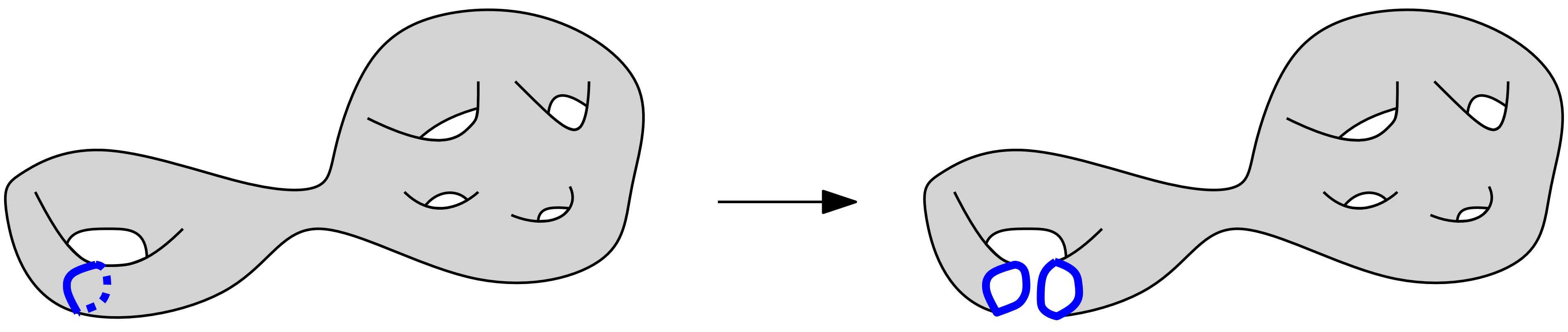
The method for hyperbolic surfaces

Let X_g be the number of closed, simple geodesics in a random hyperbolic WP surface of genus g with length $\in [a, b]$



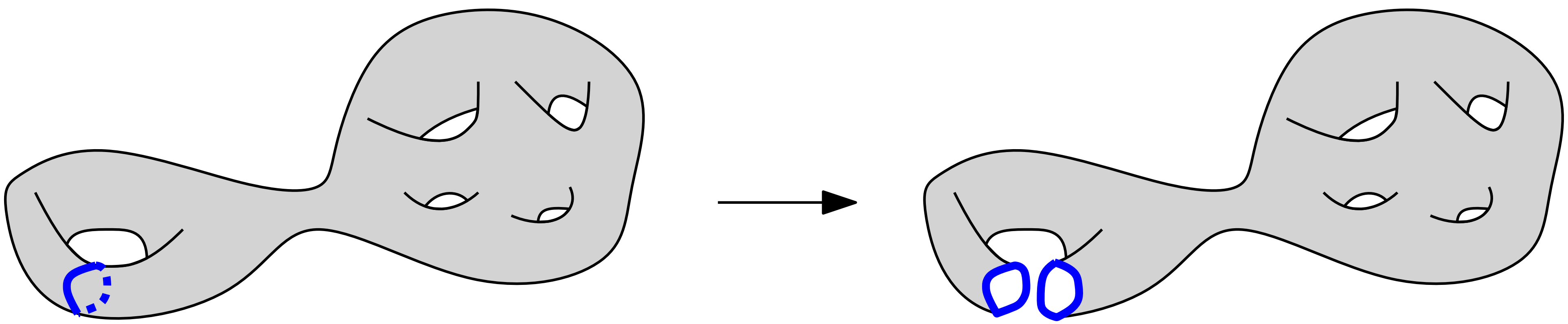
The method for hyperbolic surfaces

Let X_g be the number of closed, simple geodesics in a random hyperbolic WP surface of genus g with length $\in [a, b]$



The method for hyperbolic surfaces

Let X_g be the number of closed, simple geodesics in a random hyperbolic WP surface of genus g with length $\in [a, b]$



$$\mathbb{E}[X_g] \approx \int_{\ell=a}^{b} \frac{vol_{WP}(\mathcal{M}_{g-1,2}(\ell,\ell))}{vol_{WP}(\mathcal{M}_g)} \mathrm{d}\ell$$

use Mirzakahani's large genus estimates for WP volumes

Method for maps: the magic bijection

C-decorated tree: tree with a permutation of its vertices, with only odd cycles.

