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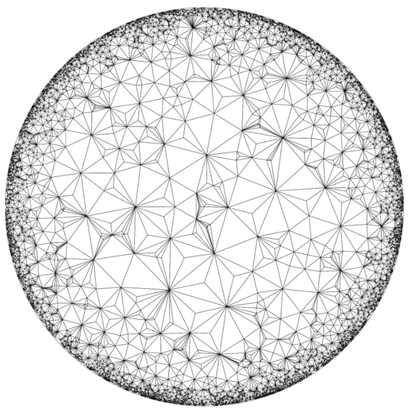

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(CNRS – IRIF – Université Paris Cité)

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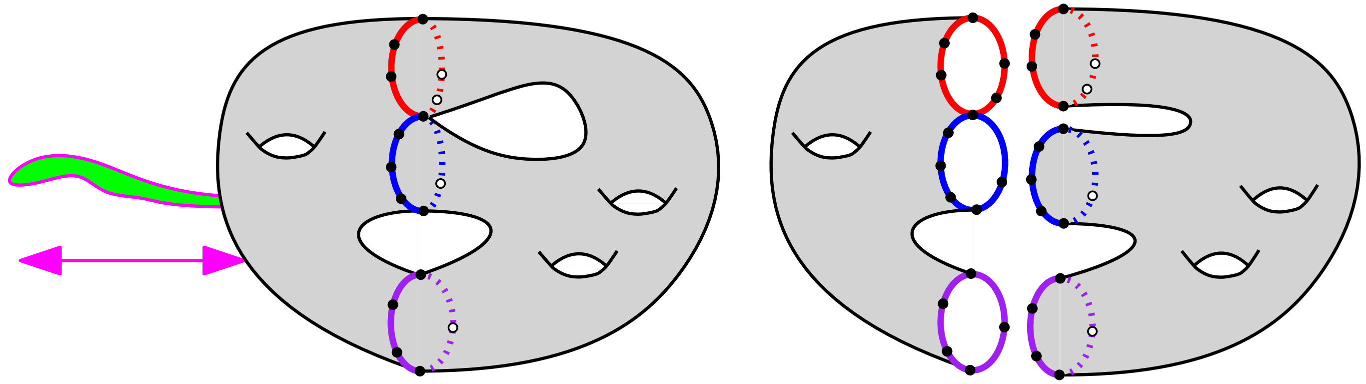

Oxford Discrete Mathematics and Probability Seminar, May 2024 (online)

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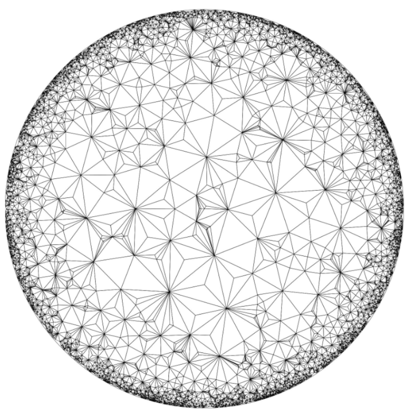

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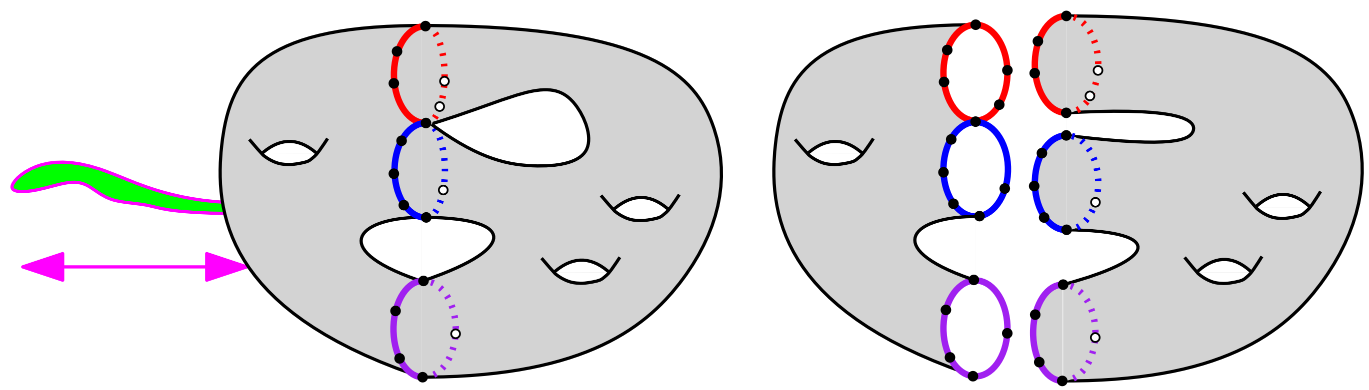

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What is this talk about

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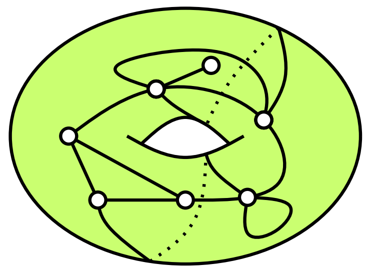
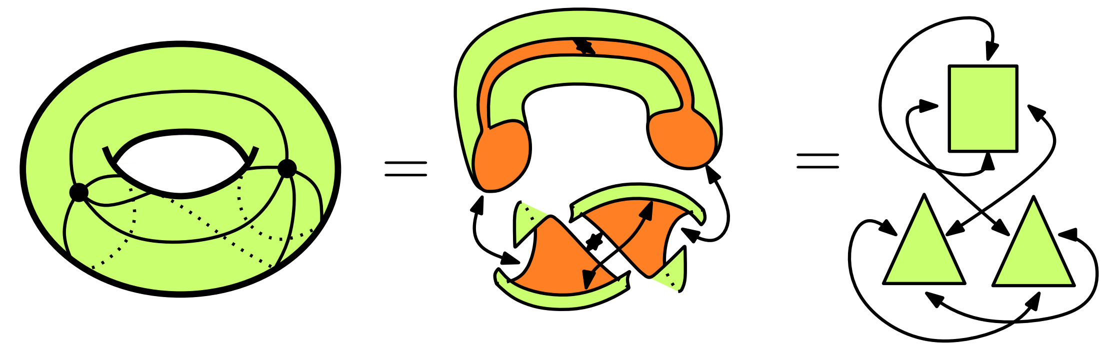
- I am a combinatorialist. Today I'll try to do an introduction about random maps and what we are interested to ask/say about them. Statements will be mostly probabilistic in nature, but combinatorics (and counting) plays a key role everywhere.

Maps

(Random) Combinatorial maps

A map of genus $g \geq 0$ is a finite graph properly embedded on the oriented compact genus-g surface, with polygonal faces, considered up to homeomorphism.





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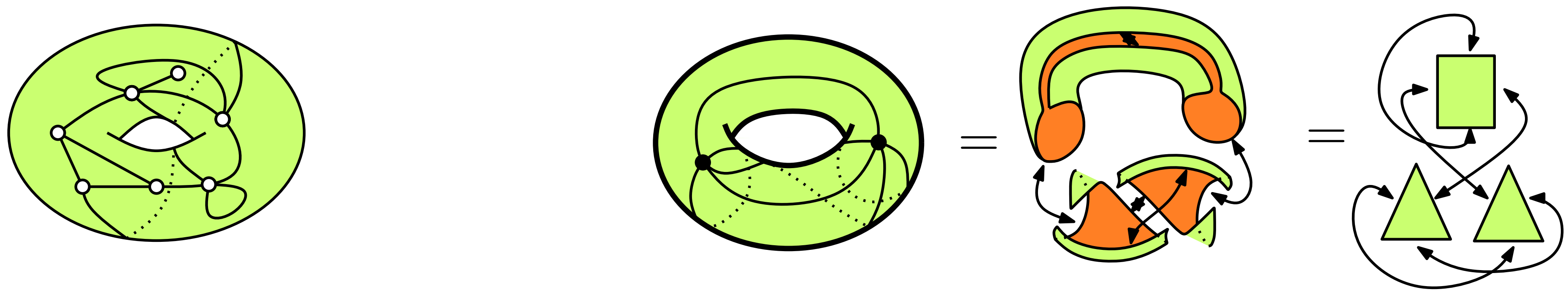

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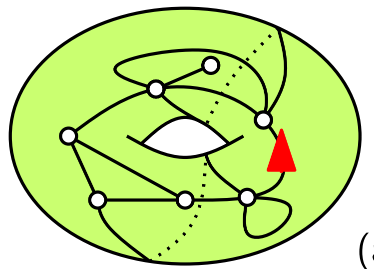
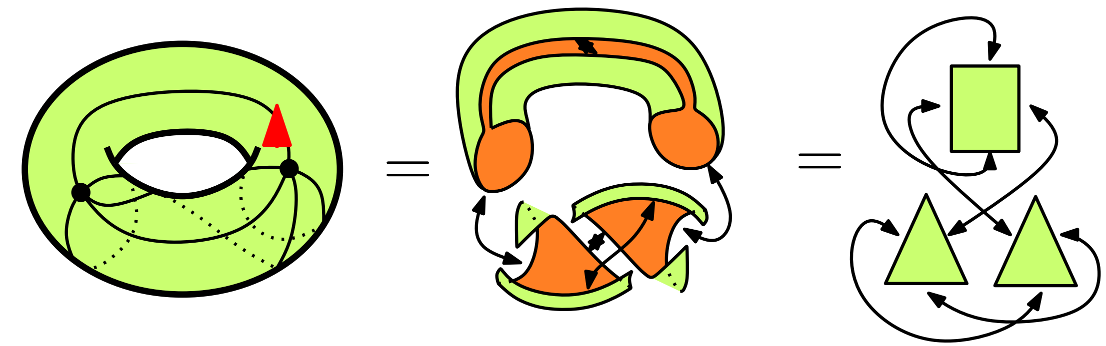


(all my maps are rooted)



