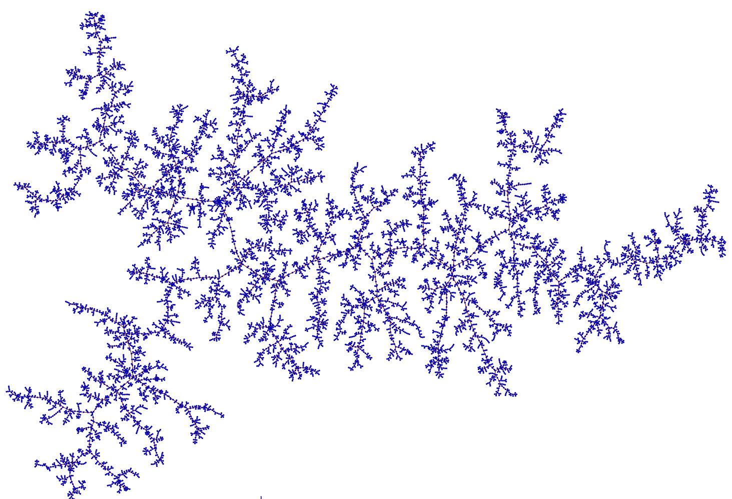
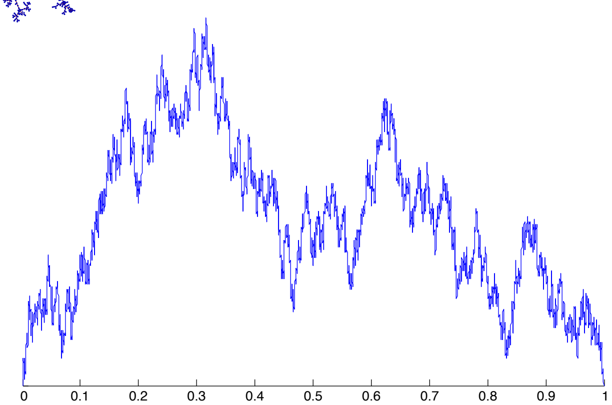
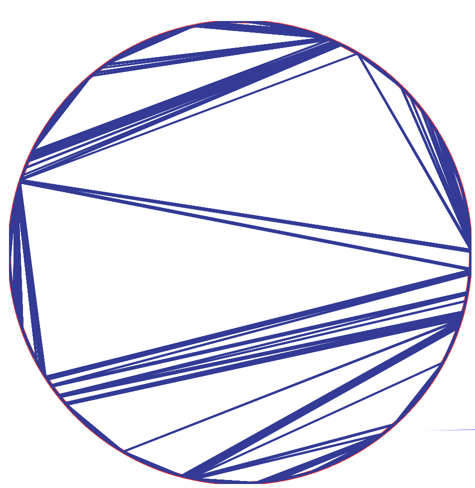
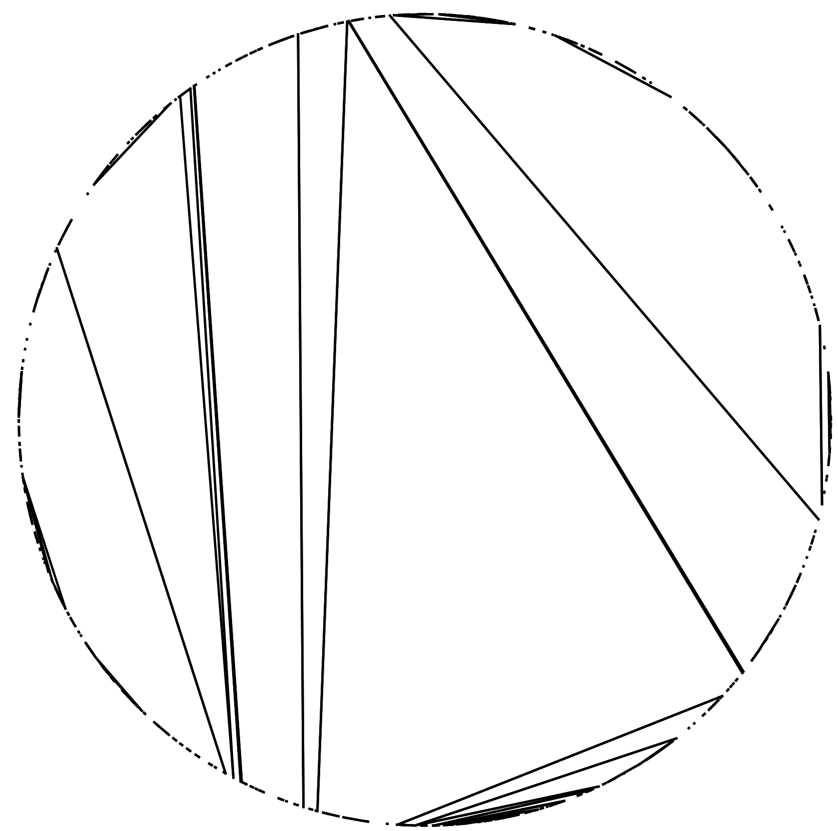
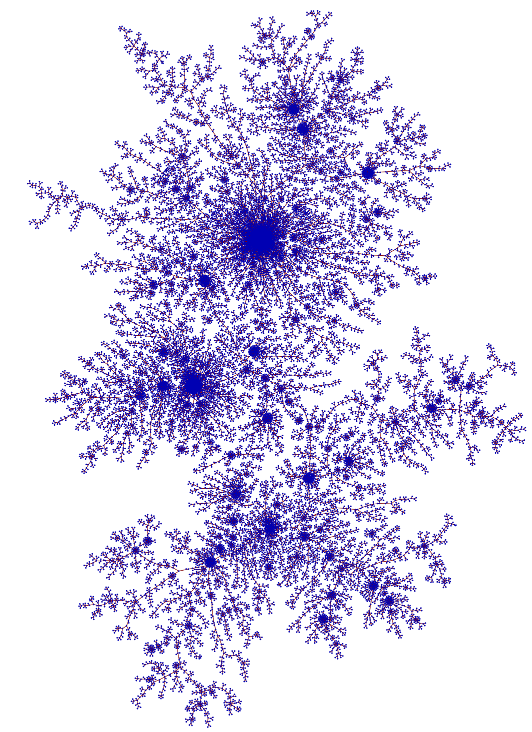
The geometry of random minimal factorizations of a long cycle

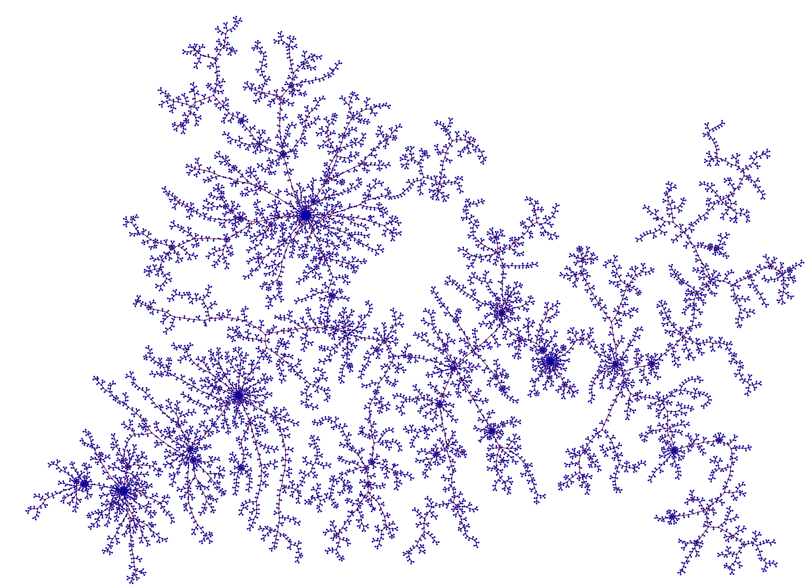












Igor Kortchemski

(with Valentin Féray)

CNRS & École polytechnique

Outline

I. Random triangulations of polygons

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II. Minimal factorizations

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III. Main steps of the proof

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Triangulations

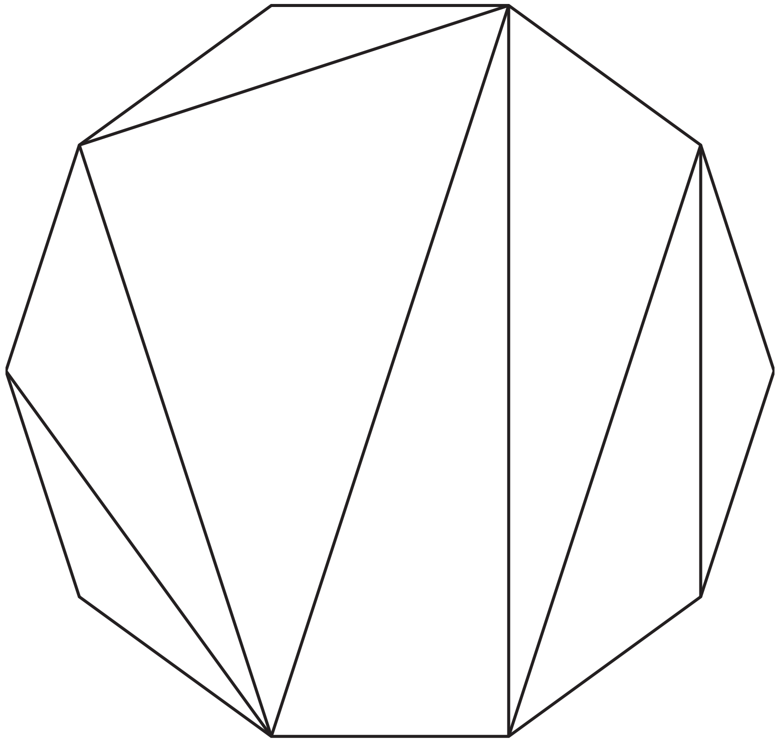


Figure: A triangulation of a polygon with 10 vertices.

