A Subspace Theorem for Manifolds

joint work with Nicolas de Saxcé

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I. Schmidt's subspace theorem

Let $L_1, \dots, L_d$ be d linearly independent linear forms on $\mathbb{R}^d$ with coefficients in $\overline{\mathbb{Q}}$.

Theorem: There are proper rational subspaces $V_1, \dots, V_r$ such that $\forall \varepsilon > 0$ the solutions $v \in \mathbb{Z}^d$ of the inequality

$$\left| \prod_{i=1}^{d} L_i(v) \right| \leq \frac{1}{\|v\|^{\varepsilon}}$$

are all contained in $\bigcup_{i=1}^{r} V_i$, except for finitely many of them.

Remarks:

(a) $r \leq r(d)$

(b) There is no known effective bound (in terms of ε, $\{L_i\}_i$) on the height nor even the number of exceptional solutions.

(c) The V_i's are explicit, independent of ε. They may be necessary even when $\operatorname{Ker} L_i \cap \overline{\mathbb{Q}}^d = 0$ $\forall i$

<u>exple</u>: $(x - y\sqrt{2} + z\sqrt{3})(x - y\sqrt{2} - z\sqrt{3})\,x$

then $\{z = 0\}$ is one of the V_i's

(d) In fact the V_i's depend only on which rational translates $B\sigma B g$, $g \in GL_d(\mathbb{Q})$ of a Bruhat cell $B\sigma B$ L belongs to...

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_d \end{pmatrix} \in GL_d(\overline{\mathbb{Q}})$$

Example: Take $L_1(p,q) = q$

$L_2(p,q) = q\alpha - p$

for $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$. Then

$\forall \varepsilon > 0 \ \exists c > 0 \ \forall p, q \in \mathbb{Z}$

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{2+\varepsilon}}.$$

This is <u>Roth's theorem</u>.

more generally:

If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent, then $\forall \varepsilon > 0 \ \exists c > 0$ s.t. $\forall v \in \mathbb{Z}^n \setminus \{0\}$

$$|\alpha_1 v_1 + \dots + \alpha_n v_n| \geq \frac{c}{\|v\|^{n-1+\varepsilon}}$$

pf: apply the Subspace Thm to

$L_1(v) = \alpha_1 v_1 + \dots + \alpha_n v_n$

$L_i(v) = v_i \quad i \geq 2$.

Induct on dimension. $\square$

II. Diophantine approximation on manifolds:

Theorem (Kleinbock-Margulis '98):

Let $M \subseteq \mathbb{R}^n$ be an connected analytic submanifold. Assume M is not contained in a hyperplane.
Then for Lebesgue almost every $x \in M$

$\forall \varepsilon > 0 \ \exists c > 0$ s.t. $\forall v \in \mathbb{Z}^n \setminus \{0\}$

$$|x_1 v_1 + \dots + x_n v_n| \geq \frac{c}{\|v\|^{n-1+\varepsilon}}$$

(solved a conjecture of Sprindzuk)

example: $M = \{ (1, x, \dots, x^d), \ x \in \mathbb{R} \}$

Thm $\Rightarrow$ for a.e. x, $\forall \varepsilon, \exists c$,

$$|P(x)| \geq \frac{c}{H(P)^{d+\varepsilon}}$$

for all $P \in \mathbb{Z}[x]$, $\deg P \leq d$, $P \neq 0$.

$H(P) :=$ max | coeffs of P |.

Def: $M \subseteq \mathbb{R}^n$ is called extremal if Lebesgue a.e. point on M has the same diophantine exponent as a.e. point on $\mathbb{R}^n$, namely $n-1$.

Kleinbock inheritance principle '04

M is extremal iff the linear span of M in $\mathbb{R}^n$ is extremal.

In general $\exists \beta = \beta(M) \in \mathbb{R}$ s.t. for Lebesgue a.e $x \in M$ $\beta(x) = \beta$,

where $\beta(x) := \inf \{ \gamma \mid \exists c > 0 \ \forall v \in \mathbb{Z}^n \setminus \{0\} \ |x_1 v_1 + \dots + x_n v_n| \geq \frac{c}{\|v\|^\gamma} \}$

Kleinbock: $\beta(M)$ exists and depends only on the linear span of M in $\mathbb{R}^n$.

Approximation for matrices: slightly more general diophantine pb.

$Y \in Mat_{m \times n}(\mathbb{R})$. Ask for smallest β s.t.