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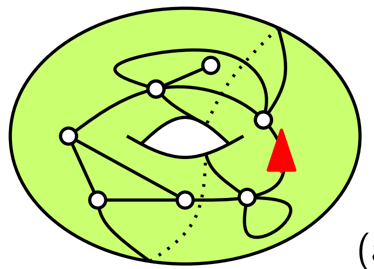

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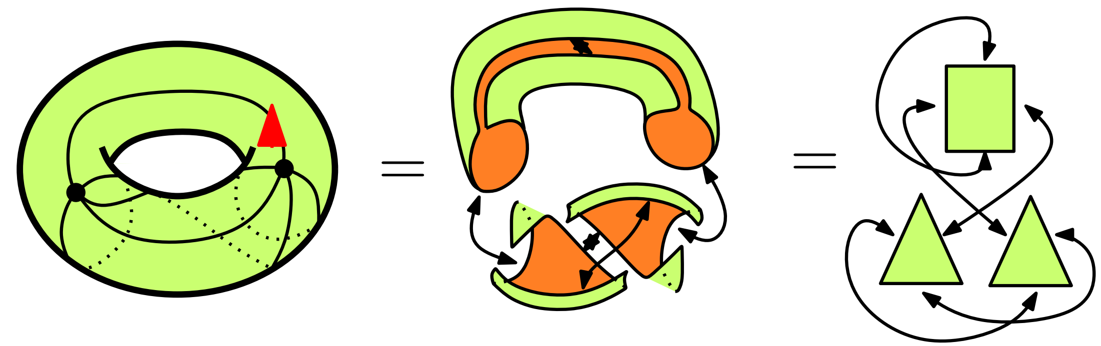

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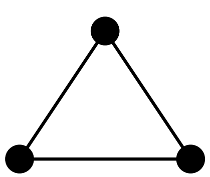
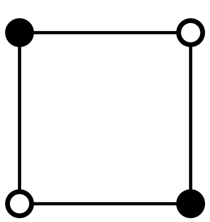
Random maps: take a map uniformly at random among the ones having genus g and size n, possibly with some face-degree constraints/weights.



Triangulations: $\mathcal{T}_{n,g} = \{$triangulations, genus g, $2n$ faces $\}$

$$\mathbf{T}_{n,g} \in_u \mathcal{T}_{n,g} \qquad \tau(n,g) = |\mathcal{T}_{n,g}|$$



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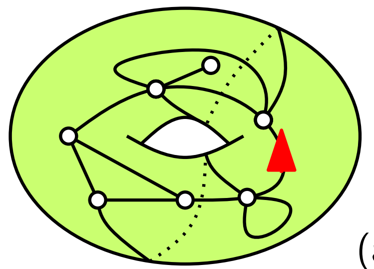

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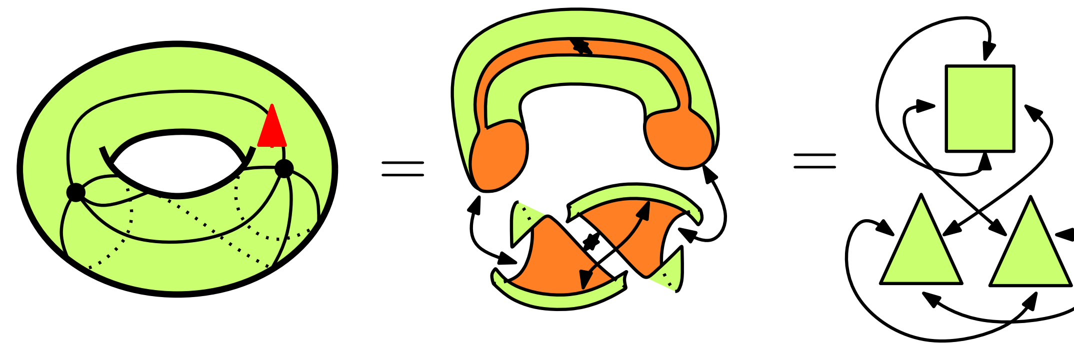

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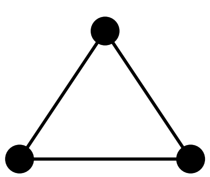

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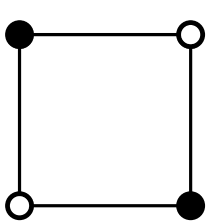

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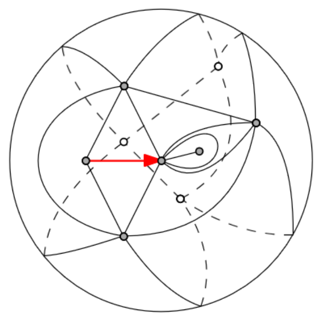
what do they look like (when n goes to infinity)? → local behaviour? → global behaviour?

Local behaviour: planar case

Local limit: [Angel-Schramm 2000's:] When n goes to infinity,

$$\mathbf{T}_{n,0} \longrightarrow UIPT \text{ in distribution for the local limit topology}$$



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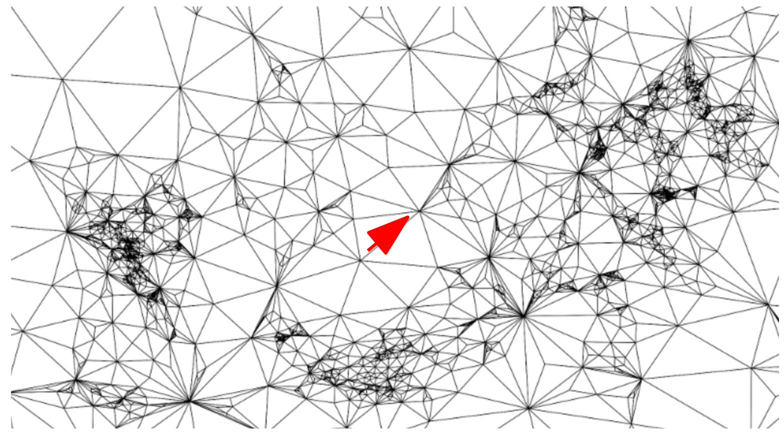

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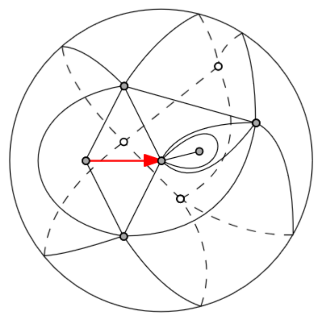

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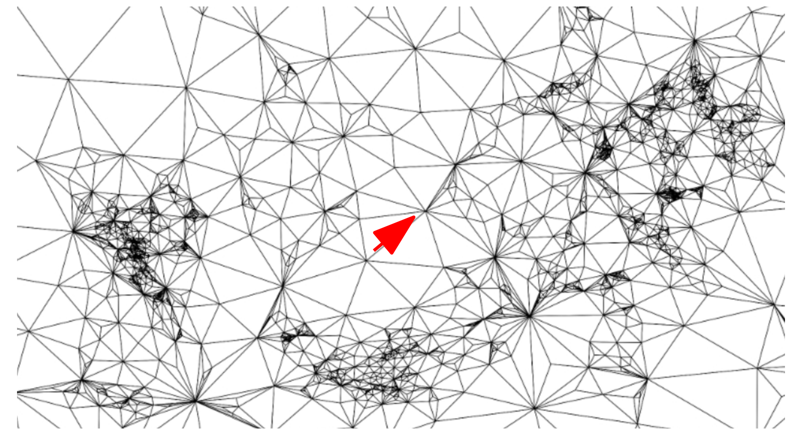

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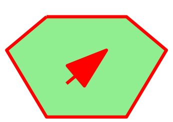
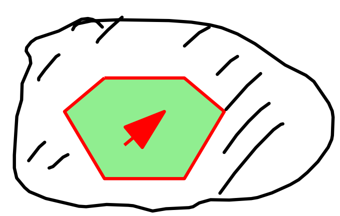
Observables for this topology are finite neighbourhoods of the root:

Fix a test map:



perimeter p
m triangles

$\mathbb{P}\Big(\mathbf{T}_{n,0}$ looks like



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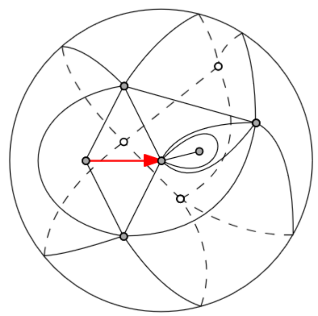

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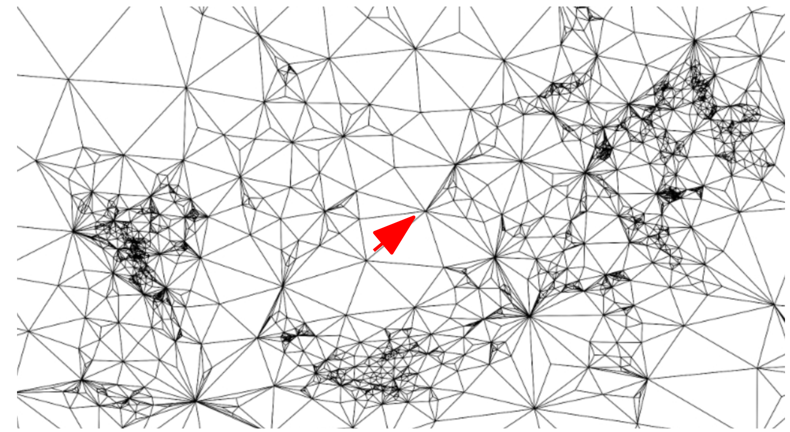

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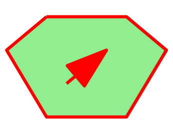

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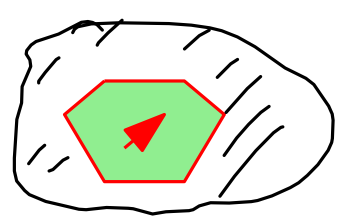
$\mathbb{P}\Big(\mathbf{T}_{n,0}$ looks like  $\Big) = \dfrac{\tau_p(n-m,0)}{\tau(n,0)} \overset{n\to\infty}{\longrightarrow}$ something explicit.