Triangulations

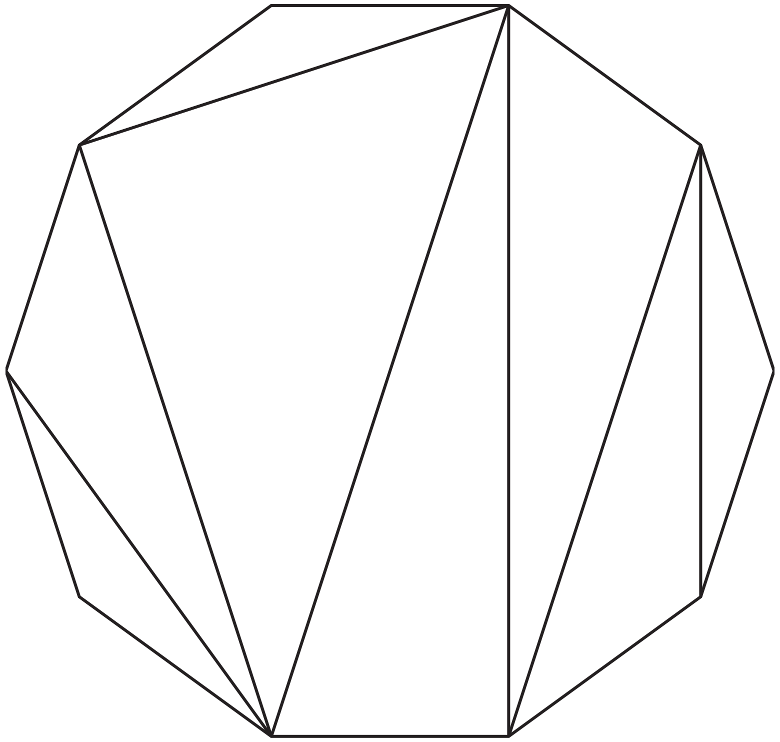
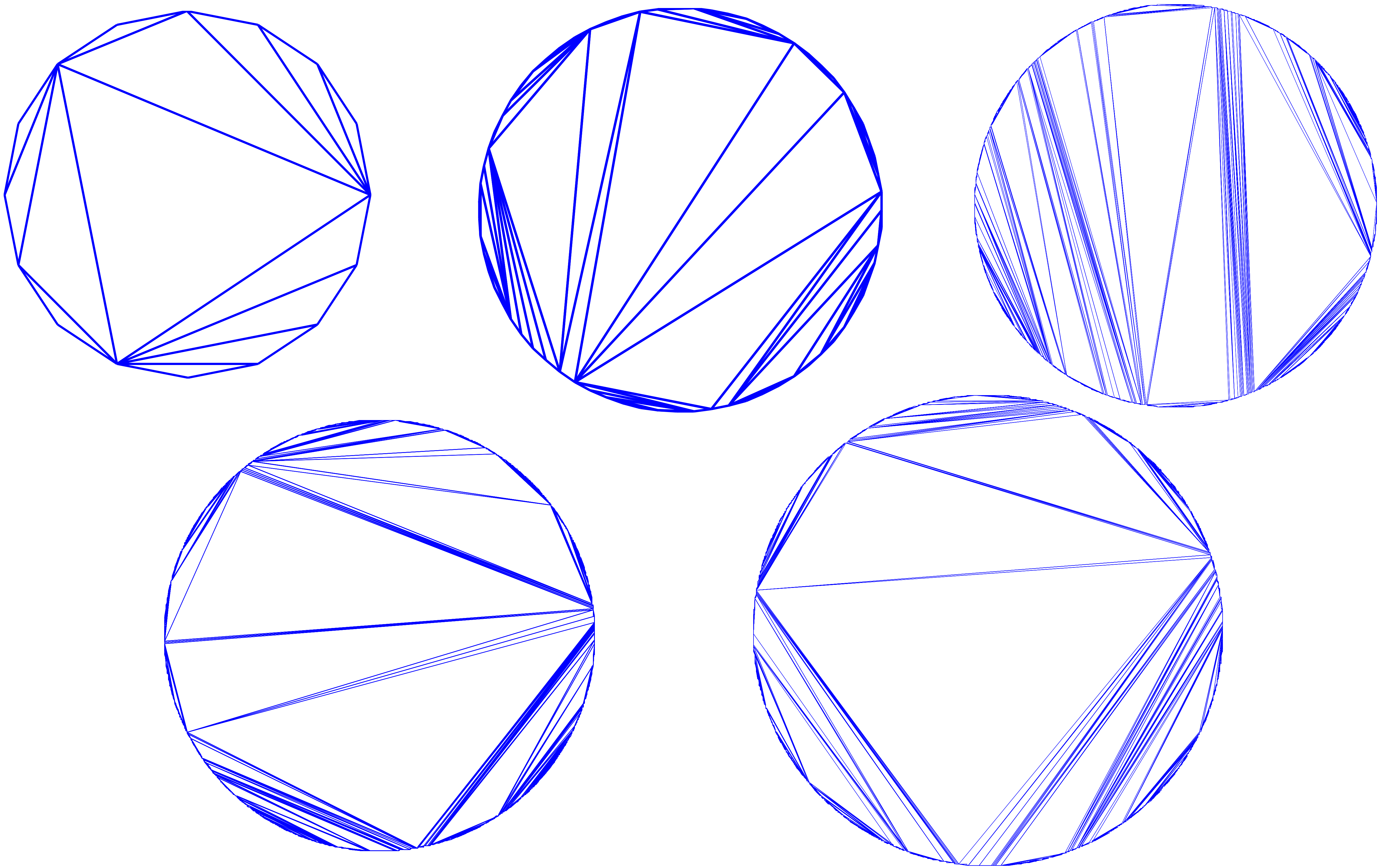


Figure: A triangulation of a polygon with 10 vertices.

Question: What does a large typical triangulation of the polygon whose vertices are $e^{\frac{2i\pi j}{n}}$ $(j = 0, 1, \ldots, n-1)$ look like?

Typical triangulations



What space for triangulations?

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$\rightsquigarrow$ Aldous: compact subsets of the unit disk.

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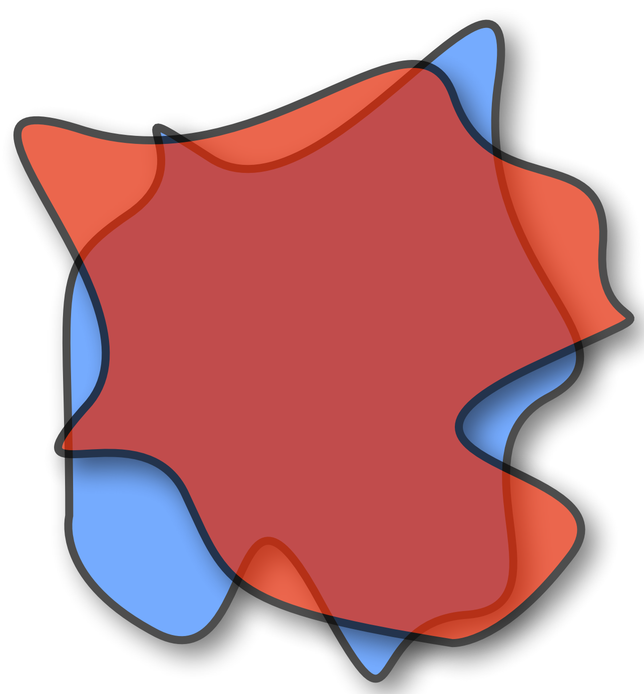

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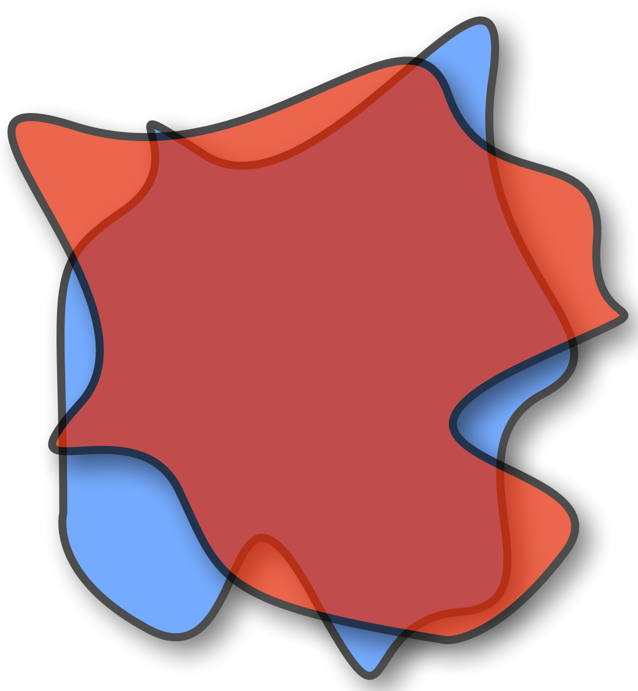

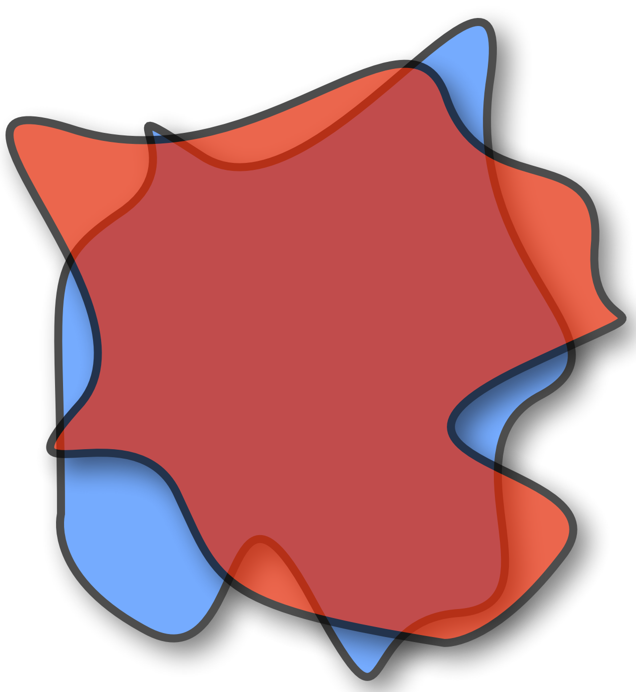

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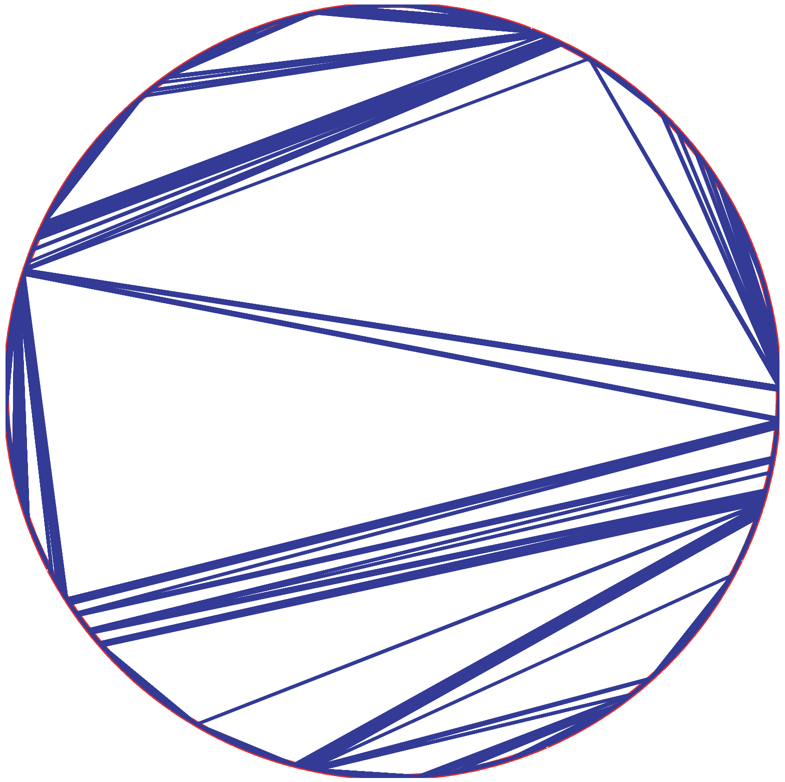