$\forall \varepsilon > 0 \ \exists c > 0 \quad \|Yv\| \geq \frac{c}{\|v\|^\beta}$

$\forall v \in \mathbb{Z}^n \setminus \{0\}$

(for a.e. Y the exponent is $\frac{n}{m} - 1$)

Now $M \subseteq Mat_{m \times n}(\mathbb{R})$ manifold

$\rightarrow$ can define $\beta(M) :=$ almost sure exponent.

Problems: (a) find necessary and sufficient conditions for extremality of a manifold...

(b) in the non-extremal case, can one compute the best exponent?

(c) same for strong extremality

Def (multiplicative exponent):

$$[\![v]\!] := \left[\prod_{i=1}^{d} |v_i|^+\right]^{1/d}, \quad |x|^+ = \max\{|x|, 1\}$$

Ask for smallest β^* s.t. $\forall_\varepsilon \; \exists c > 0$

$$\left|\prod_{i=1}^{m} Y_i(v)\right|^{1/m} \geq \frac{c}{[\![v]\!]^{\beta^*}}$$

Note: $\beta^* \geq \beta$.

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} \in M_{m,n}(\mathbb{R})$$

III. A subspace theorem for manifolds

There is a formal similarity between the statements and part of the proofs of Schmidt's Subspace Theorem on the one hand, and the Kleinbock-Margulis theorem on the other hand.

For example, the corollary to Schmidt's theorem says that if $x \in \overline{\mathbb{Q}}^n$ and the x_i's are $\mathbb{Q}$-linearly independent, then $\beta(x) = n+1$, i.e. x is extremal.

Slogan: worst $\leftrightarrow$ algebraic $\leftrightarrow$ random

The K-M result is the manifold analogue to Schmidt's corollary. This strongly suggests that it ought to be a consequence of a more general statement.

Indeed our main result is precisely such a statement.

Main theorem (B. - de Saxcé):

Let $M \subseteq GL_d(\mathbb{R})$ be a connected analytic manifold. Assume that the Zariski closure of M is defined over $\overline{\mathbb{Q}}$. Then there are proper rational subspaces $V_1, \dots, V_r$ in $\mathbb{Q}^d$ such that for Lebesgue almost every $L \in M$ and for all $\varepsilon > 0$, the solutions $v \in \mathbb{Z}^d$ of the inequality

$$\left| \prod_{i=1}^{d} L_i(v) \right| \leq \frac{1}{\|v\|^{\varepsilon}}$$

are contained in $\bigcup_{i=1}^{r} V_i$, except finitely many.

N.B. here L_i : i^{th}-row of L

Remarks: (a) When M is a point this is just the original Subspace Theorem.

(b) The exceptional subspaces depend only on the Schubert closure of M over $\mathbb{Q}$. This is the intersection of all $\overline{B\sigma B g}$, $g \in GL_d(\mathbb{Q})$, B = Borel, $\sigma \in \mathrm{Sym}(d)$ permutation.

(c) $r(d) \leq d^{2d}$ (but we suspect this can be improved.), Schmidt had d^{2d^2}.

(d) If $\forall W_1, W_2$ rational subspaces in $\mathbb{Q}^d$ there is $L \in M$ s.t.

$$\dim(LW_1 \cap W_2) \leq \frac{\dim W_1 \cdot \dim W_2}{d}$$

then there are no exceptional subspace. We say in this case that M is semistable.

(e) Define the skeleton S_M of M as the set of permutations σ s.t. $M \subseteq \overline{B\sigma B g}$ for some $g \in GL_d(\mathbb{Q})$. Then $r(d)$ is controlled by:

\# minimal elements for Bruhat order in S_M.

(f) What are the Schubert closed varieties? (cf. Schubert calculus).

IV. Dynamics in the space of lattices

Both the proof of Schmidt's theorem and the Kleinbock-Margulis result use the geometry of numbers. In fact the proof of Schmidt's cleverly combines the diophantine arguments of Roth with some geometry of numbers. The K-M result is a consequence of the so-called quantitative non-divergence estimates, that give upper bounds on the time spent near the cusp by a unipotent orbit on the space of lattices

$$SL_n(\mathbb{R})/SL_n(\mathbb{Z}).$$

As for the proof of Schmidt's theorem, the proof of our manifold version goes through the proof of a stronger "parametric version".