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One of the reasons you can do this is because you can count.

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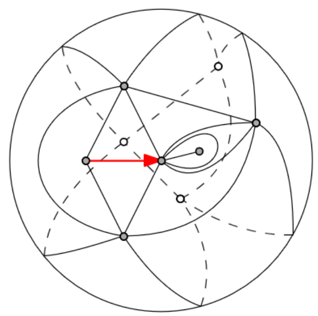
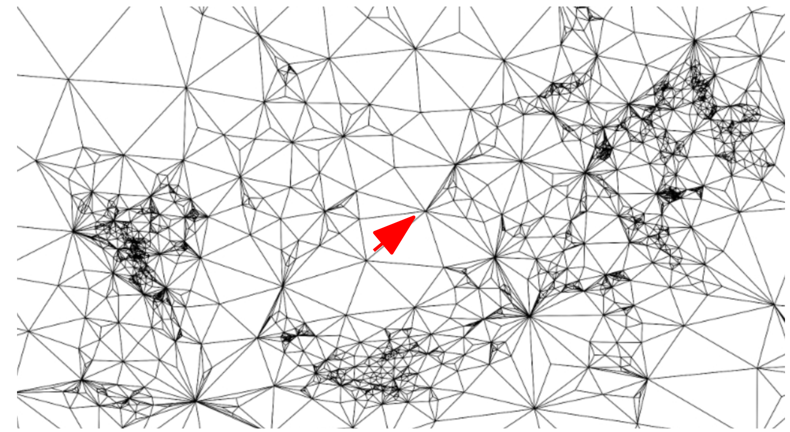
 $\longrightarrow$ 

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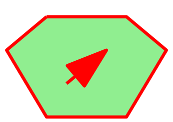

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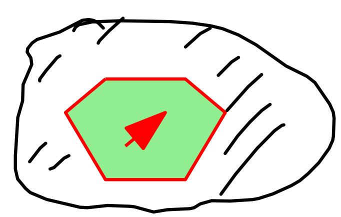
$\mathbb{P}\Big(\mathbf{T}_{n,0}$ looks like  $\Big) = \dfrac{\tau_p(n-m,0)}{\tau(n,0)} \xrightarrow{n\to\infty}$ something explicit.

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Similar behaviour in any fixed genus (the local behaviour is not affected by g).

Fixed genus case: global properties?

[Chassaing-Schaeffer 2004], [C.2010] For $g \geq 0$ (fixed) one has

$\mathrm{diam}(\mathbf{Q}_{n,g}) \approx n^{1/4}$

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One can do much more and show the convergence of the whole metric space

genus 0: $(\mathbf{Q}_{n,0}; \frac{d_{gr}}{n^{1/4}}) \xrightarrow{GH} (\mathbf{Q}_{\infty,0}; d_{\infty}^{(0)})$

Brownian map [Le Gall '11, Miermont '11].

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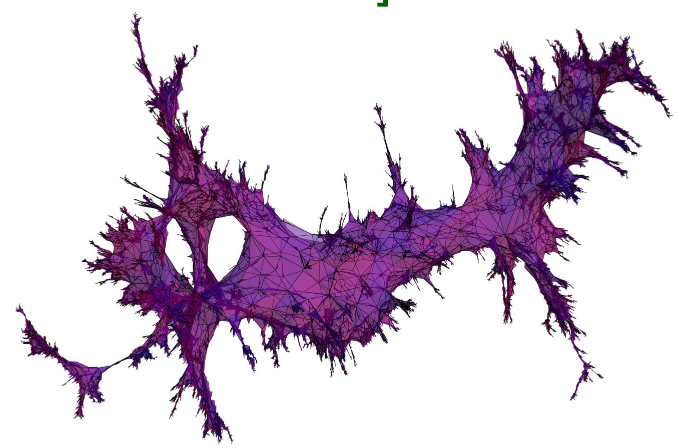

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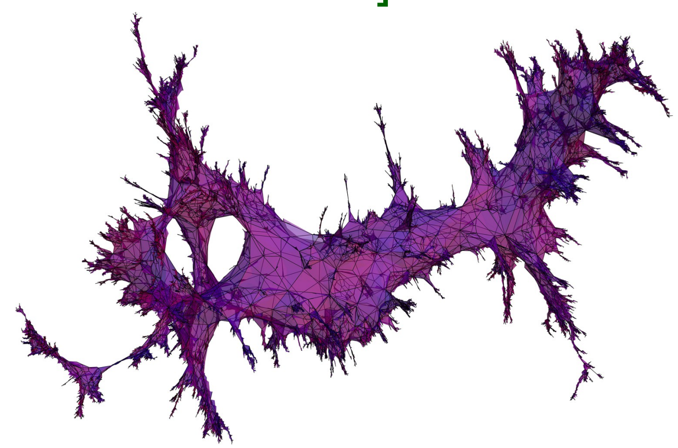
(the GH-distance, for Gromov-Haussdorf, is a distance that enables you to compare two compact metric spaces and say "how different" they are one from the other")



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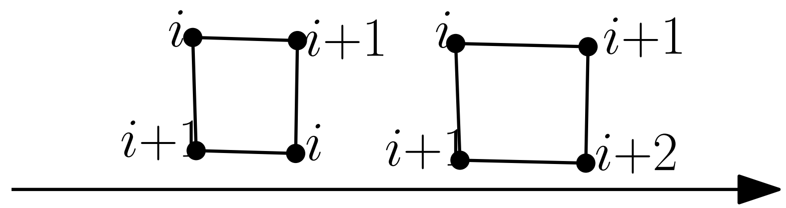
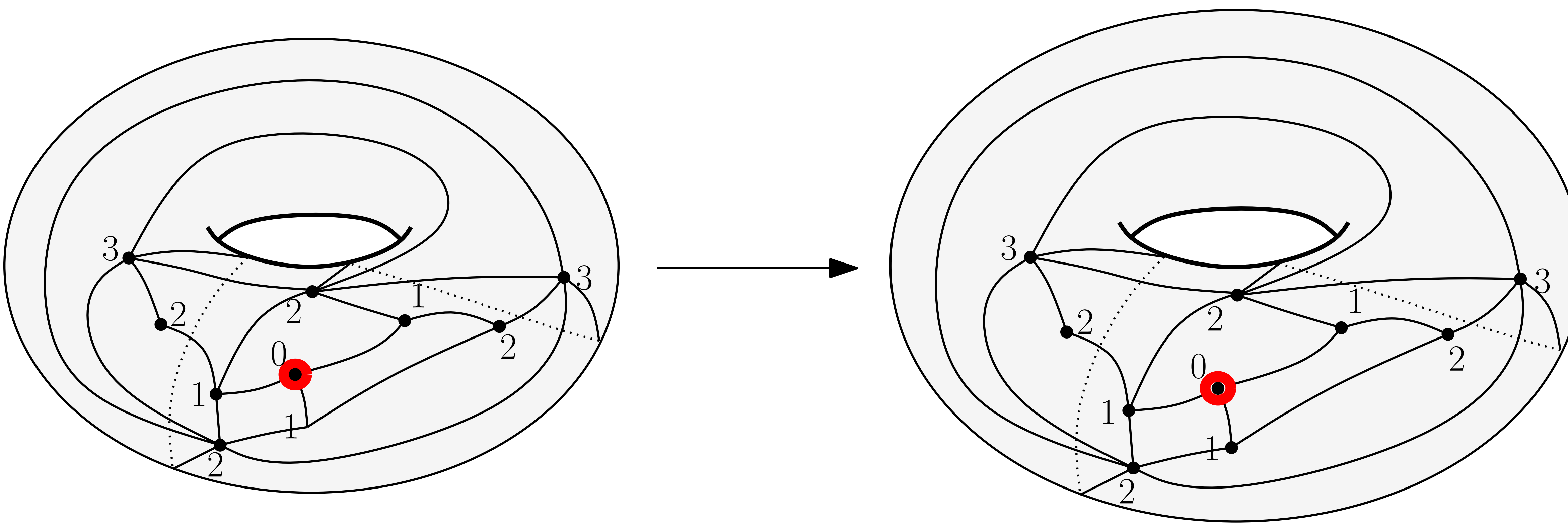
Cori-Vauquelin-Schaeffer-Marcus bijection (I)

The following construction has been designed over decades by combinatorialists with no other motivation in mind than its intrinsic beauty:

bipartite quadrangulations
n faces (root + pointed vx)
genus g



one face map, n edges, equipped with a 1-Lipshitz integer labelling of their vertices!