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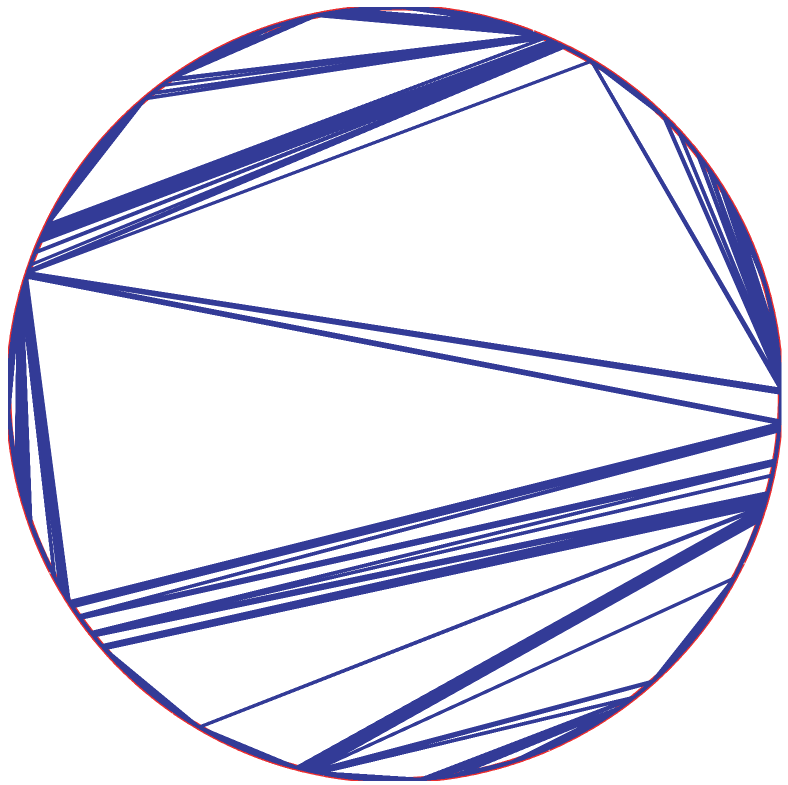

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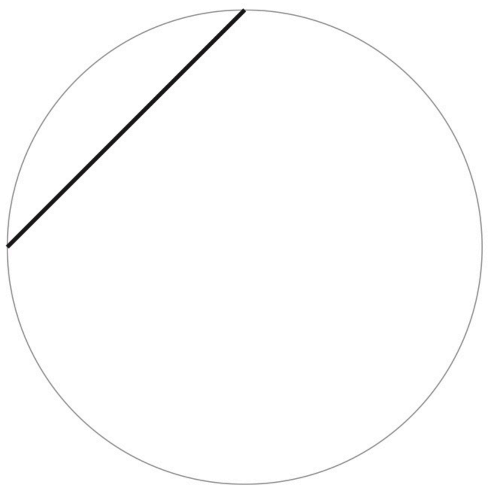

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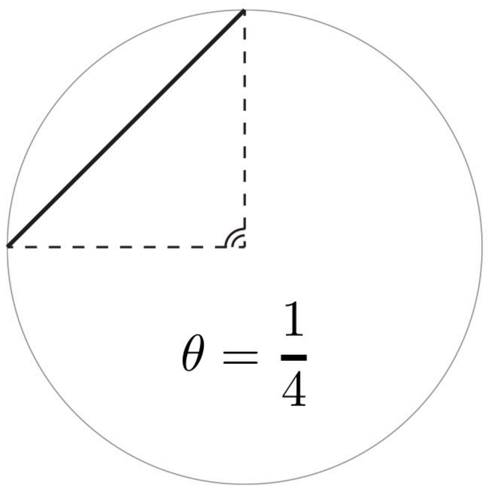

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It is the probability measure with density:

$$\frac{1}{\pi}\frac{3\theta - 1}{\theta^2(1-\theta)^2\sqrt{1-2\theta}}\mathbf{1}_{\frac{1}{3}\leqslant\theta\leqslant\frac{1}{2}}d\theta.$$

(Aldous, Devroye–Flajolet–Hurtado–Noy–Steiger)

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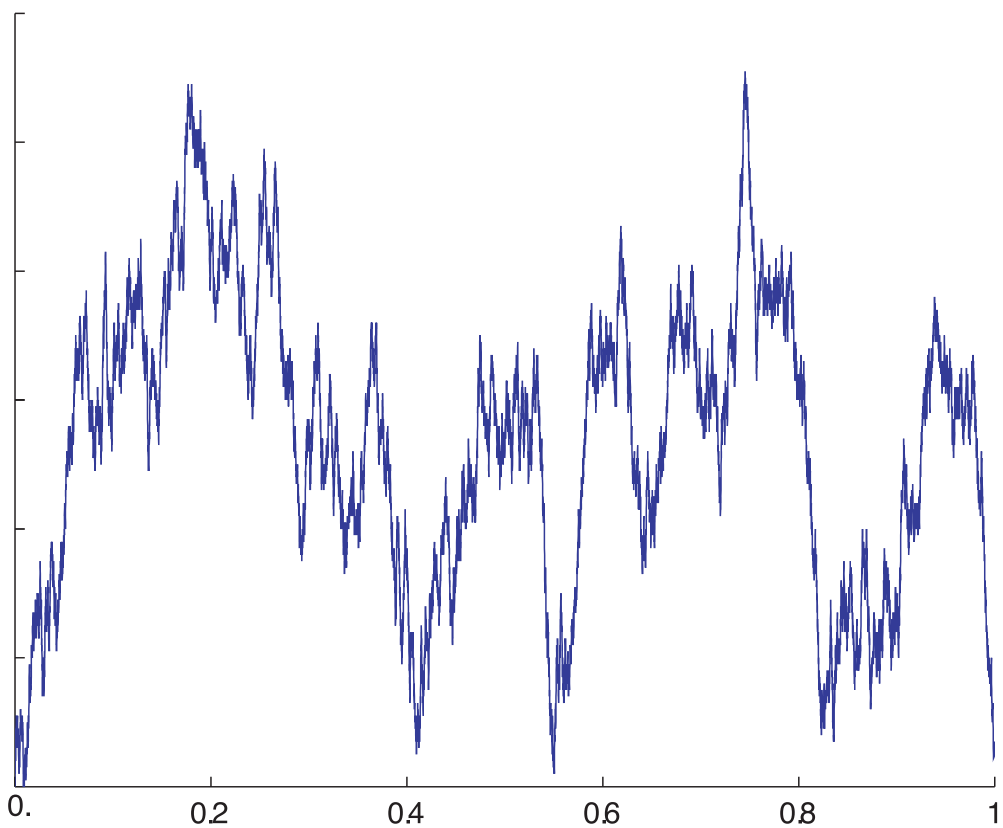