Some notation: $a_t = \exp(At)$, $t > 0$
a diagonal flow
$A = \mathrm{diag}(A_1, \dots, A_d) \in GL_d(\mathbb{R})$

For a lattice (= full rank discrete subgroup) Δ in $\mathbb{R}^d$

let $\lambda_i(\Delta) := \inf\{\lambda > 0 \mid B(0,\lambda) \cap \Delta$ has i linearly indep^t vectors$\}$

the successive minima of Δ for the Euclidean norm.

Main theorem, parametric version:

Some assumptions ($M \subseteq GL_d(\mathbb{R})$ analytic submanifold, $Zar(M)$ defined over $\overline{\mathbb{Q}}$).

Then there are numbers $\Lambda_1 \leq \dots \leq \Lambda_d$ such that for Lebesgue a.e. $L \in M$

$$\lim_{t \to +\infty} \frac{1}{t} \log \lambda_i(a_t L \mathbb{Z}^d) = \Lambda_i$$

Moreover there are rational subspaces $0 = V_0 \leq V_1 \leq \dots \leq V_d = \mathbb{Q}^d$ s.t. $\forall \varepsilon > 0$ $\forall i = 1, \dots, d$ for a.e. $L \in M$ for all $t > t(L)$,

$$\forall v \in \mathbb{Z}^d \quad \|a_t L v\| \leq e^{t(\Lambda_i - \varepsilon)} \Rightarrow v \in V_{i-1} \qquad (*)$$

In other words: $(*)$ means that the successive minima of $a_t L \mathbb{Z}^d$ are attained in the V_i's.

Interpretation: The lattice $a_t L \mathbb{Z}^d$ has a fixed asymptotic shape as $t \to +\infty$ (up to rotation, and small exponential error). And this shape is independent of L.

Remark: By Minkowski's 2nd theorem

$$\lambda_1(\Delta) \cdot \ldots \cdot \lambda_d(\Delta) \asymp \mathrm{covol}(\Delta)$$

$$\Rightarrow \Lambda_1 + \dots + \Lambda_d = \mathrm{trace}(A)$$

Corollary: If (a_t) is unimodular, i.e. $\operatorname{tr}(A)=0$
Then $\exists\, V \subsetneq \mathbb{Q}^d$ proper rational subspace, s.t. $\forall \varepsilon>0$ for a.e. $L \in M$ $\forall t > t(L)$ $\forall v \in \mathbb{Z}^d$
$$\|a_t L v\| \leq e^{-\varepsilon t} \Rightarrow v \in V$$

Rk: It is easy to deduce the subspace theorem (in its non-parametric form) from this corollary, by considering all possible flows (a finite ε-net of flows suffices).

Rk: When $M=\{\text{point}\}$, the above parametric subspace thm was first formulated and obtained by Faltings-Wüstholz. It can however be deduced from Schmidt's.

IV. Slopes and Harder-Narsimhan filtrations

The rational filtration $V_1 \leq \ldots \leq V_d$ arises naturally as a Harder-Narasimhan filtration associated to a certain submodular function on $\operatorname{Gross}(\mathbb{Q}^d)$ we now describe.

Expansion Rate of a subspace: For $V \leq \mathbb{R}^d$ a subspace we define
$$\tau(V) := \lim_{t \to +\infty} \frac{1}{t} \log \|a_t \underline{v}\|$$
where $\underline{v} \in \Lambda^{\dim V} \mathbb{R}^d$ represents V
(i.e. $\underline{v} = v_1 \wedge \ldots \wedge v_k$ for some basis $\{v_i\}_i$ of V)

We also set $\tau(\{0\}) = 0$.

A key property of τ is that it is submodular, namely $\forall V, W$ subspaces

$$\tau(V \cap W) + \tau(V+W) \leq \tau(V) + \tau(W)$$

For a bounded set M we set

$$\tau_M(V) = \max_{L \in M} \tau(LV)$$

If Zar M is irreducible, then τ_M is submodular as well.