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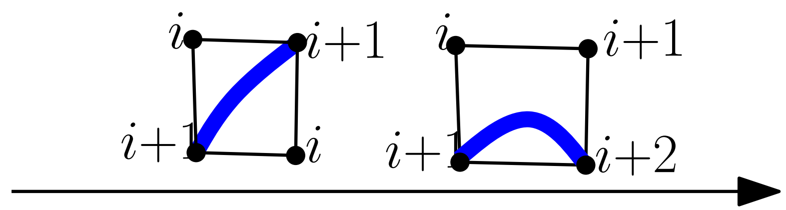

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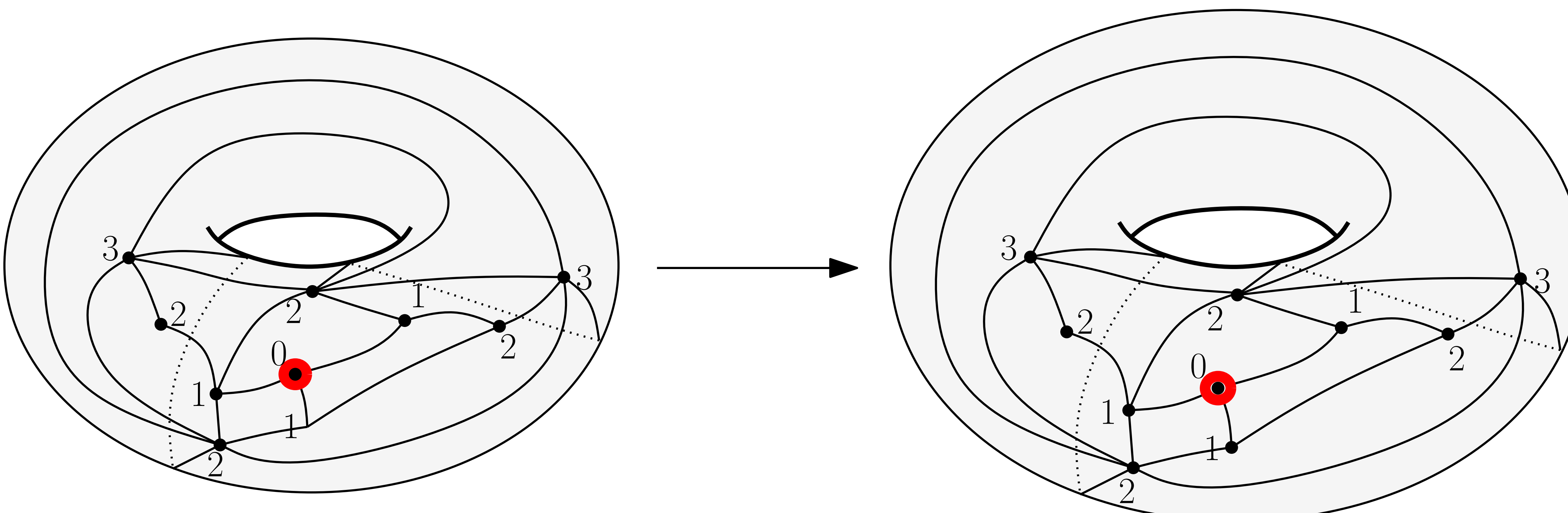

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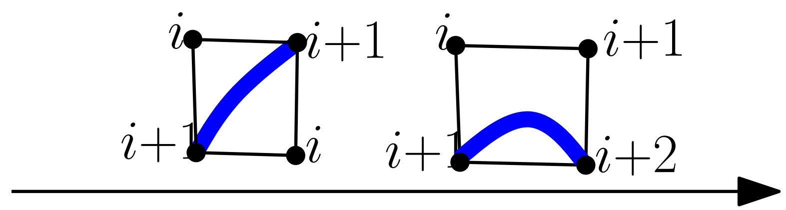

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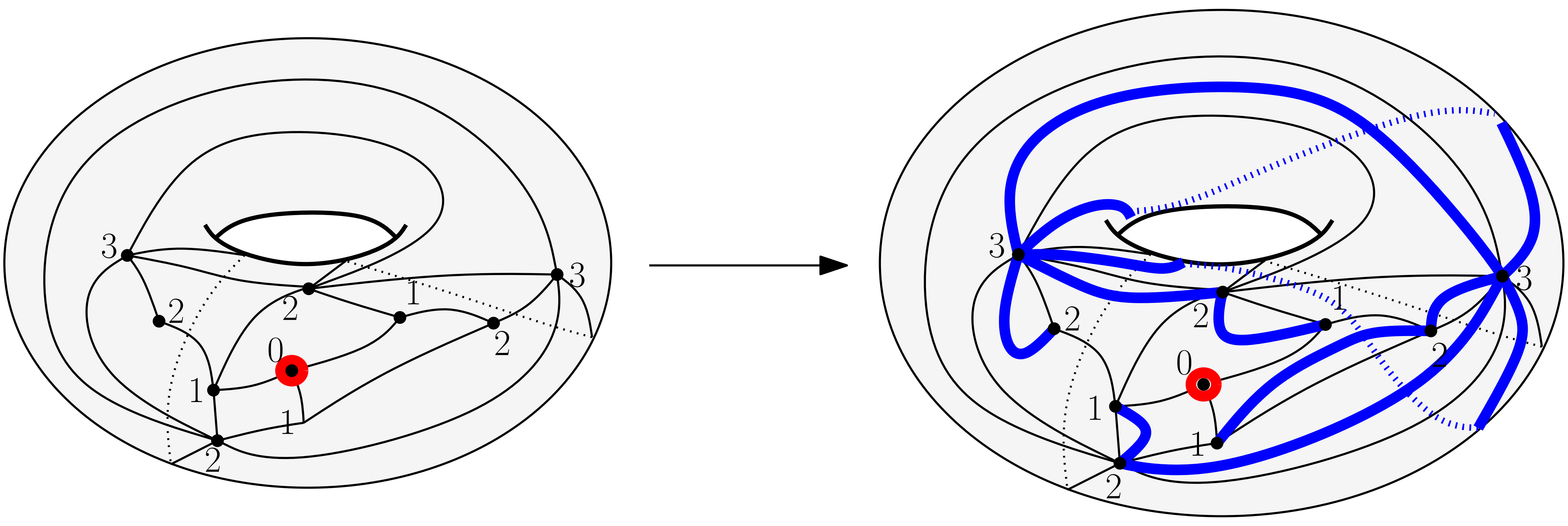

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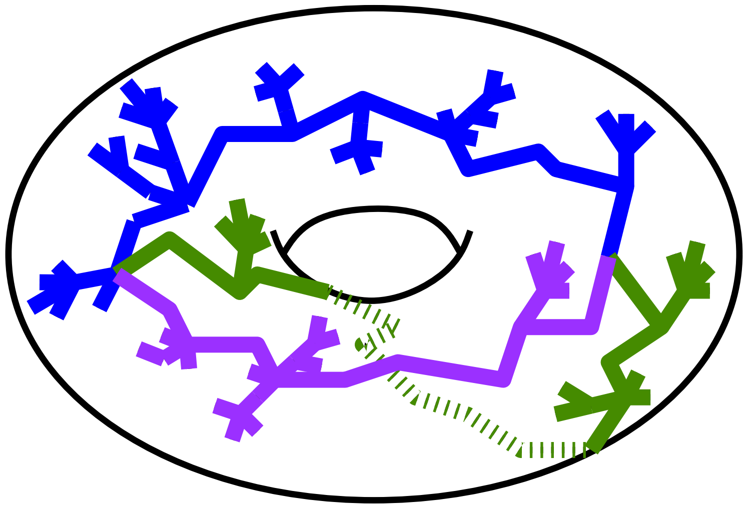

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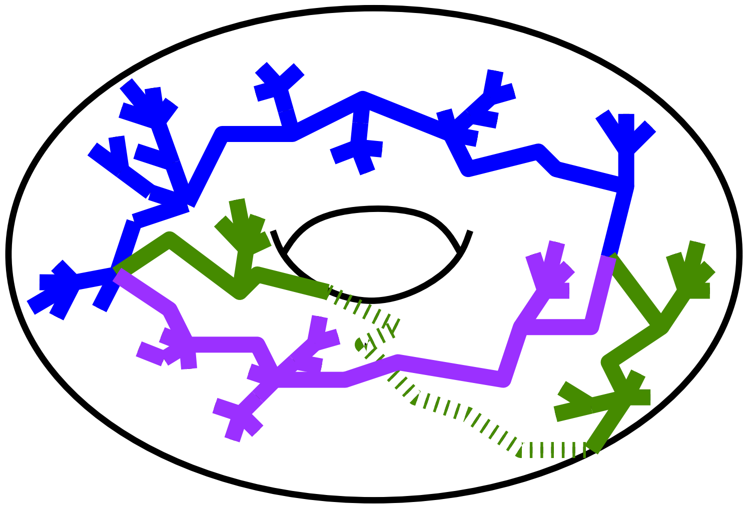
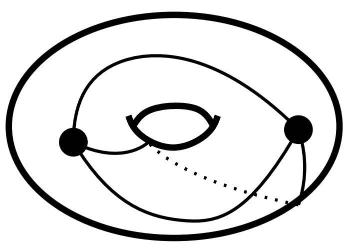


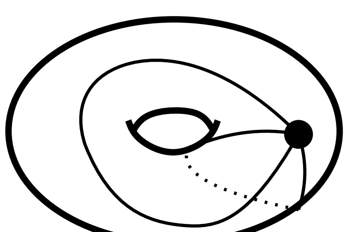
A one face map is formed by a kernel with K branches

Max. case : $K = 6g-3$ branches (cubic kernel)

$K = 3$



$K = 2$



(only two kernels for $g = 1$)

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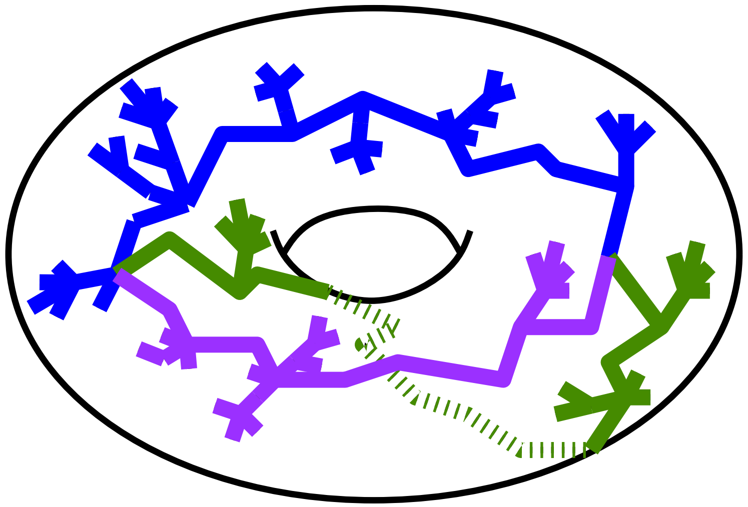

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These branches are random trees and they typically have diameter $\approx \sqrt{n}$

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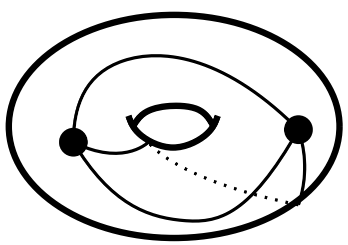