The **underlying graph** of a C-decorated tree is obtained by merging vertices who belong to the same cycle of the permutation.

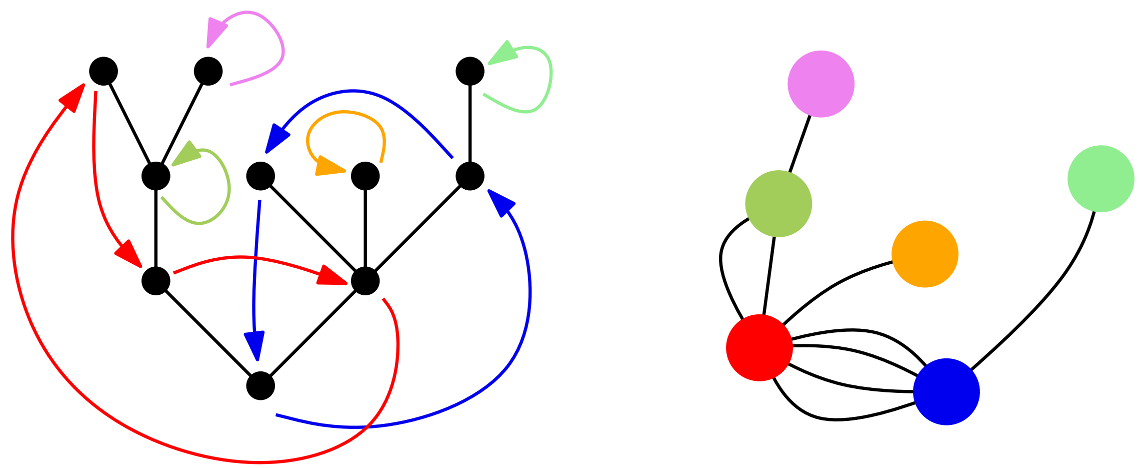


image : G. Chapuy

Method for maps: the magic bijection

C-decorated tree: tree with a permutation of its vertices, with only odd cycles.

The **underlying graph** of a C-decorated tree is obtained by merging vertices who belong to the same cycle of the permutation.

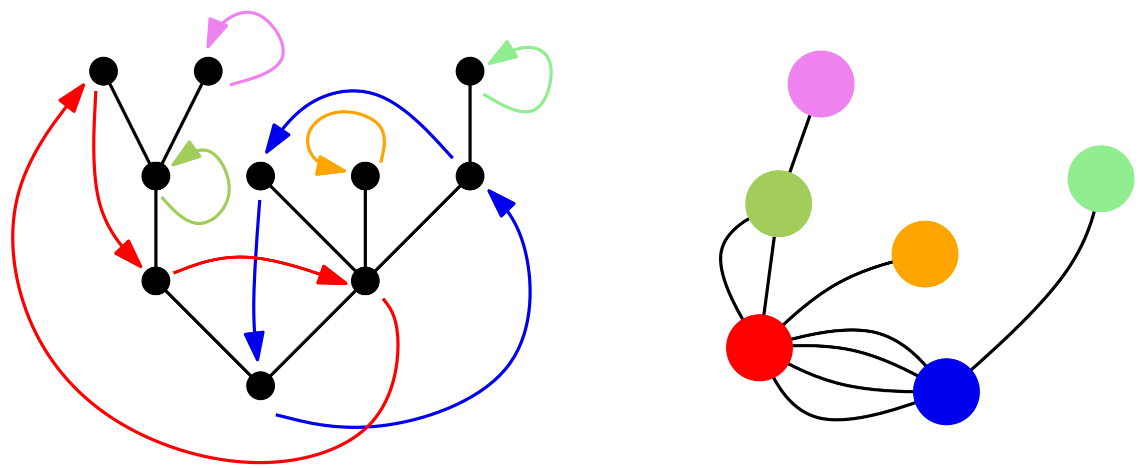


image : G. Chapuy

$\mathbf{T}_{n,g}$: Random uniform C-decorated tree with n edges and $n+1-2g$ cycles in the permutation.