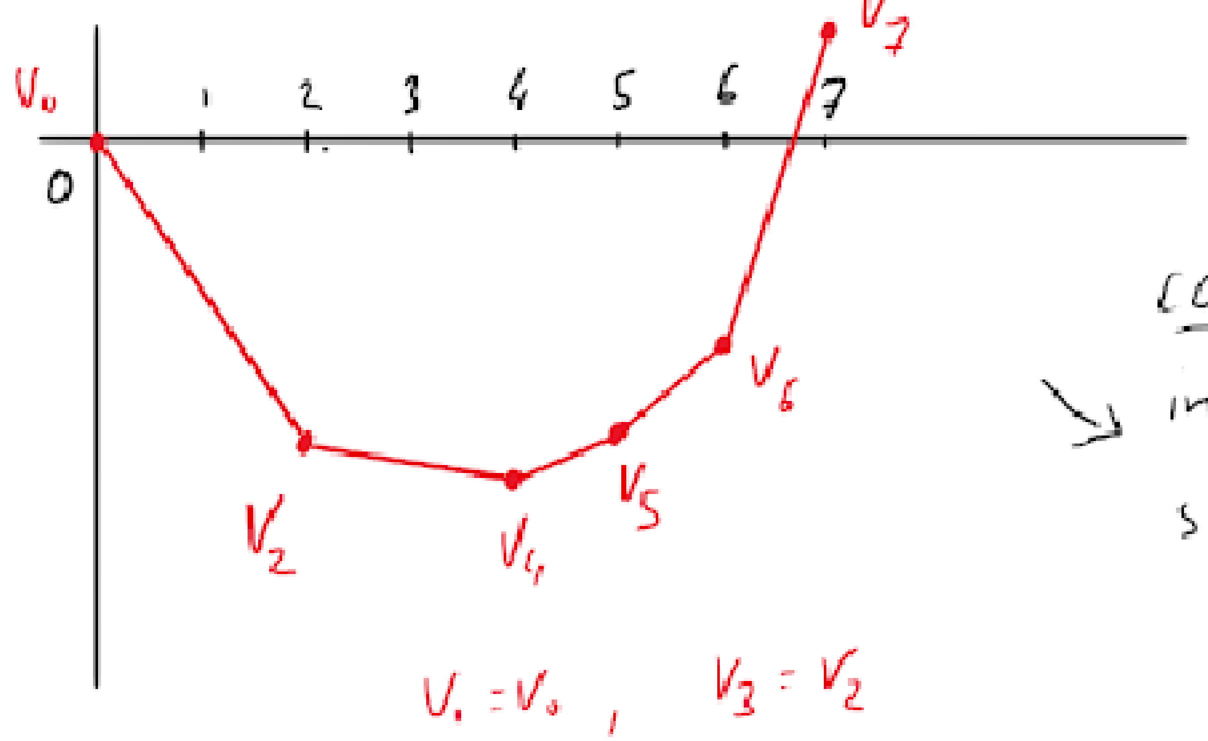
Submodularity lemma: If $\varphi : \mathrm{Gross}(k^d) \to \mathbb{R}$ is submodular (k a field $\leq \mathbb{R}$), then there is a subspace $V \leq k^d$ minimizing

$$\frac{\varphi(V) - \varphi(0)}{\dim V}$$

which contains all other such subspaces. (so it is unique).

Def: We say that k^d is semistable for φ if that subspace is k^d.

Def (Grayson polygon) In $[0,d]\times\mathbb{R}$ consider the convex hull of the points $(\dim V, \varphi(V))$ for all subspaces $V \in \mathrm{Gross}(K^d)$. The lower boundary of it is called the Grayson polygon of φ.



convexity: ↘ increasing slopes

Proposition: For each "breakpoint" (k, f) of the polygon, there is a unique subspace V with $\dim V = k$, $\varphi(V) = f$.
Moreover the resulting subspaces are nested.
They form the Harder-Narasimhan filtration $V_1 \leq \dots \leq V_d$ associated to φ.

Rk: K^d is semistable $\Leftrightarrow$ the filtration is trivial
i.e. $V_1 = \dots = V_{d-1} = 0$, $V_d = K^d$.

$$\Leftrightarrow \boxed{\frac{\varphi(W)}{\dim W} \geq \frac{\varphi(K^d)}{d} \quad \forall W \subset \mathrm{Gross}(K^d)}$$

V. Quantitative non-divergence:

For $L \in GL_d(\mathbb{R})$ we set

$$\hat{\mu}_k(L) := \sum_{i \leq k} \log \lambda_i(L\mathbb{Z}^d)$$

$$\mu_k(L) := \liminf_{t \to +\infty} \frac{1}{t} \hat{\mu}_k(a_t L)$$

Theorem (existence and inheritance of exponents):

As before $M \subseteq GL_d(\mathbb{R})$ analytic. Then $\exists\, \mu_k = \mu_k(M) \in \mathbb{R}$ s.t. for a.e. $L \in M$

$$\mu_k(L) = \mu_k = \sup_{L \in Zar(M)} \mu_k(L)$$

Theorem (Kleinbock-Margulis):

Let $B \subseteq \mathbb{R}^n$ a ball, $h : B \to SL_d(\mathbb{R})$ a function belonging to a fixed finite dimensional space $\mathcal{F}$ of analytic maps.