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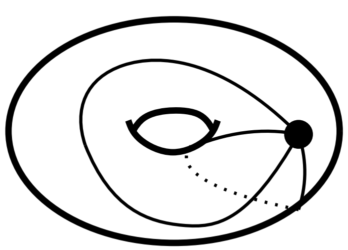

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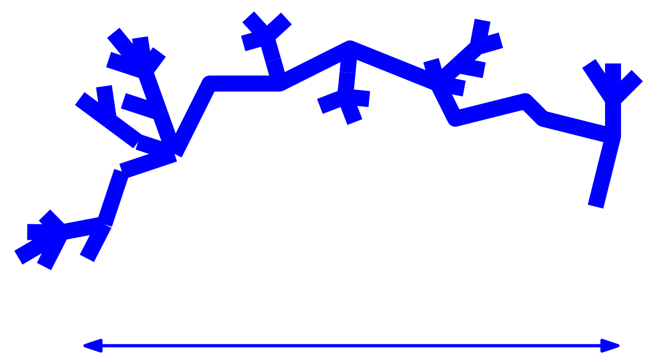

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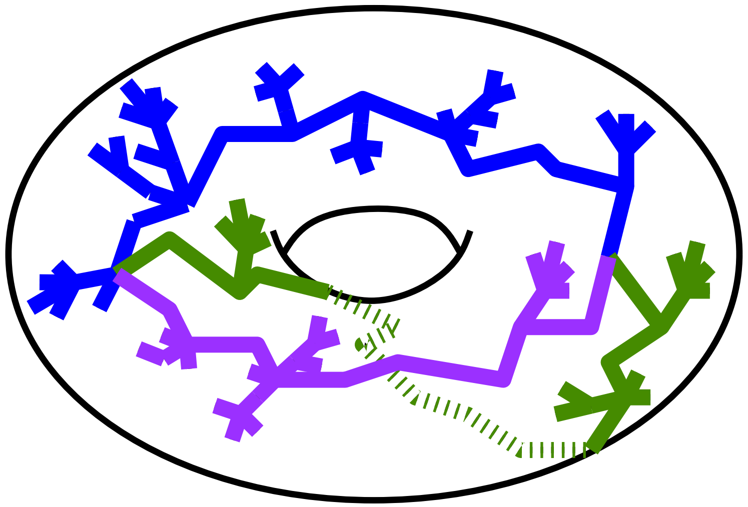

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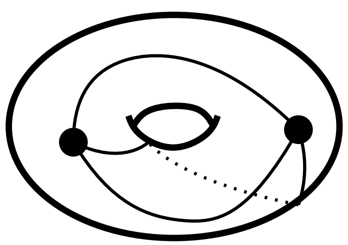

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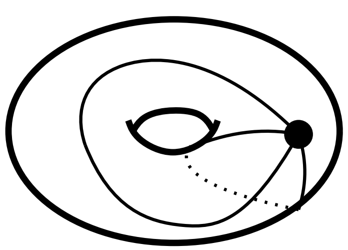

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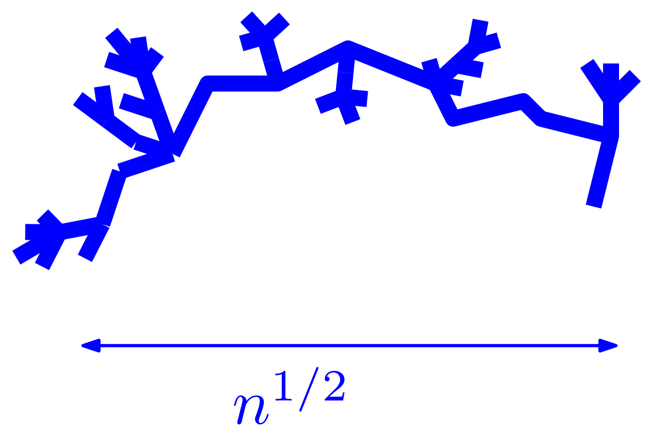

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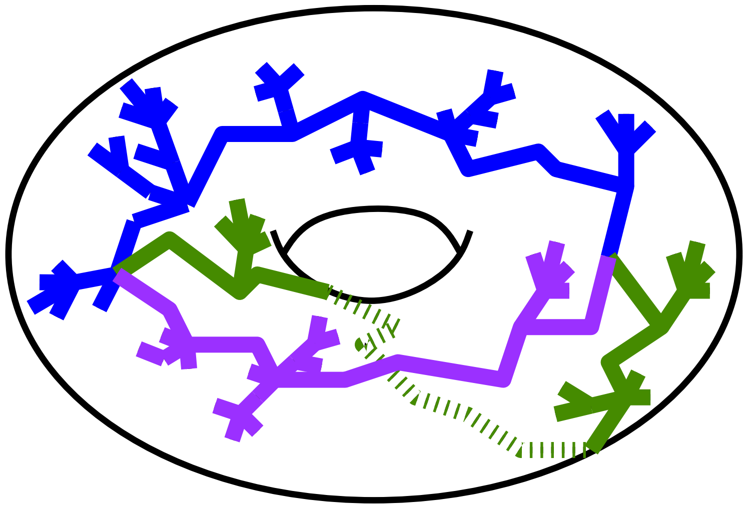

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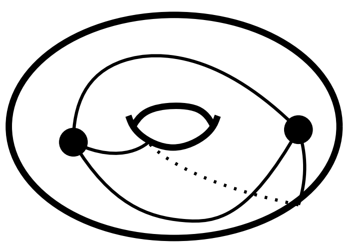

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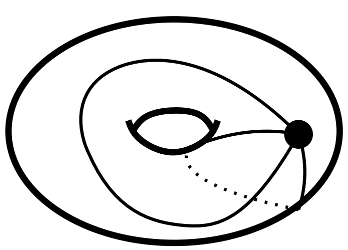

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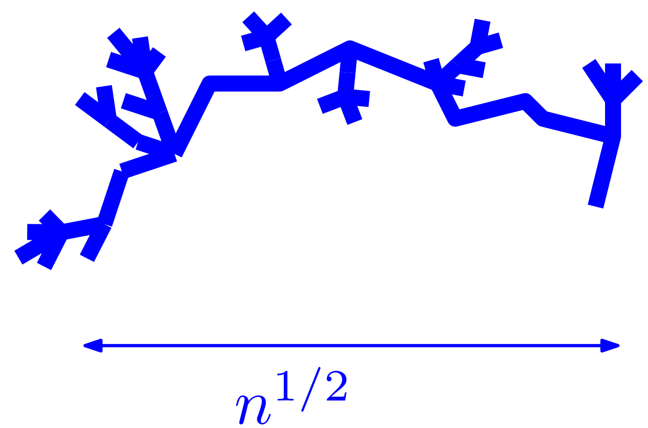

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Fixed genus maps: things we don't really know

[Bender et Canfield 1986:] For $g \geq 0$ (fixed) the number of maps of size n satisfies

$$|\mathcal{Q}_{n,g}| \sim t_g n^{\frac{5}{2}(g-1)} 12^n \qquad n \longrightarrow \infty.$$

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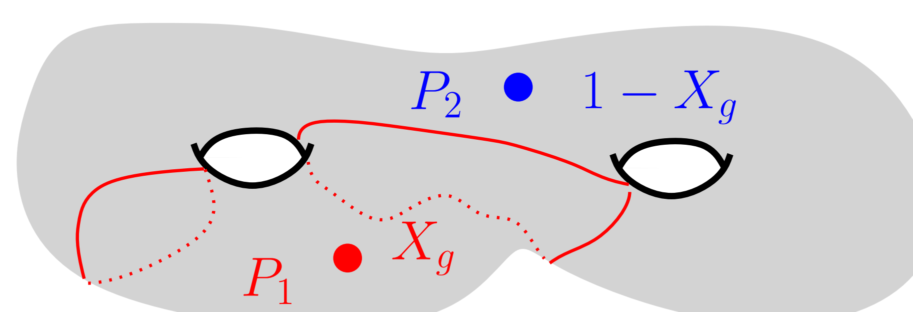
Conjecture [Ch. '17] Pick two points uniformly on a Brownian surface of genus g

Let X_g = fraction of points in the Voronoï cell of P_1 vs P_2

Then X_g is uniform on $[0,1]$???

(the fact that $\mathbf{E}X_g^2 = \frac{1}{6}$ is known and is "bijectively/surgerically equivalent" to the double scaling limit above)



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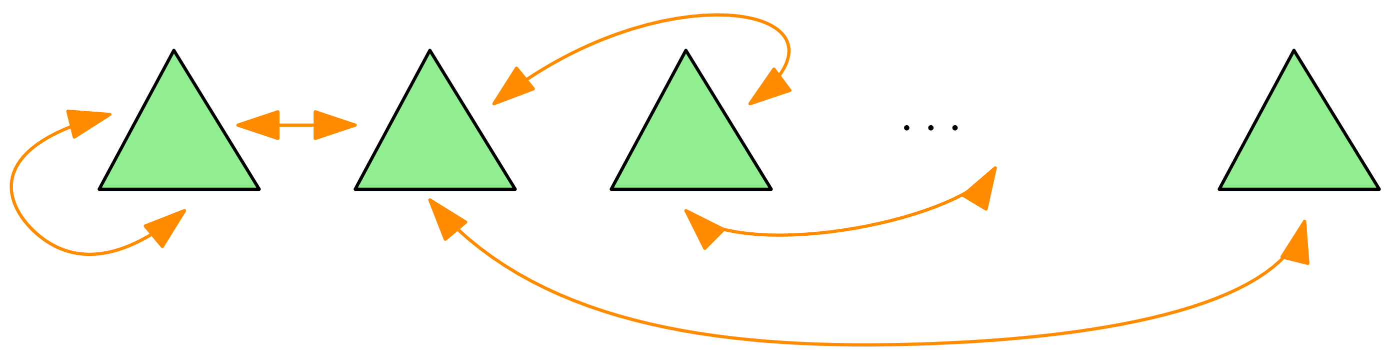