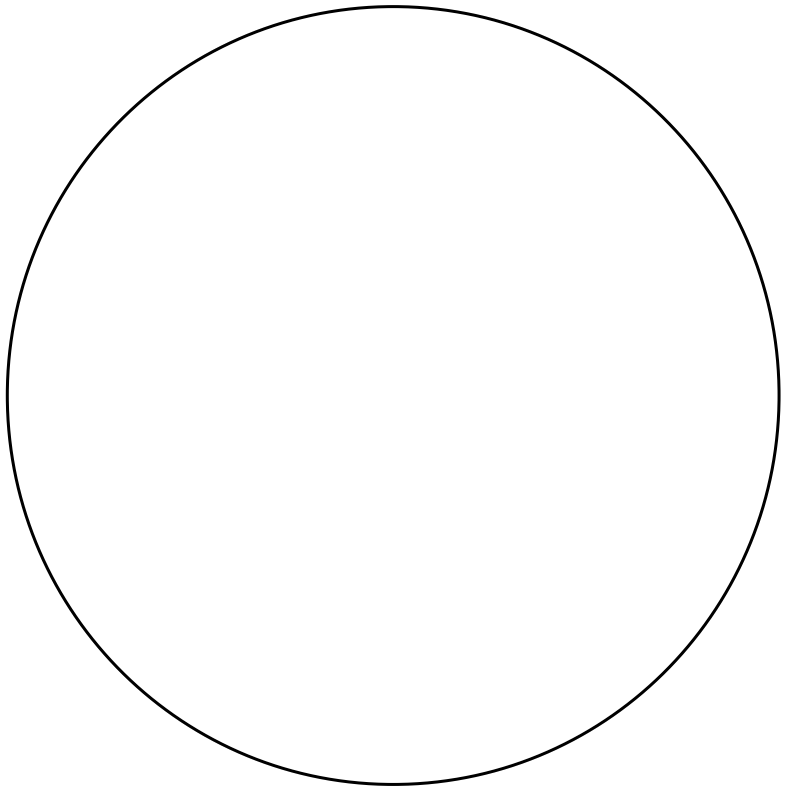

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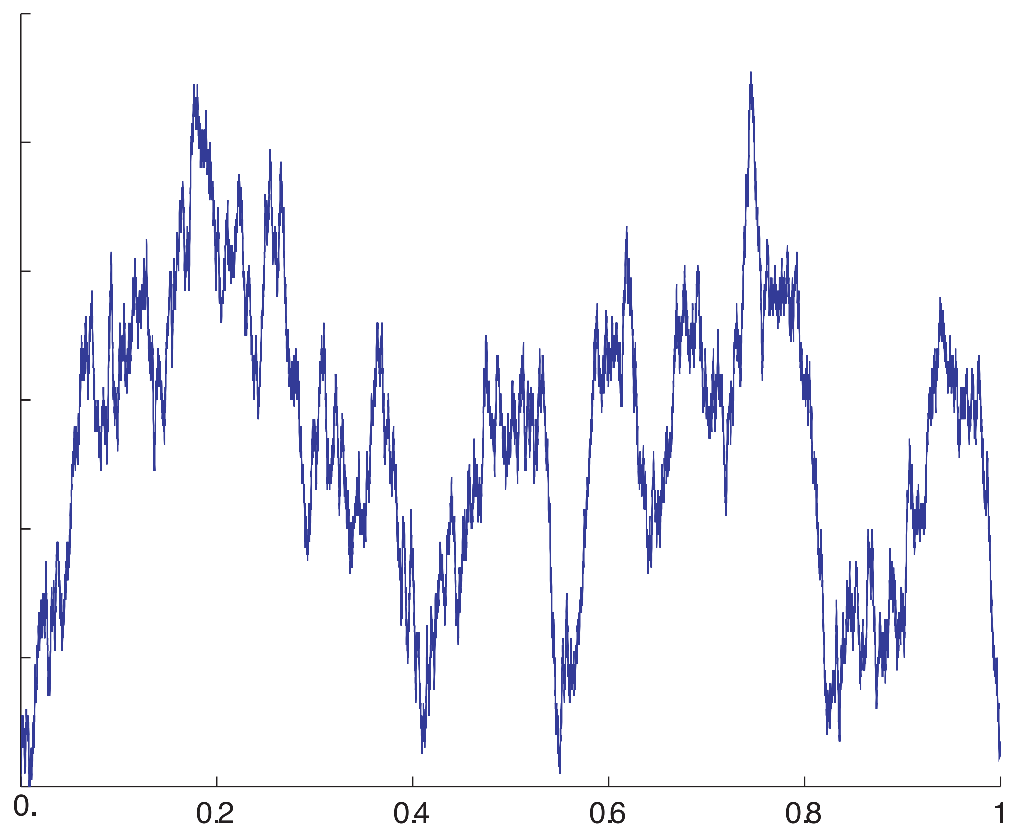

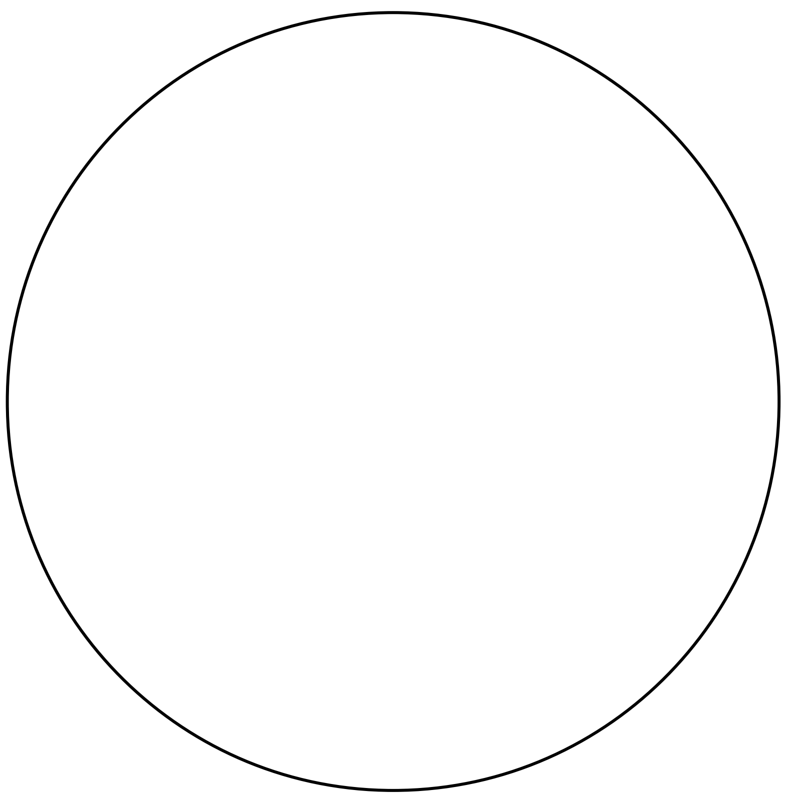

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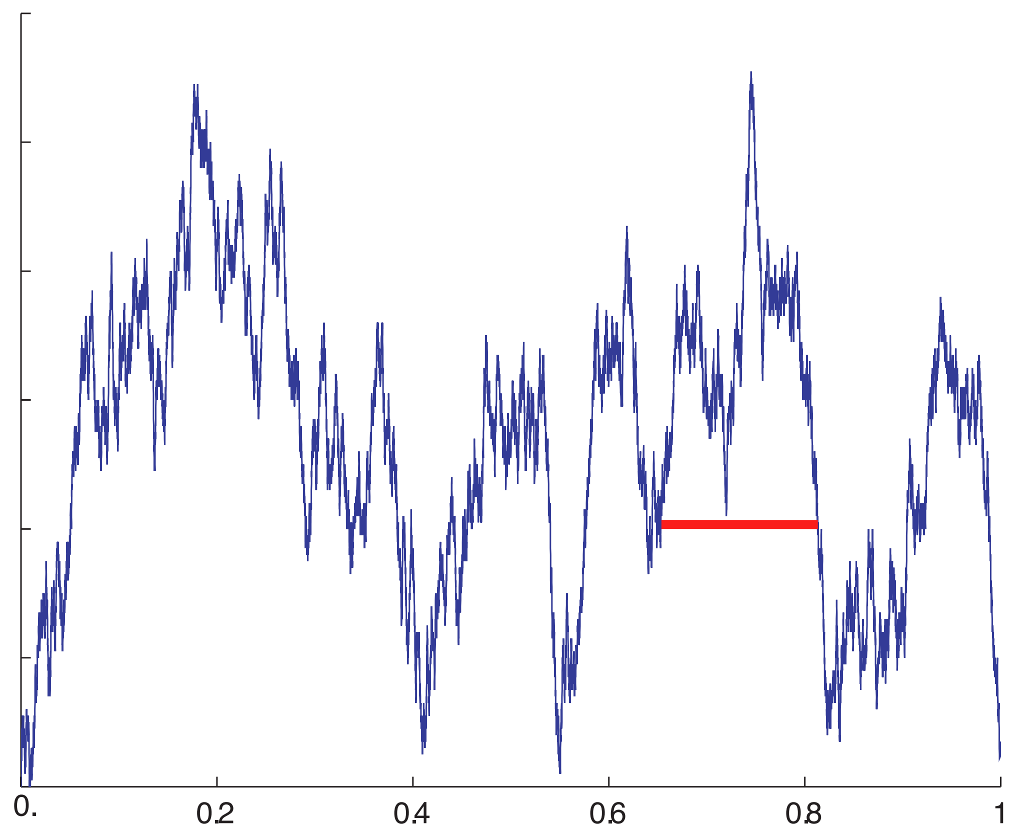

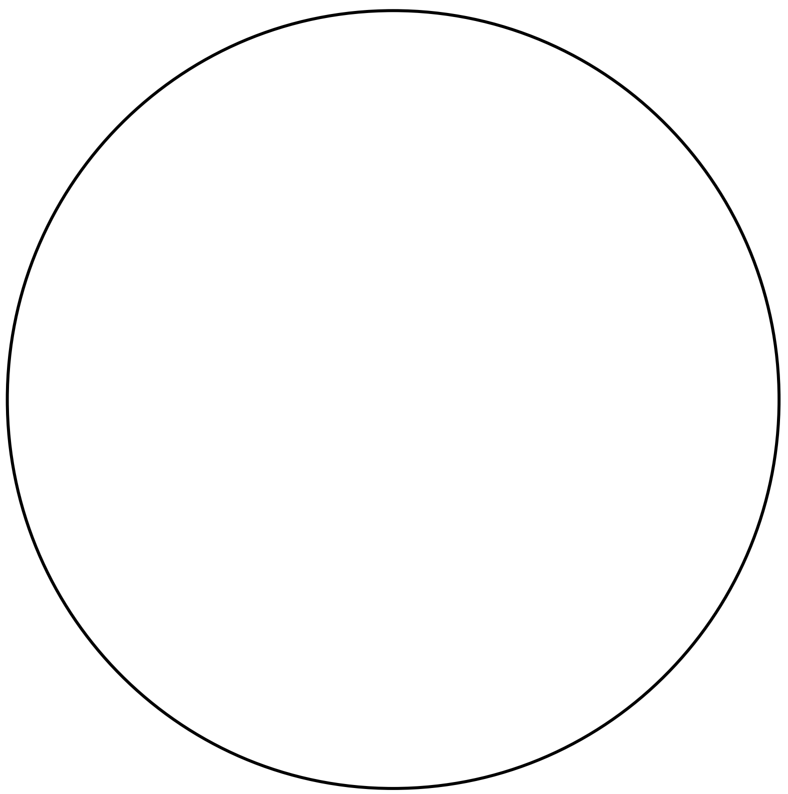

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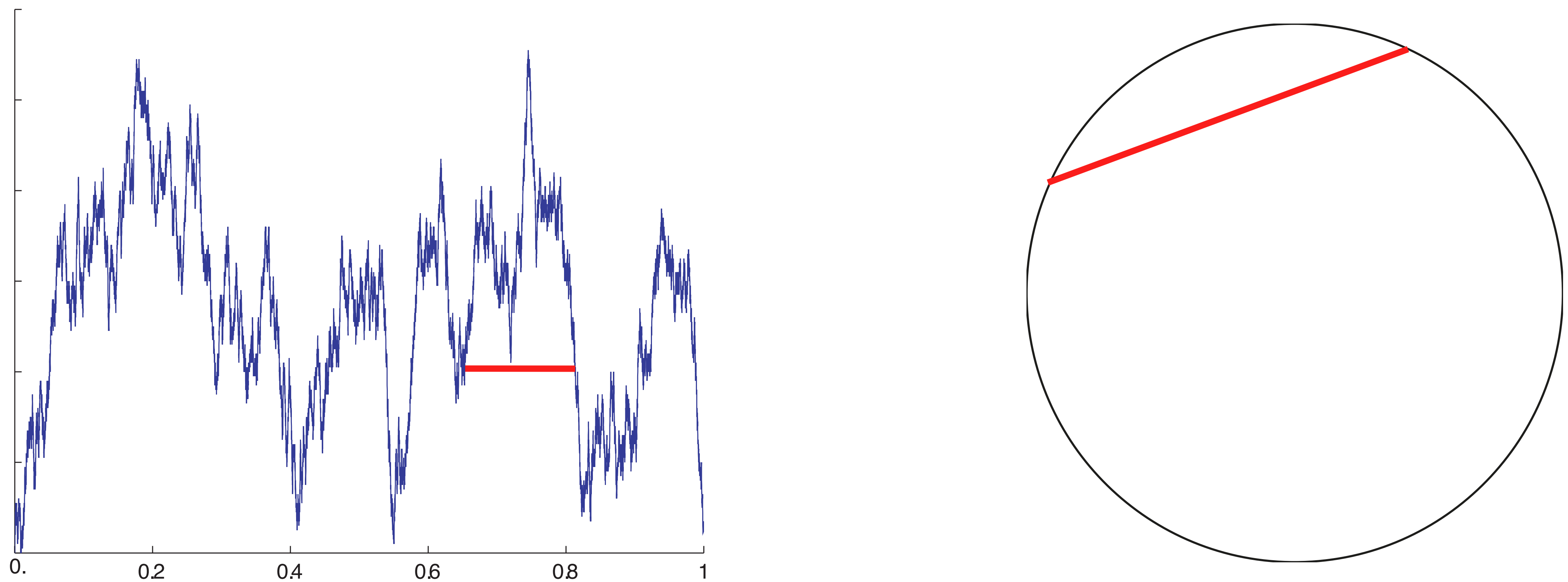