Then $\exists\, C, \alpha > 0$ s.t. $\forall \varepsilon > 0$ $\forall \rho \in (0,1)$

if $\sup_{y \in B} \| h(y) v_1 \wedge \dots \wedge v_k \| \geq \rho^k$ $\forall v_1, \dots, v_k \in \mathbb{Z}^d$

Then $\left| \{ y \in B \mid \lambda_1(h(y)\mathbb{Z}^d) < \varepsilon \} \right| \leq C \left(\frac{\varepsilon}{\rho} \right)^{\alpha} |B|$

slighlty generalized version (Lindenstrauss-Margulis-Mohammadi-Shah and de Saxcé):

Let $\varphi : \mathrm{Grass}(\mathbb{Q}^d) \to \mathbb{R}$

$\varphi(V) = \log \sup_{y \in B} \|h(y) \underline{v}\|$

Let $\{(k, \varphi_k)\}_{k=1,\dots,d}$ the associated Grayson polygon.

$\exists\, C, \alpha > 0$ s.t.

$$\left|\left\{ y \in B \;\middle|\; \exists k \quad \hat{\varphi}_k(h(y)\mathbb{Z}^d) \leq \varphi_k + \log \varepsilon \right\}\right| \leq C \varepsilon^{\alpha} |B|$$

Proposition (comparison of polygons):

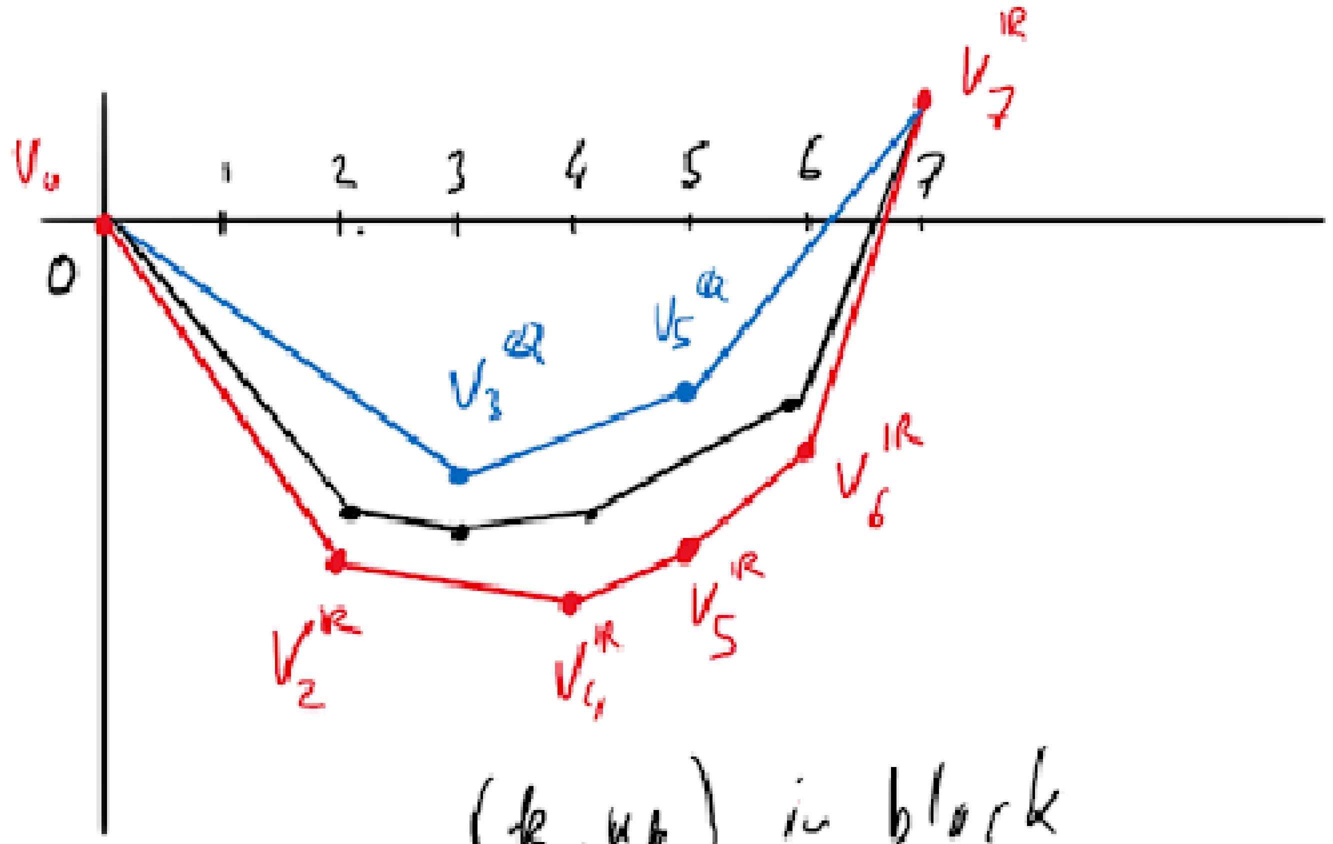
Let $\mathcal{P}^{\mathbb{R}}$ and $\mathcal{P}^{\mathbb{Q}}$ the Grayson polygons for the submodular functions