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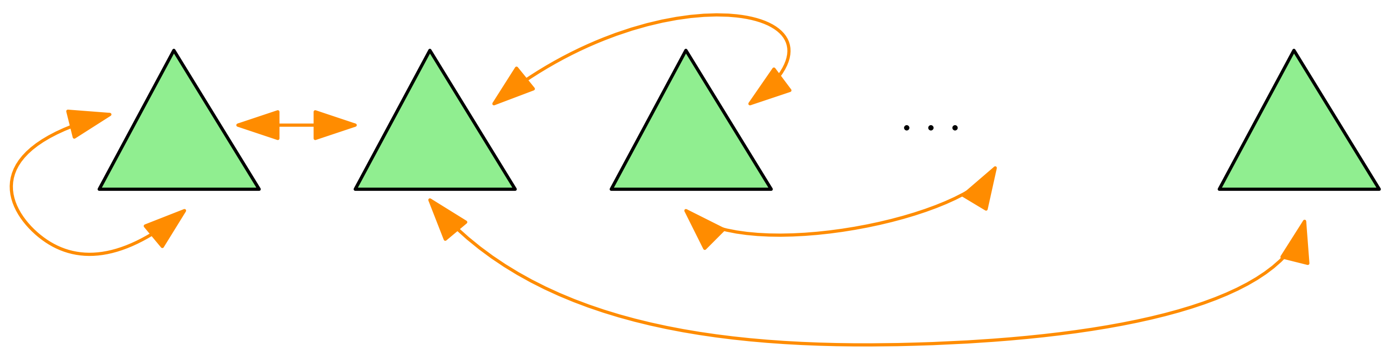

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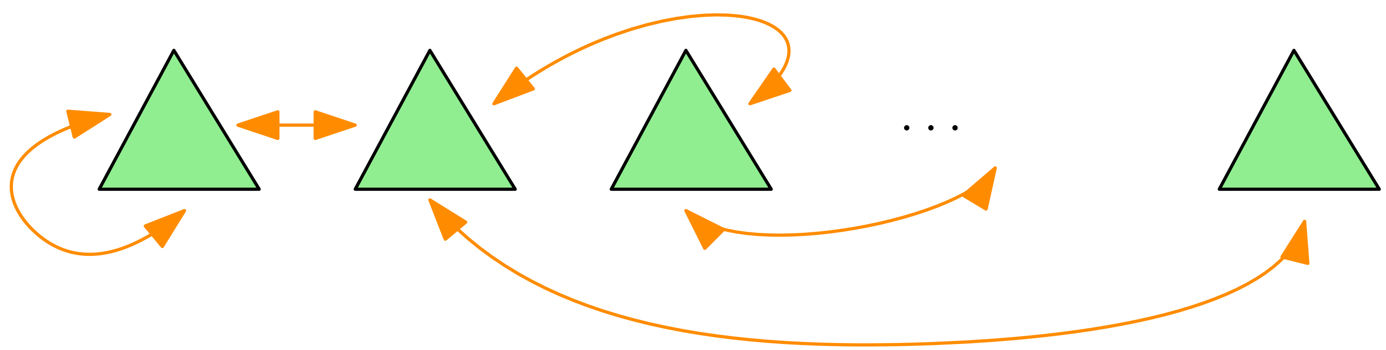

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$f = 2n, e = 3n,$ $v = n + 2 - 2g,$

Average degree $\sim \frac{6}{1-2\theta} > 6.$

The high-genus regime

Now we fix $\theta > 0$ and we consider maps of genus $g_n \sim \theta n$. This is called the high-genus regime.

This model is fun because it is difficult:

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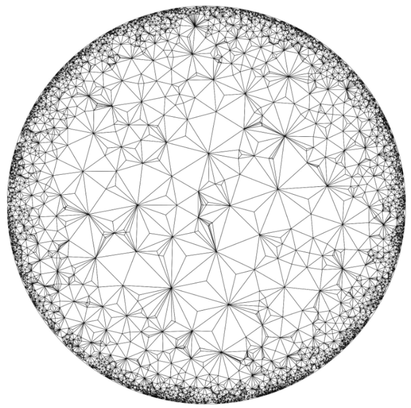
$\rightarrow$ the combinatorial results do not exist in the general case (e.g. triangulations). Until recently high-genus triangulations were just good for science-fiction....

A breakthrough: the local limit in high-genus!

[Budzinski-Louf 2019] Proof of the Benjamini-Curien conjecture" When n goes to infinity and $g \sim \theta n$, $\theta \in (0, \frac{1}{2})$ $\mathbf{T}_{n,g} \longrightarrow PSHT(\lambda(\theta))$ for the local limit topology



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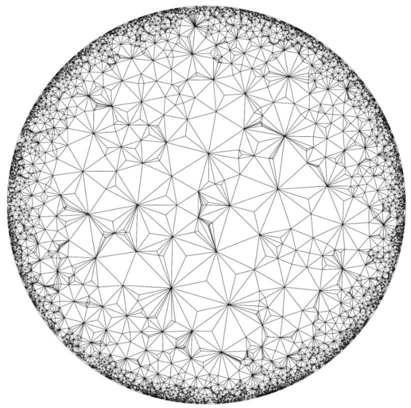

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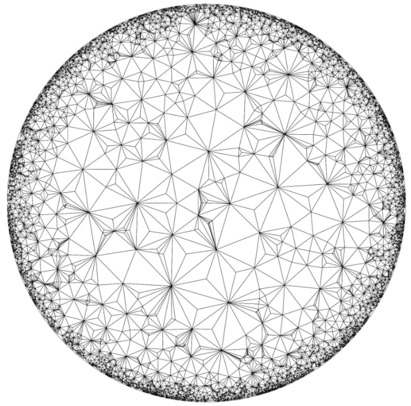

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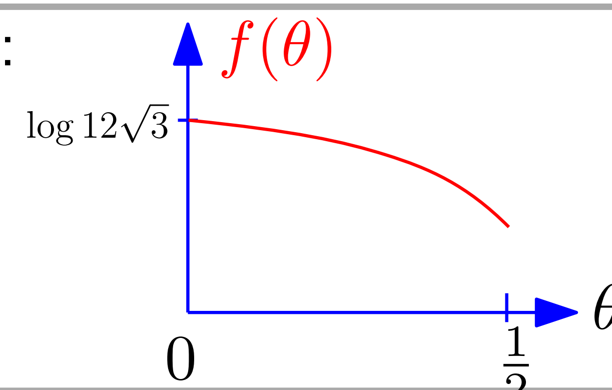
[Budzinski-Louf 2019] Counting corollaries:

$$\tau(n, g) = n^{2g} exp(f(\theta)n + o(n))$$



This is far from a true equivalent ($e^{o(n)}$ can be big!) but the best one can do!

Our new result: global properties in high genus

[Budzinski-Ch-Louf 2023$^+$] When n goes to infinity and $g \sim \theta n$, $\theta \in (0, \frac{1}{2})$

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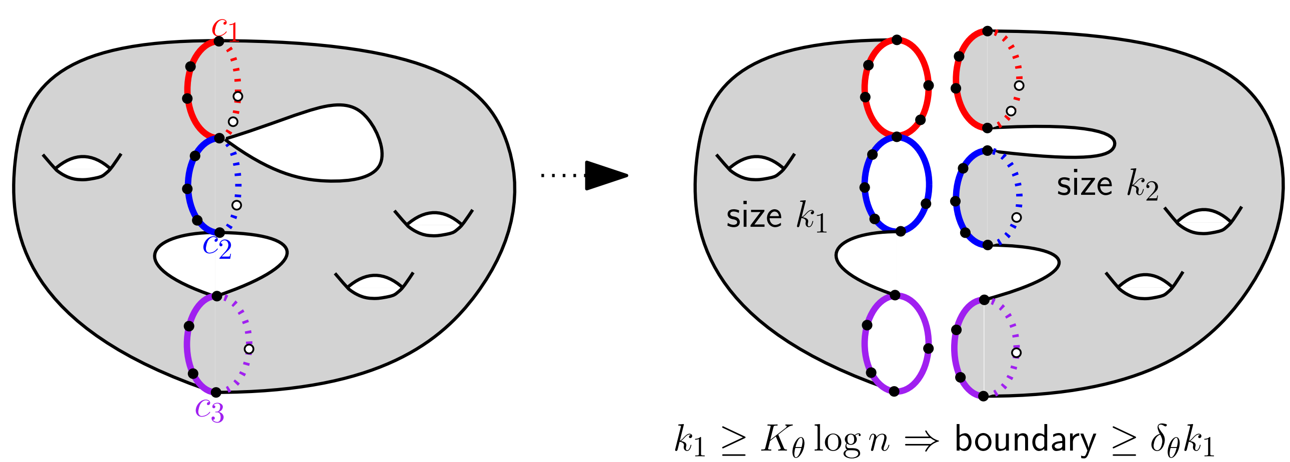