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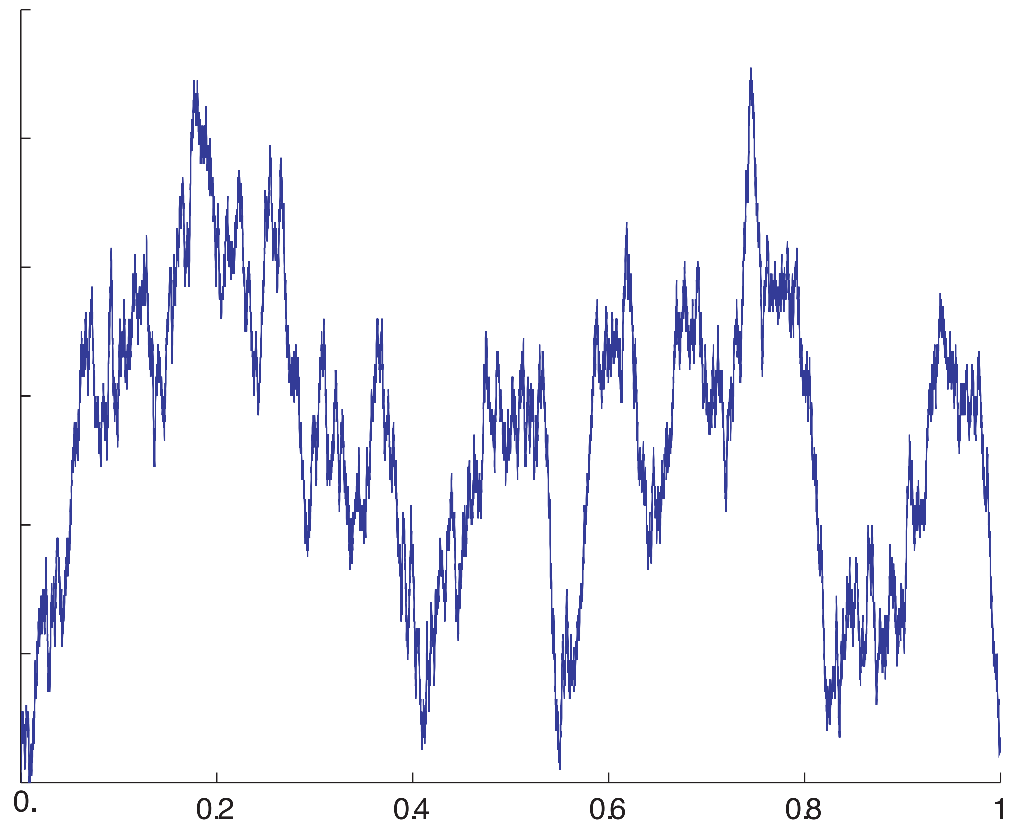

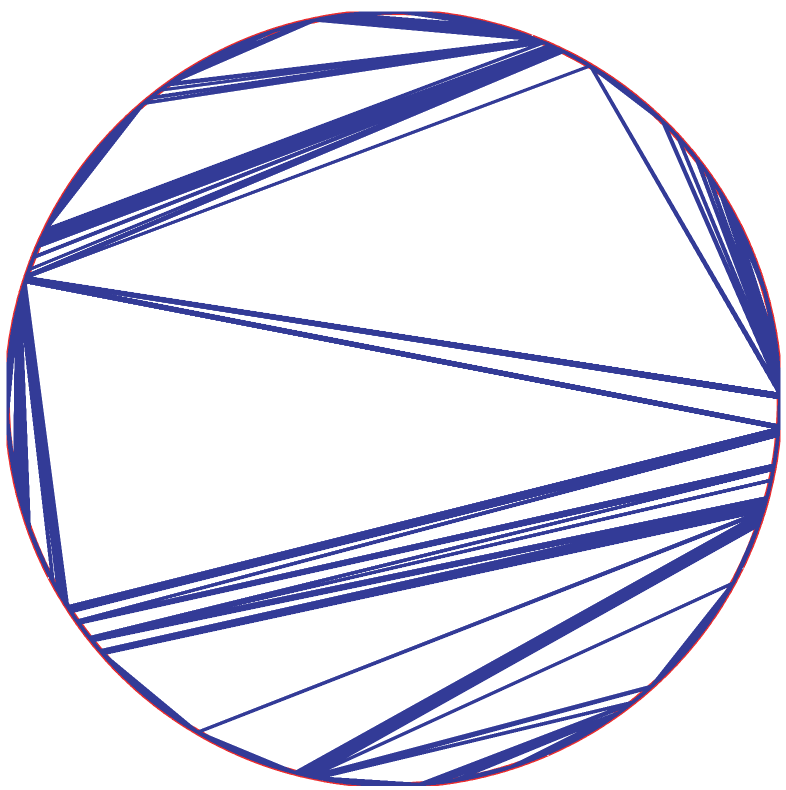

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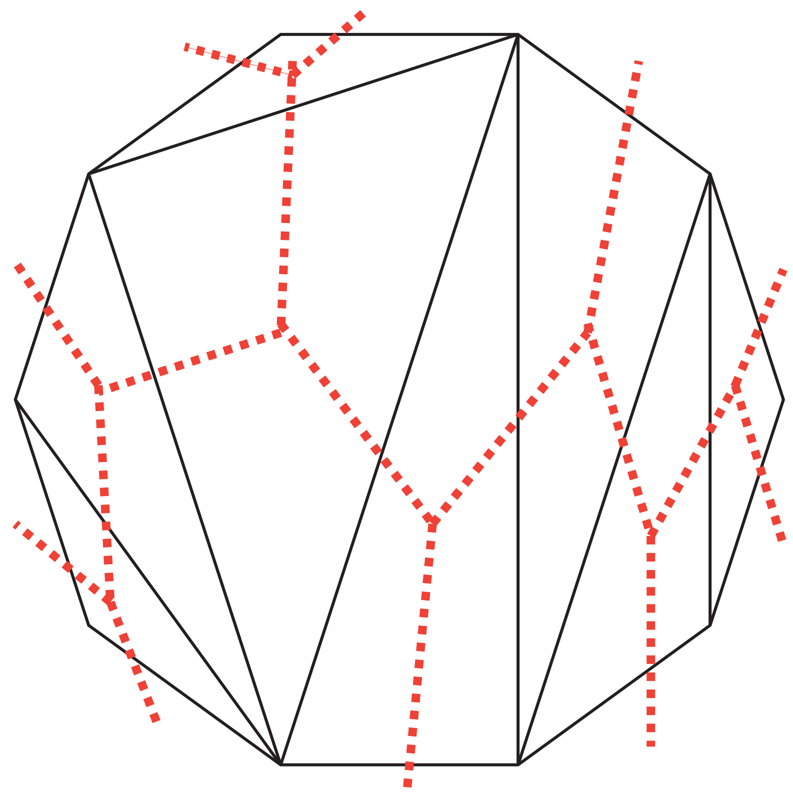
Why?

Idea of the proof



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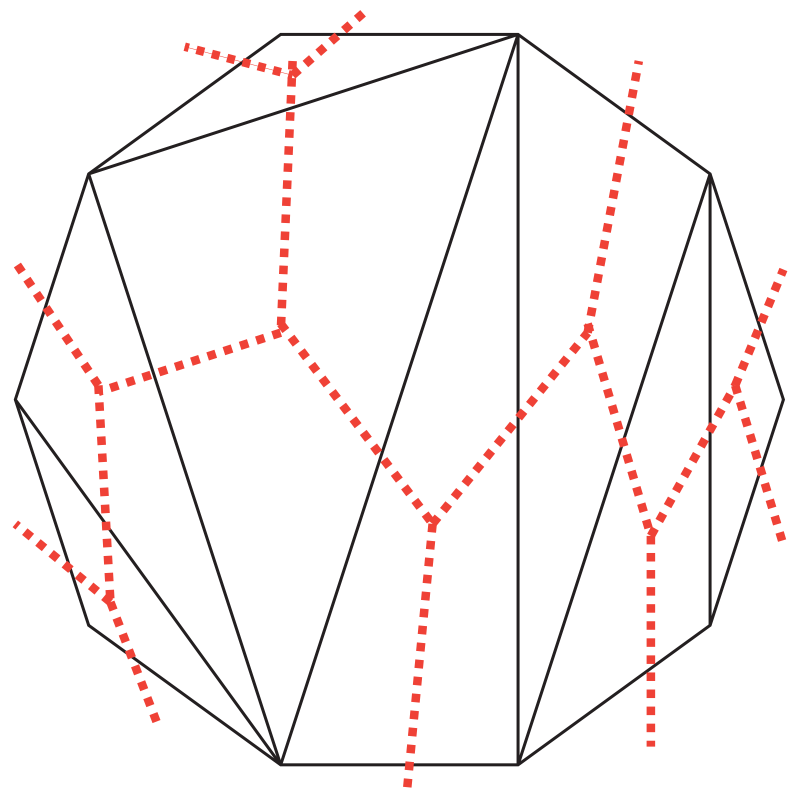
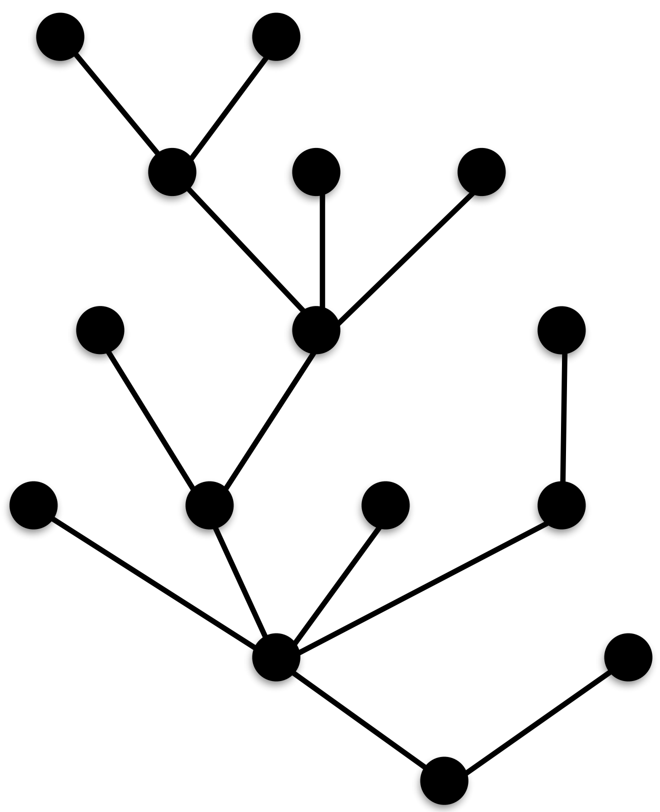