Method for maps: the magic bijection

C-decorated tree: tree with a permutation of its vertices, with only odd cycles.

The **underlying graph** of a C-decorated tree is obtained by merging vertices who belong to the same cycle of the permutation.

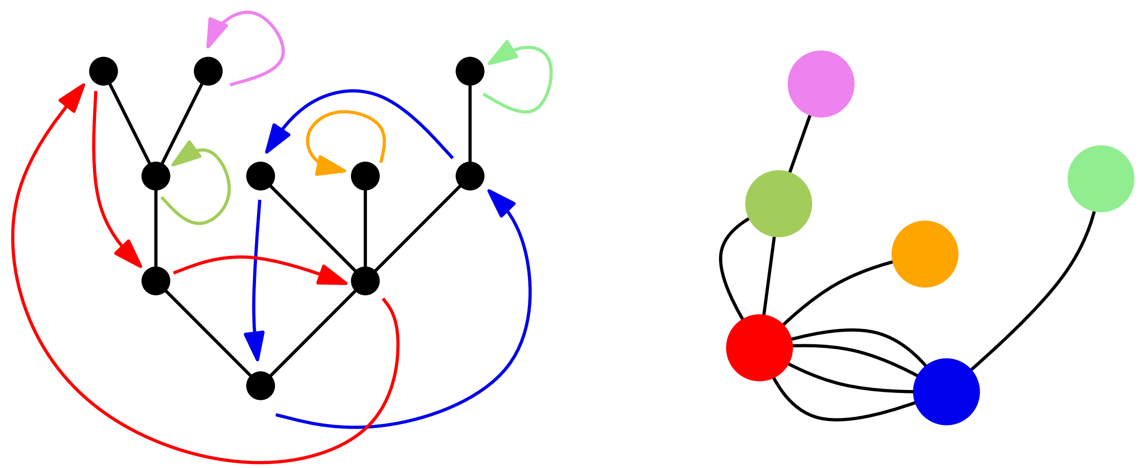


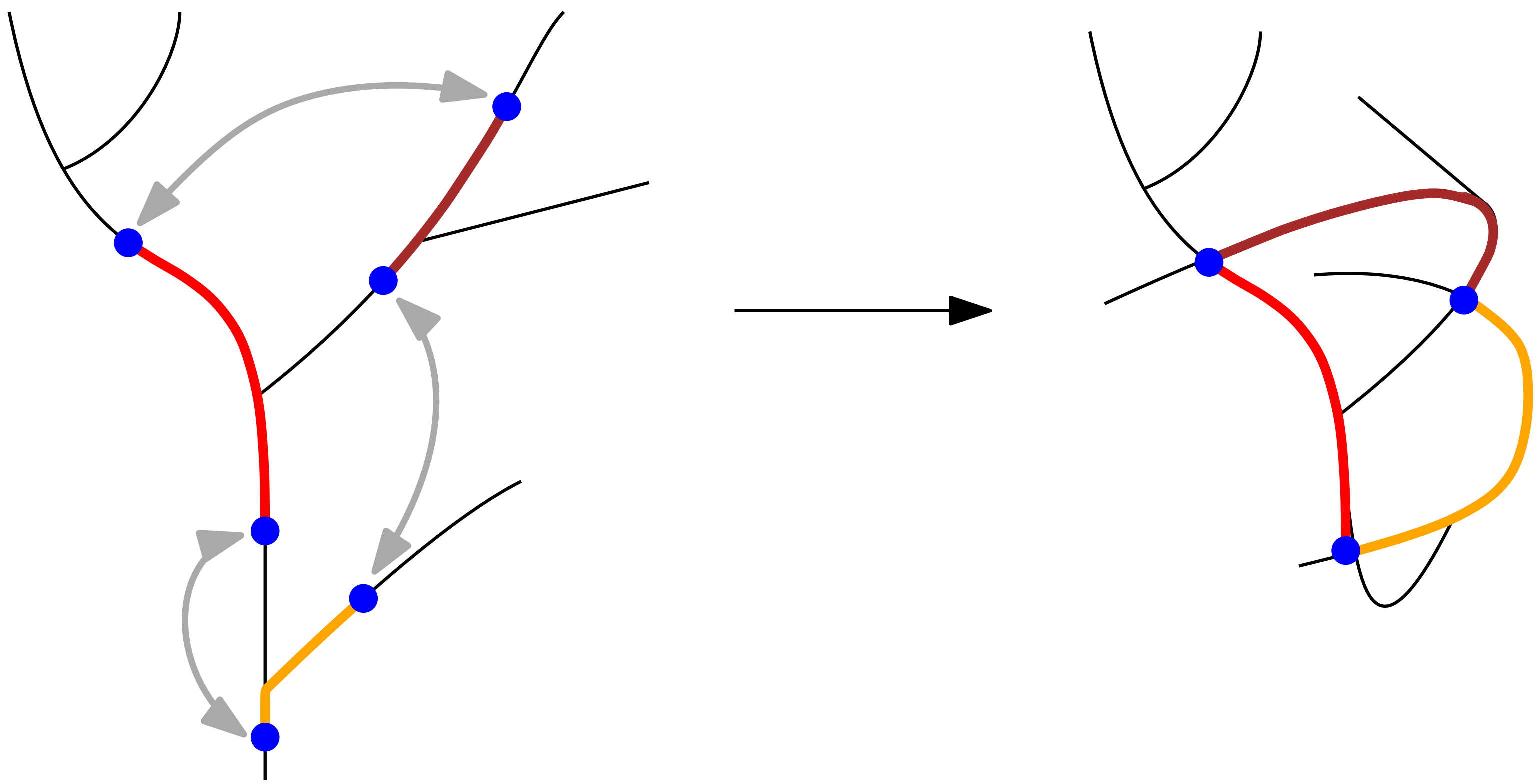
image : G. Chapuy

$\mathbf{T}_{n,g}$: Random uniform C-decorated tree with n edges and $n+1-2g$ cycles in the permutation.

Bijection [Chapuy–Féray–Fusy '13] (probabilistic version): the underlying graphs of $\mathbf{U}_{n,g}$ and $\mathbf{T}_{n,g}$ have the same law.

What is a cycle ?

A cycle in (the underlying graph of) $\mathbf{T}_{n,g}$ is a list of paths $p_1, p_2, \ldots, p_\ell$ such that end(p_i) and start(p_{i+1}) are "merged by the permutation".



Additional questions

More comments and questions

- There are unicellular maps in hyperbolic surfaces $\to$ Penner's **spine construction**. Combinatorial equivalent for maps : the **Schaeffer bijection**. Does a "Chapuy–Féray–Fusy bijection" for hyperbolic surfaces exist ?

- [Budd-Curien '22] the scaling limit of the random hyperbolic sphere with many cusps is the Brownian sphere. Can you study the Benjamini–Schramm limit of the random hyperbolic g-torus with n cusps ans $n, g \to \infty$? (cf [Budzinski-L. '19])

- Conjecture ("universality of high genus enumeration"): for $n, g \to \infty$ with $\frac{n}{g} \to \theta$

$$vol_{WP}(\mathcal{M}_{g,n}) = n^{2g} \exp(nf(\theta) + o(n))$$

Thank you !