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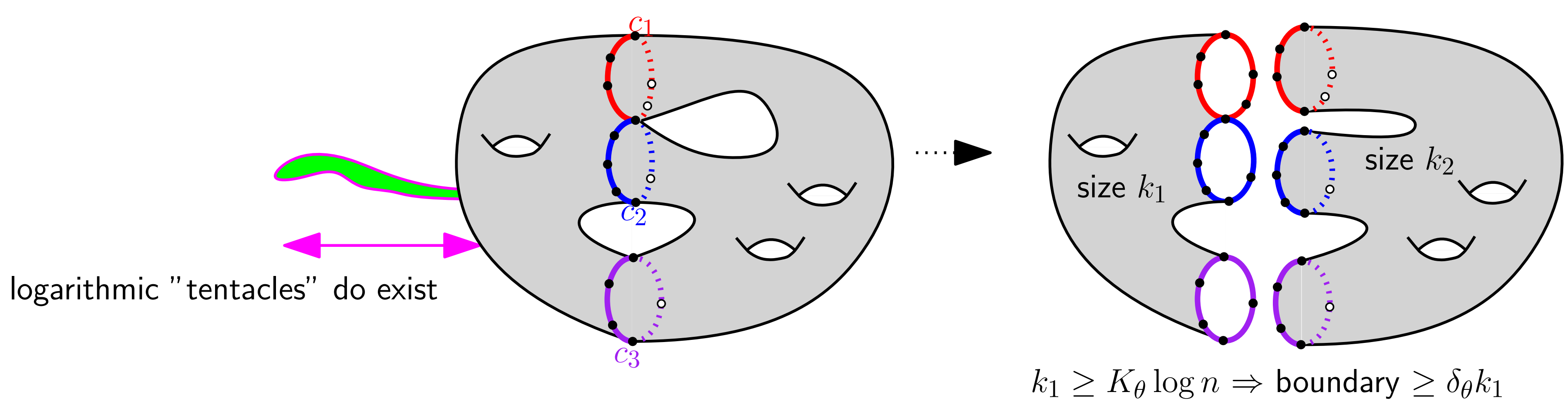
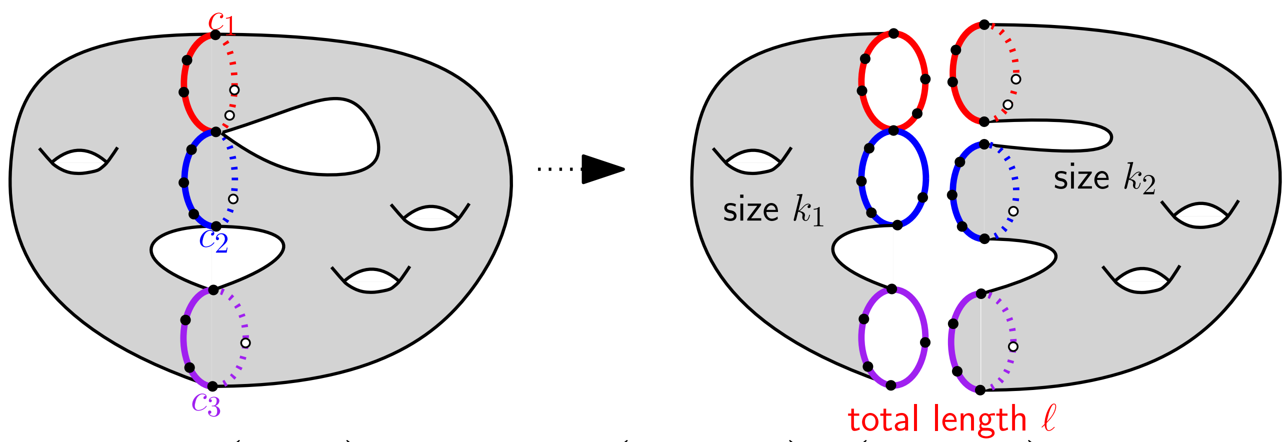


We also get the Cheeger constant: $C_\theta \frac{1}{\log_n} \leq h \leq C'_\theta \frac{1}{\log n}$ w.h.p.

where $h = \min\{\frac{|\partial A|}{|A|}, A \subset faces(\mathbf{T}_{n,g}), |A| \leq n\}$

Some elements of the proofs

- Idea behind isoperimetry: use and strengthen the counting estimates of [BL19]



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$\tau(n, g)$ versus $\tau(n_1, g_1)\tau(n_2, g_2)$

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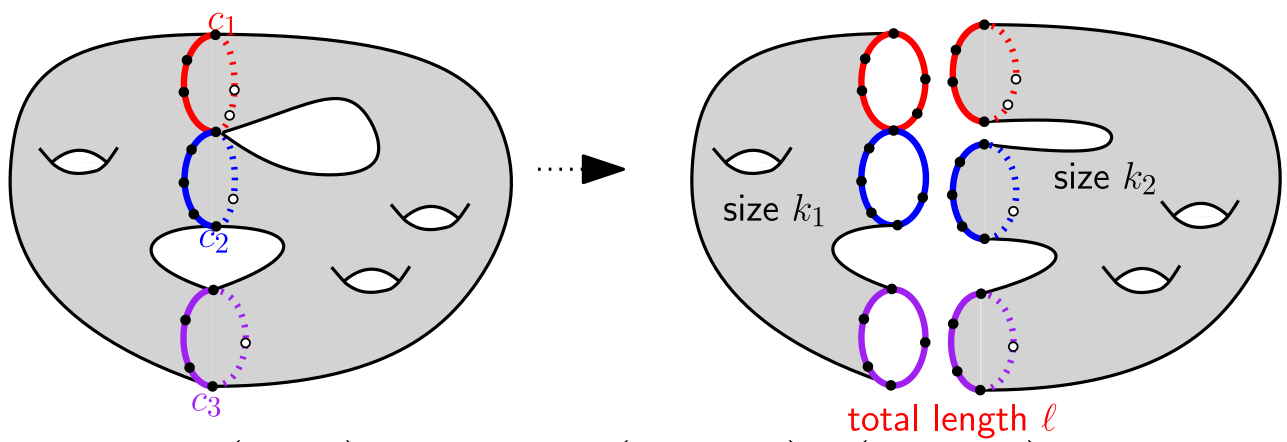

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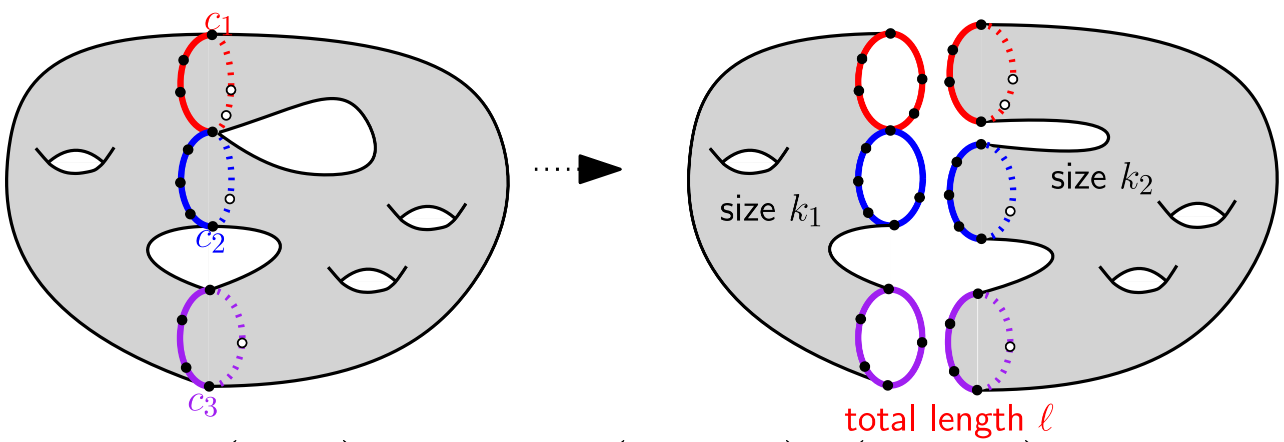

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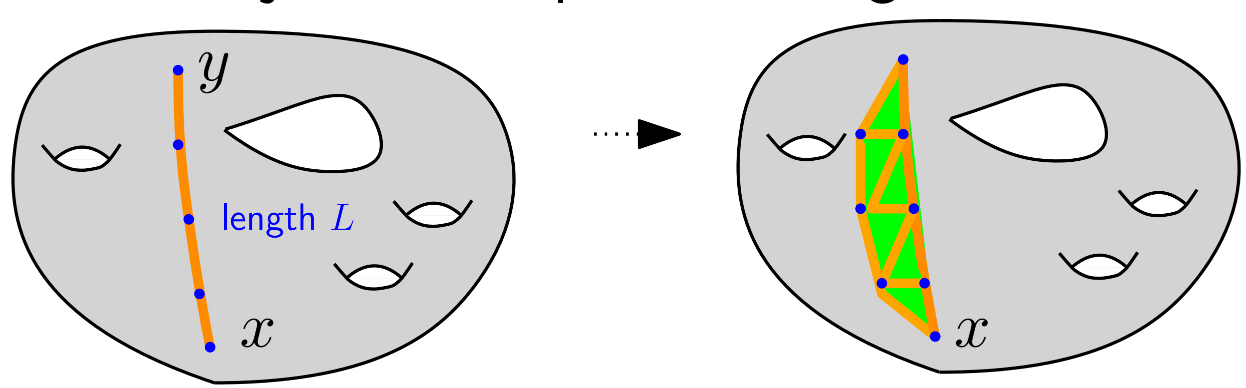
- Lower bounding the diameter: we just count paths of length L between two random points:

$$\frac{\tau(n-1,g)}{\tau(n,g)} \longrightarrow \lambda(\theta)$$

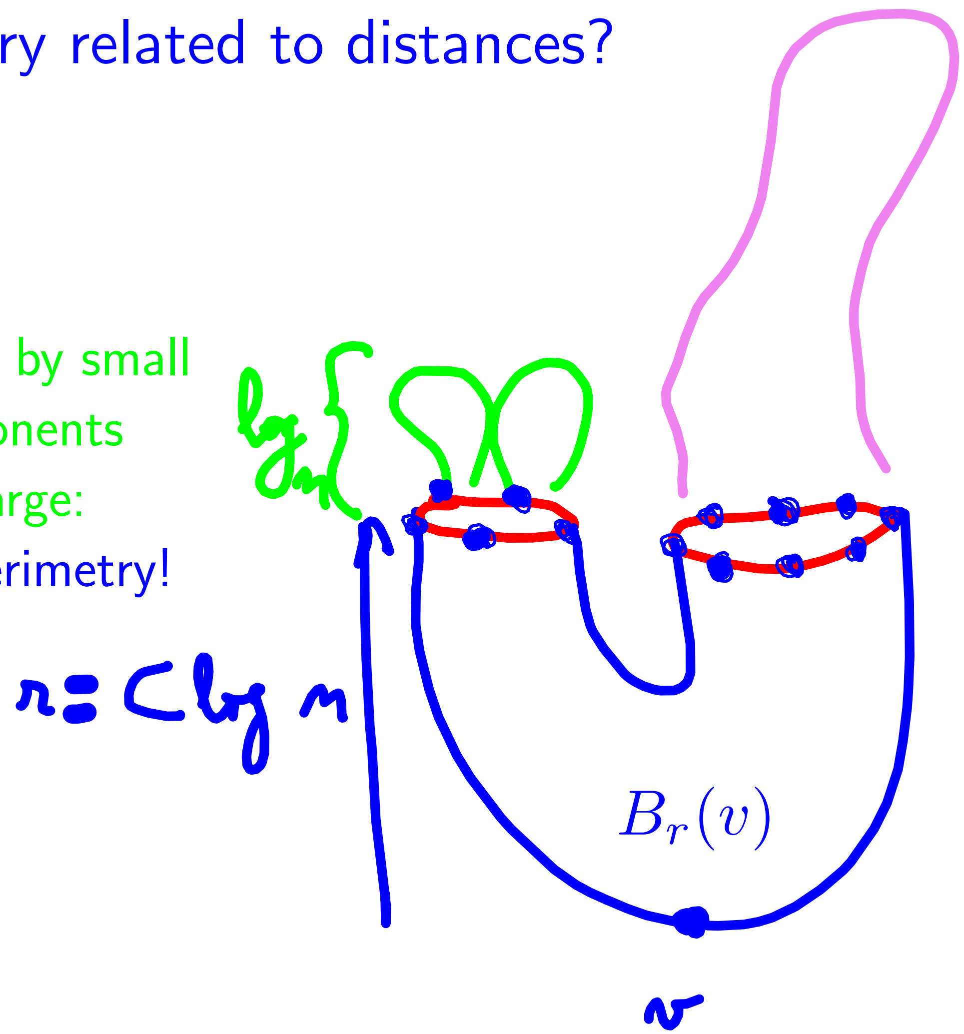


$$\mathbf{E}[\#\text{paths of length } L \text{ from } x \text{ to } y] \leq (cst)\frac{n\tau(n+L,g)}{n^2\tau(n,g)} \leq (\lambda(\theta)+\epsilon)^L n^{-1} \rightarrow 0 \text{ if } L < \epsilon \log n.$$

Some ideas

- Why is isoperimetry related to distances?

pieces separated by small boundary components cannot be too large: because of isoperimetry!



when the boundary of the ball $B_r(v)$ is comparable to the size of the ball, the growth is exponential (because $|B_{r+1}(v)| \geq \frac{1}{3}|\delta B_r(v)| + |B_r(v)|$)

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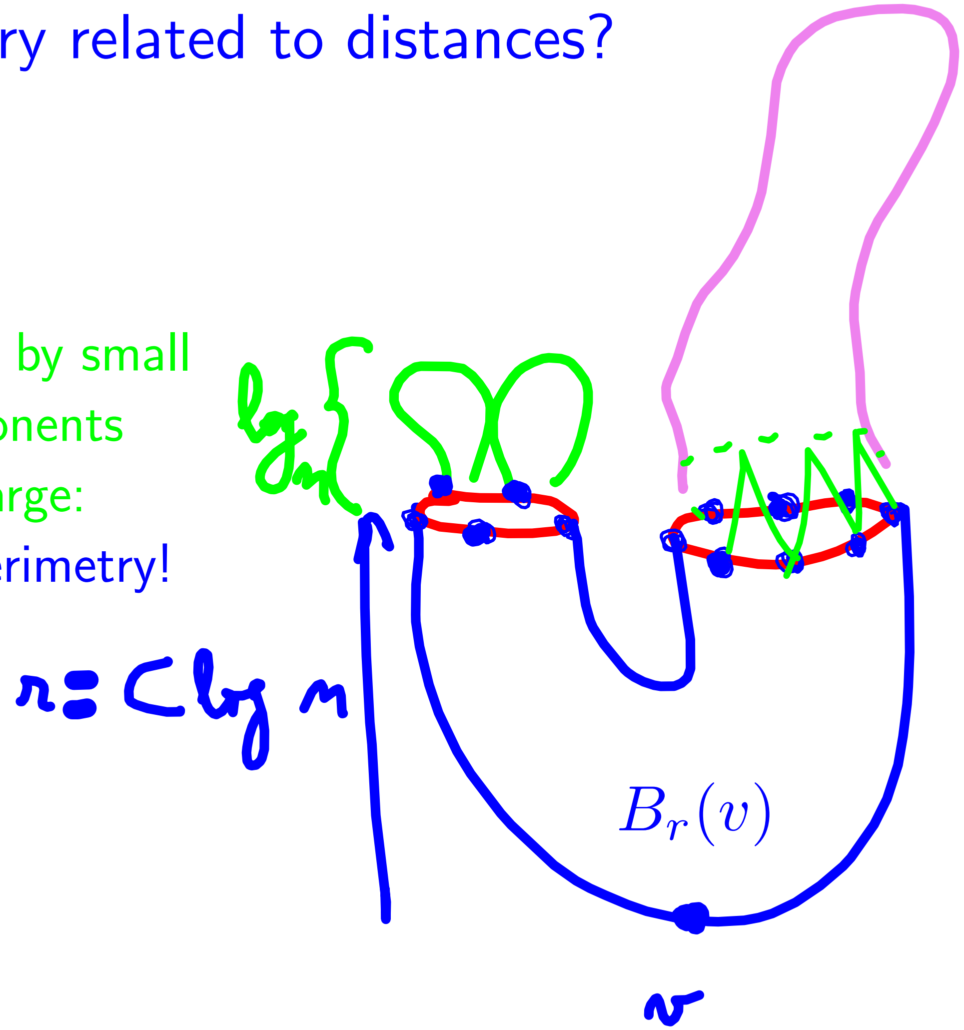

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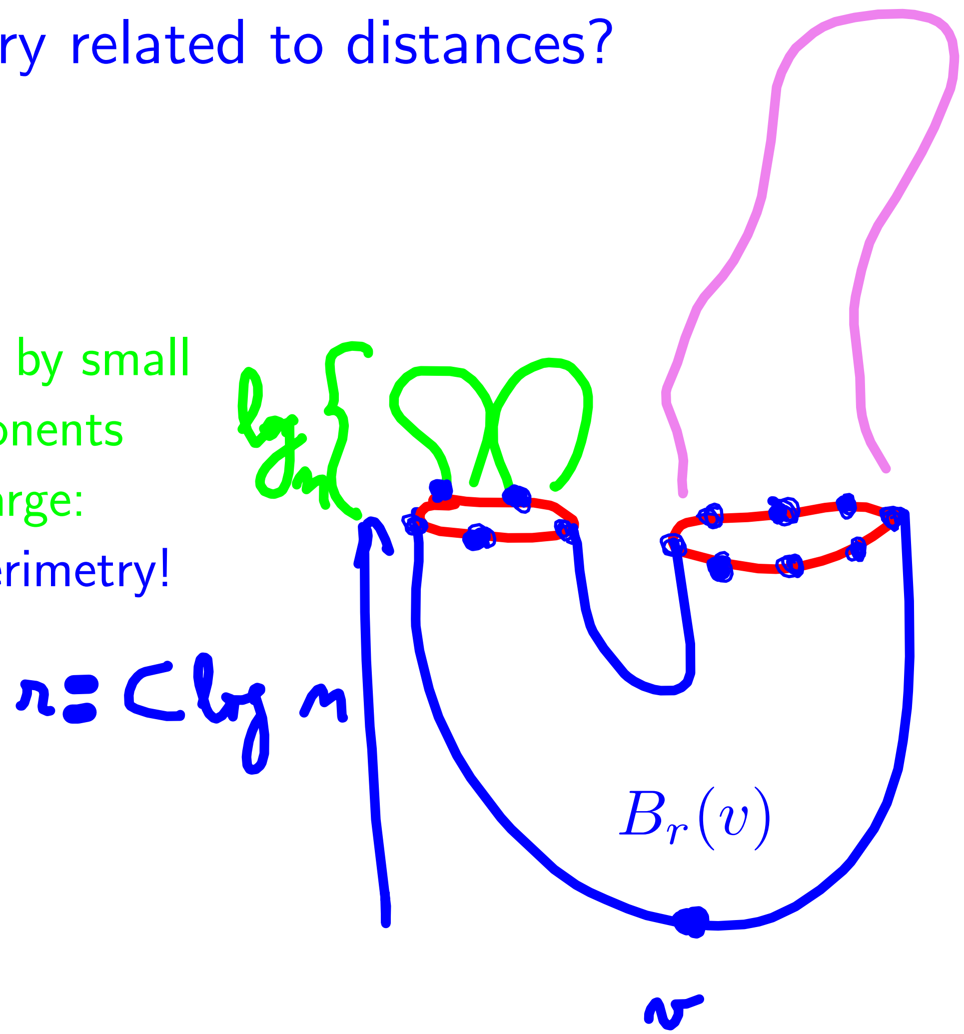

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• There are many things we cannot do. The most basic one (which would trivialize many results in this talk): Can one give an asymptotic equivalent of $\tau(n,g)$ when $g \sim \theta n$ and $n \to \infty$??? This question is frustrating because we have an explicit recurrence formula to compute these numbers.

[Goulden-Jackson'09] $\tau(n,g) = \frac{1}{3n+2} f_g^n$ with $f_g^n = \dfrac{4(3n+2)}{n+1}\Big(n(3n-2)f_{g-1}^{n-2} + \sum_{\substack{i+j=n-2\\h+k=g}} f_h^i f_k^j\Big).$

Conclusion

• This was a very biased introduction to random maps. All of them are. My only hope is that I was biased differently compared to other speakers.

• Counting results play an important role in all the proofs I didn't give. These include:

- bijections to count maps on surfaces
- Eynard-Orantin's topological recursion
- integrable hierarchies (KP, 2-Toda) for map generating functions and the ways to prove them (Fermionic Fock space, random matrices, representation theory of the symmetric group).

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• Why would a "random space" have uniform Voronoï tessellations?

THANK YOU!