⇝ "big" chords correspond to "big" subtrees.

The lexicographical order



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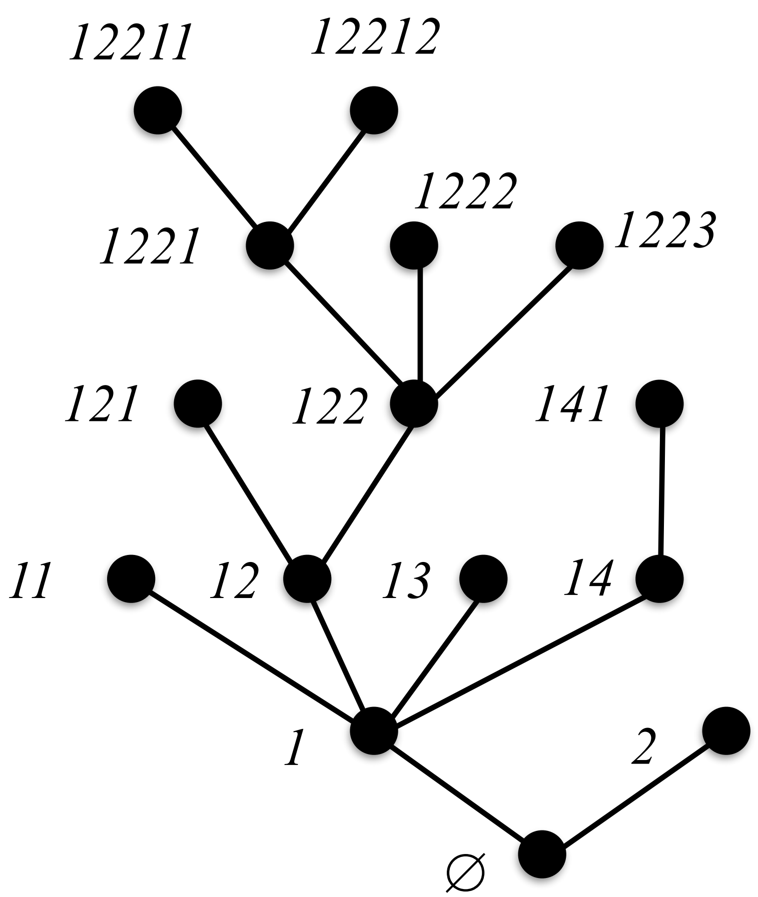

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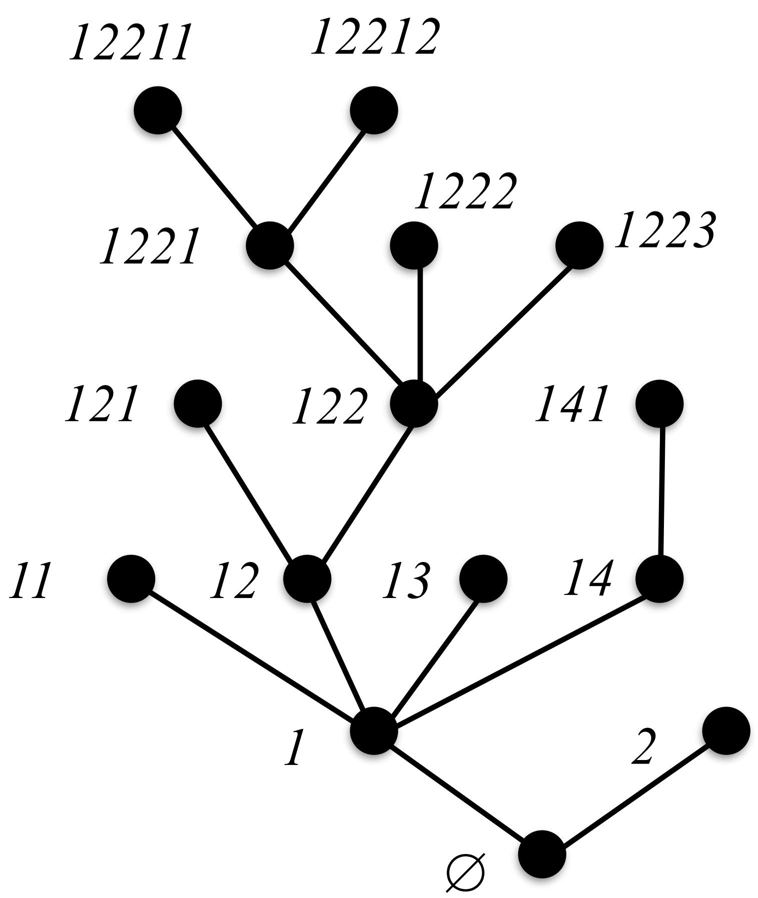

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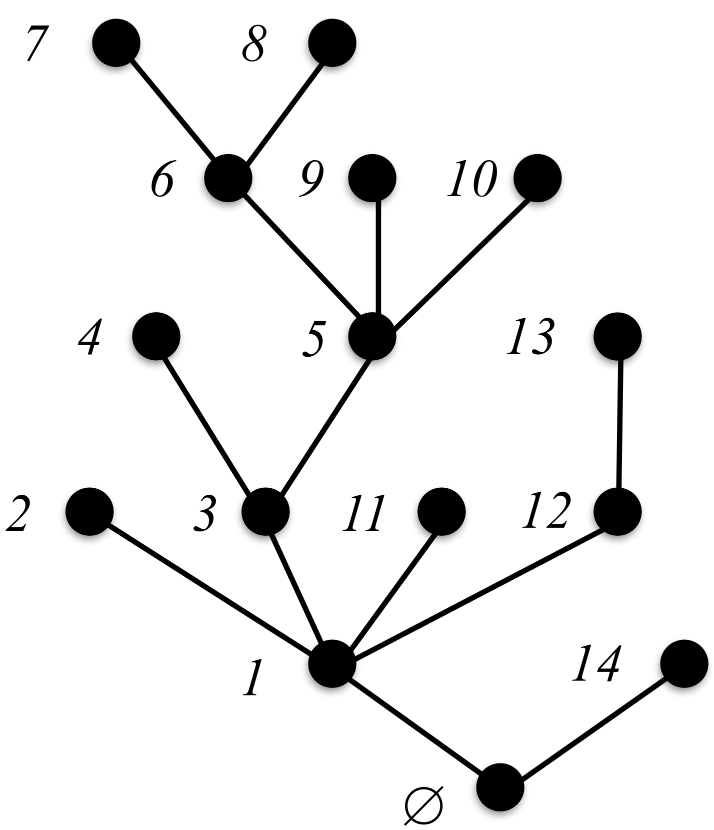

The Lukasiewicz path

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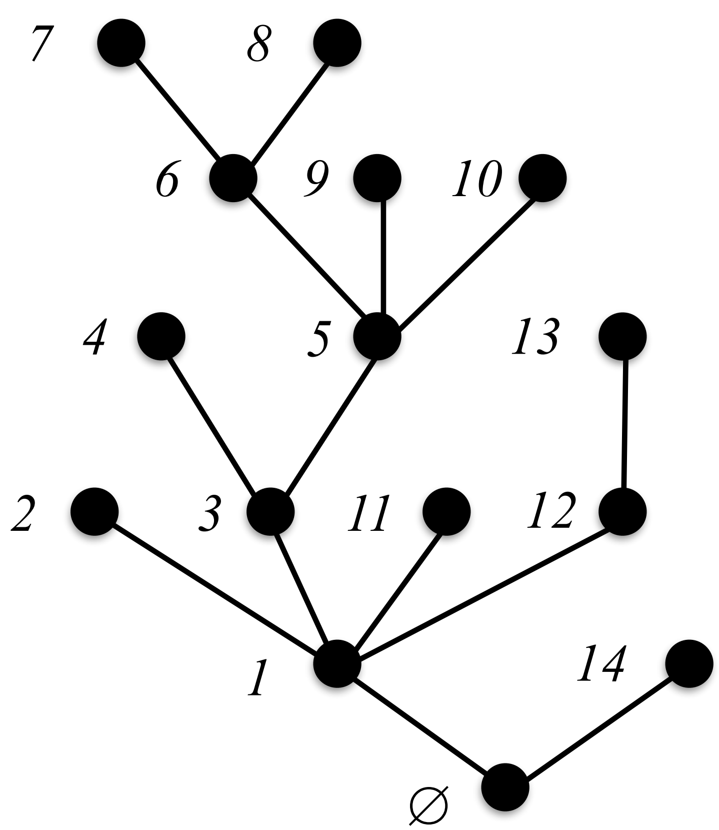