$$\tau_M^k(V) = \max_{\lambda \in M} \lim \frac{1}{t} \log \|a_t \lambda \underline{v}\|$$

$k = \mathbb{R}$ or $\mathbb{Q}$.

Let P_μ the polygon $\{(k, \mu_k(M))\}_k$

Then $\mathcal{P}^{\mathbb{R}} \leq P_\mu \leq \mathcal{P}^{\mathbb{Q}}$



(k, μ_k) in block

$\mathcal{P}^{\mathbb{R}}$, $\mathcal{P}^{\mathbb{Q}}$, $\mathcal{P}_\mu$

If $\mathrm{Zar}(M)$ is defined over $\overline{\mathbb{Q}}$, then

$$\mathcal{P}_\mu = \mathcal{P}_{\mathbb{Q}}.$$

Pf of Main thm:

pick $L \in \mathrm{Zar}(M)(\overline{\mathbb{Q}})$ generic enough so that

$$\mathcal{P}^{\mathbb{Q}}(M) = \mathcal{P}^{\mathbb{Q}}(L)$$

By Schmidt's thm $\mathcal{P}_\mu(L) = \mathcal{P}^{\mathbb{Q}}(L)$

By Quantitative Non-divergence

$$\mathcal{P}_\mu(L) \le \mathcal{P}_\mu(M) \le \mathcal{P}^{\mathbb{Q}}(M)$$

$$\Rightarrow \mathcal{P}_\mu(M) = \mathcal{P}_{\mathbb{Q}}(M). \quad \square$$

VII. Applications:

Diophantine approximation on submanifolds of matrices :

$M \subseteq \mathrm{Mat}_{m,n}(\mathbb{R})$ $\qquad \tilde{M}$ image under $Y \mapsto \begin{pmatrix} I_m & Y \\ & I_n \end{pmatrix} \in GL_d(\mathbb{R})$

$d = m + n$.

M is extremal $\iff$ $\mathbb{R}^d$ is semistable for flow

$$a_t = \exp(\mathrm{diag}(\underbrace{nt, \dots, nt}_{m \text{ times}}, \overbrace{-mt, \dots, -mt}^{n \text{ times}}))$$

M strongly extremal $\iff$ $\mathbb{R}^d$ semistable for all flows a_t with

$$\min_{i \leq m} A_i \geq \frac{1}{d} \sum A_i \geq \max_{i > m} A_i$$

Non-commutative Roth type theorems

G algebraic group over $\mathbb{Q}$, say $G \leq GL_d$

Fix $g_1, \dots, g_k \in G$. w word of length $\ell(w)$ in k letters.

How close to 1 can $w(g_1, \dots, g_k)$ get as w varies an $n \nearrow +\infty$?

$$\delta(n) := \inf \left\{ \|w - 1\| \, , \, \begin{matrix} \ell(w) \leq n \\ w \neq 1 \end{matrix} \right\}$$

Thm (B + Sarcé) If G is unipotent $\forall g_1, \dots, g_k \in G(\overline{\mathbb{Q}})$ away fr certain exceptional subvarieties defined over $\mathbb{Q}$.

$$\delta(n) \simeq 1/n^{\beta}$$

$$\beta = F(k) \, , \, F \in \mathbb{Q}[x]$$

Open Pb (Roth for GL_2) $\exists ?\, c > 0$ s.t. $\forall a, b \in GL_2(\overline{\mathbb{Q}})$

$$\delta(n) \geqslant \exp(-cn)$$

$\forall n \gg 1$.

THANK YOU!