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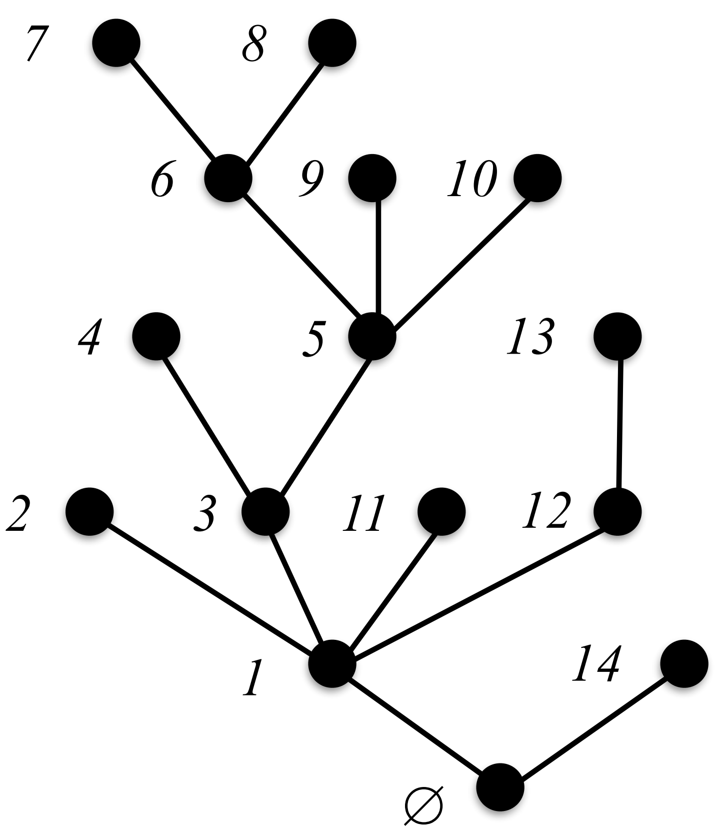

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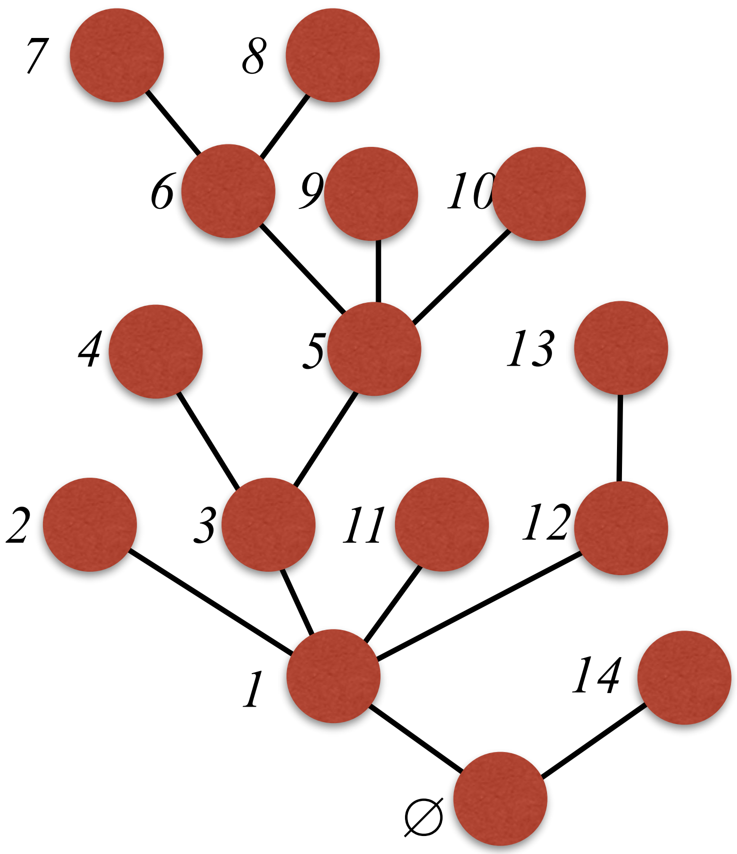

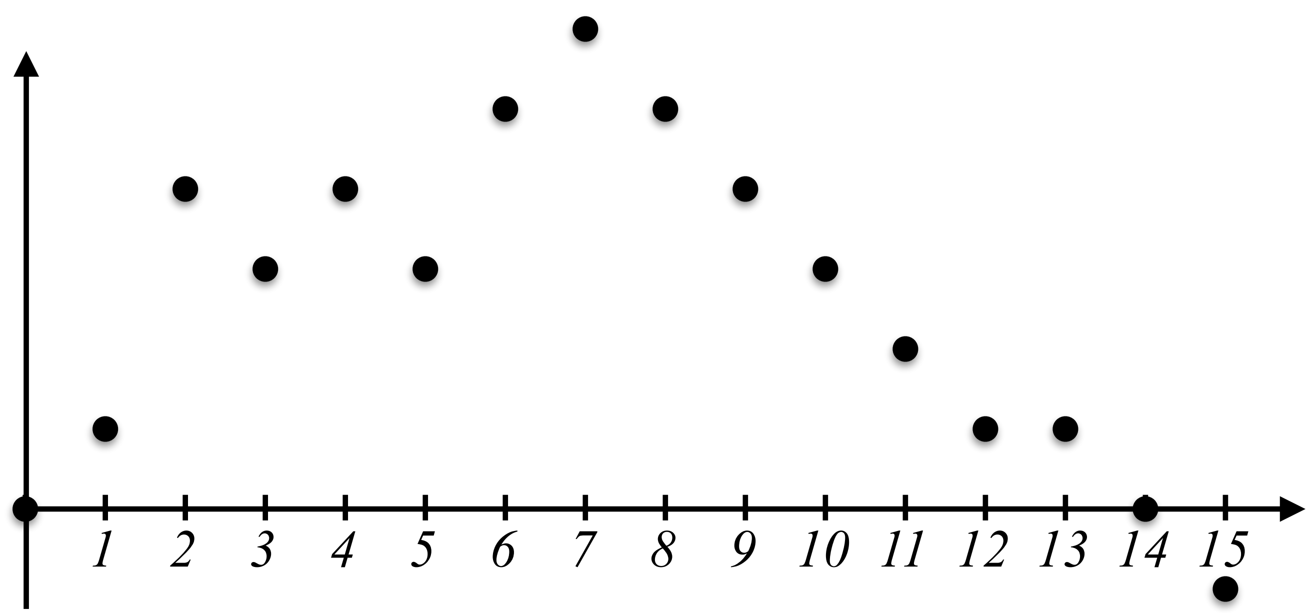

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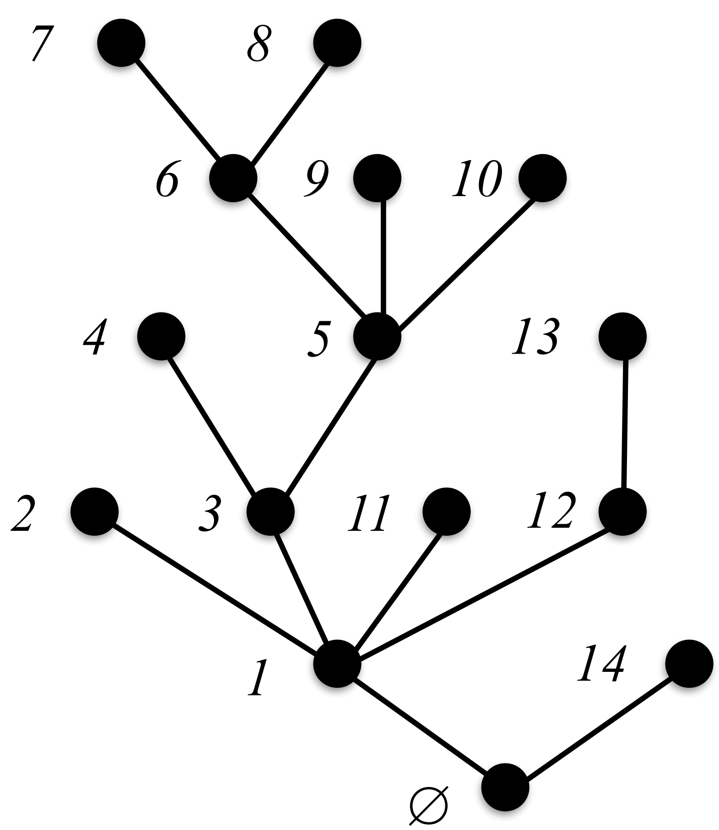

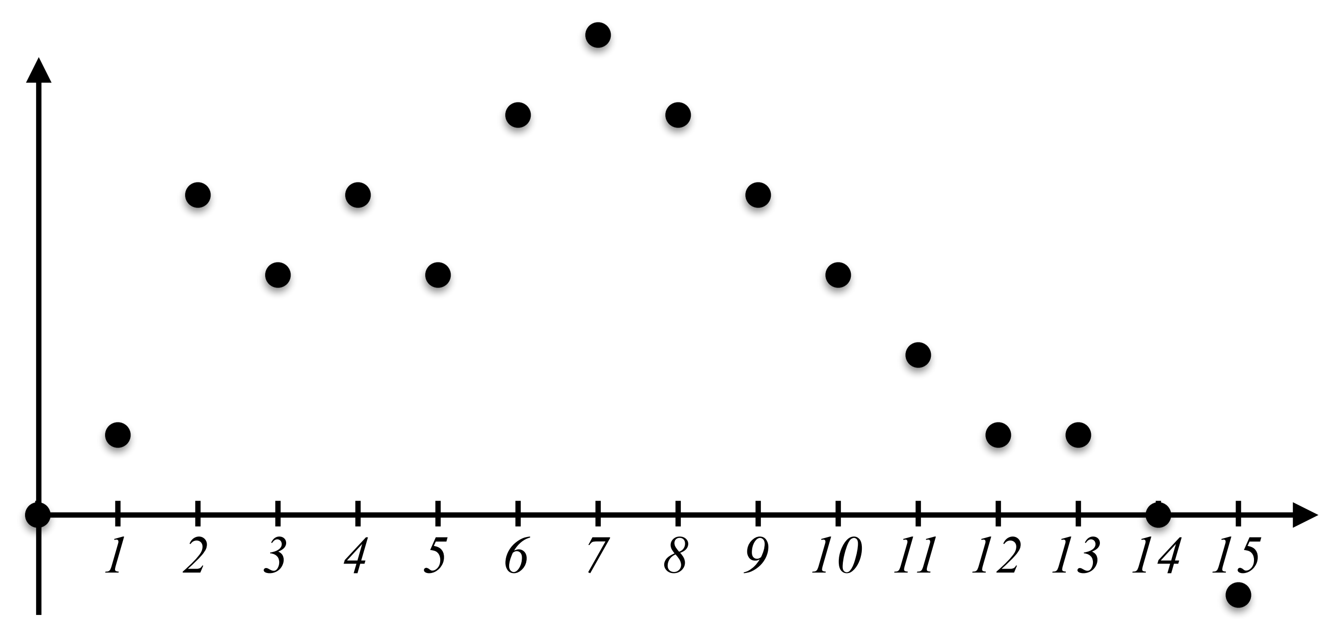

$\rightsquigarrow$ Fact: $\mathcal{W}_i(\tau) \geqslant 0$ for every $0 \leqslant i \leqslant |\tau| - 1$, and $\mathcal{W}_{|\tau|}(\tau) = -1$.

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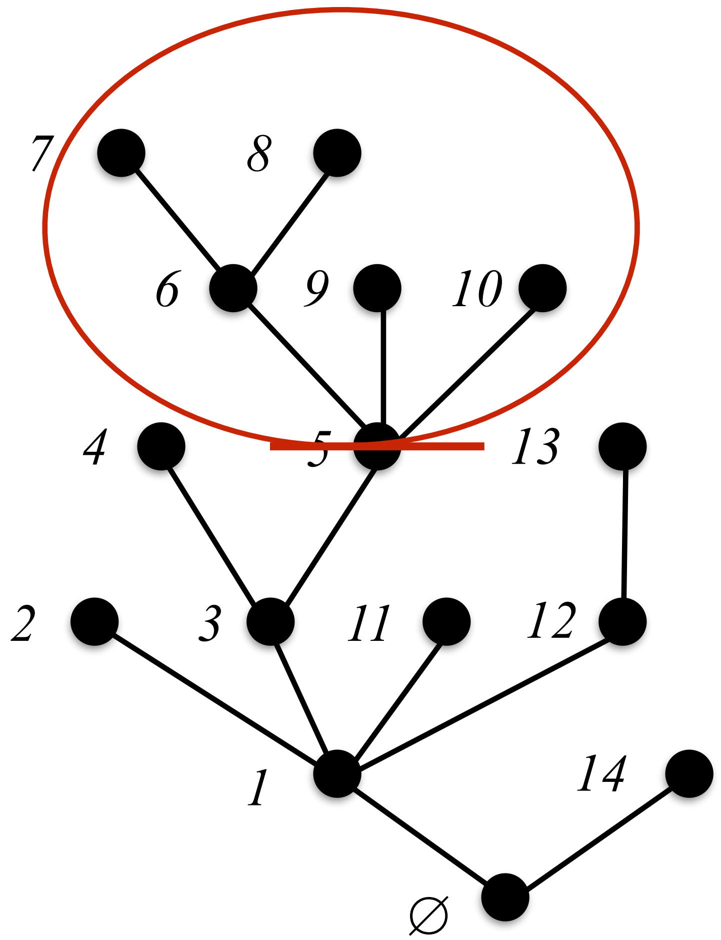

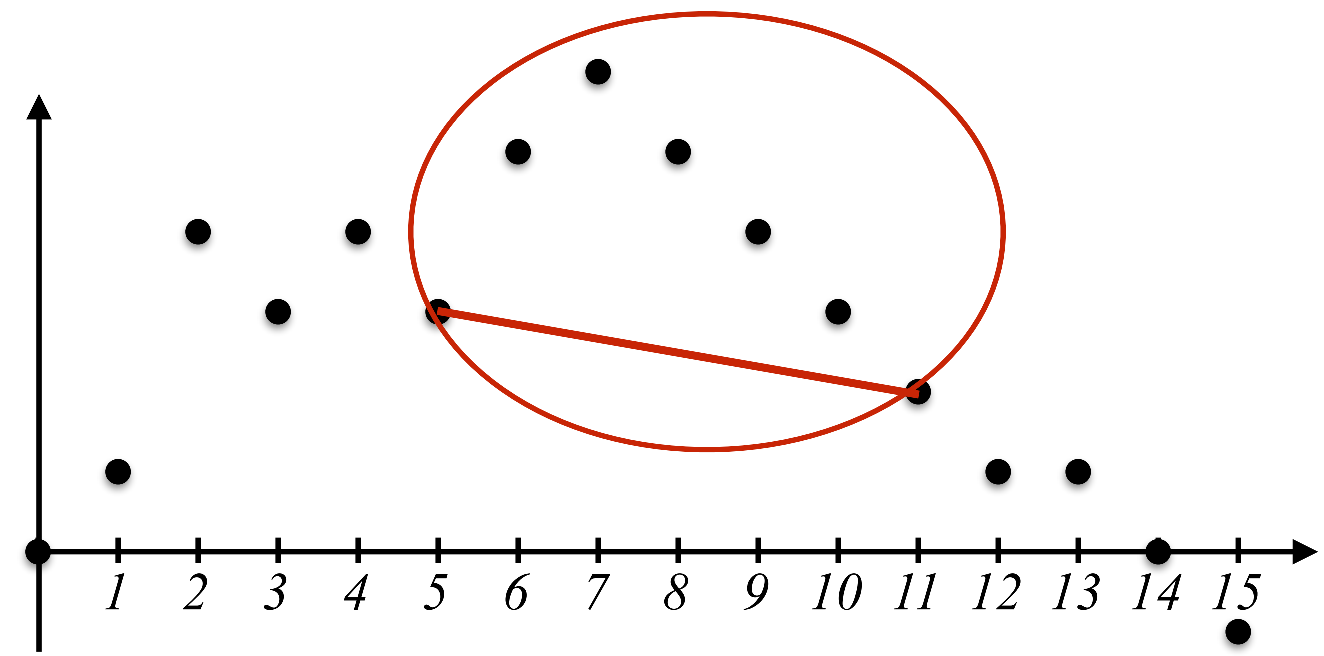

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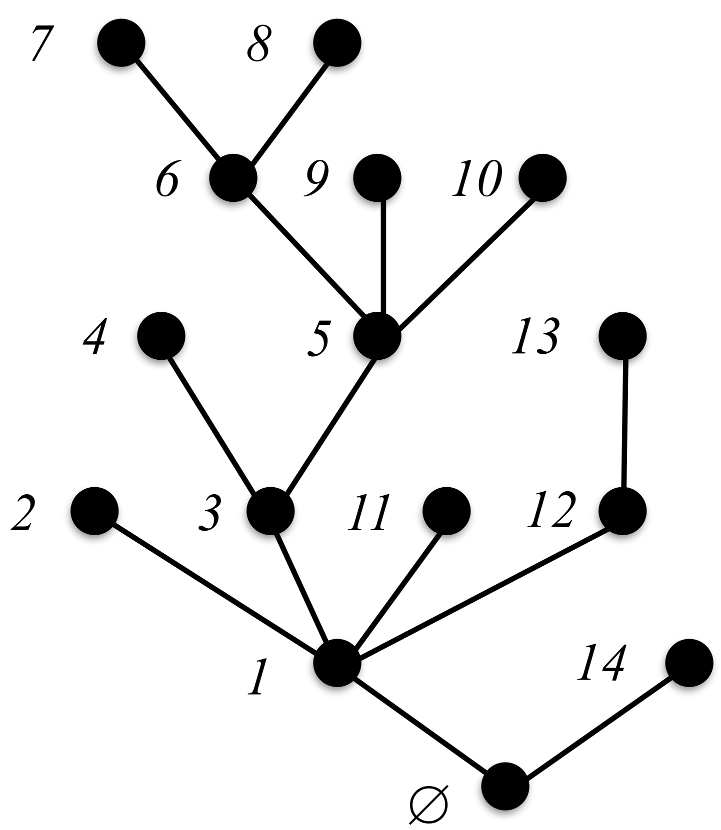

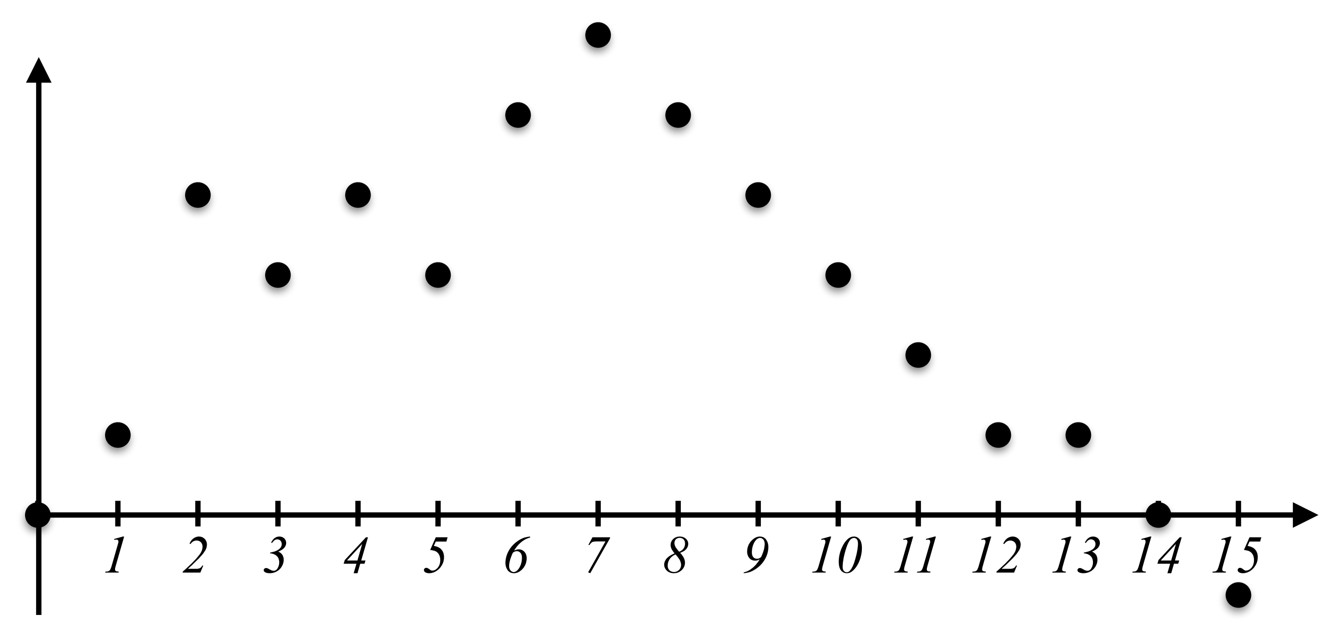

$\rightsquigarrow$ The scaling limit of the Lukasiewicz path of the dual of a uniform triangulation is the Brownian excursion.

I. Random triangulations of polygons

II. Minimal factorizations

III. Main steps of the proof

Minimal factorizations

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Minimal factorizations

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Consider the set

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$$\mathfrak{M}_n = \{(\tau_1, \ldots, \tau_{n-1}) \text{ transpositions } : \tau_1 \tau_2 \cdots \tau_{n-1} = (1, 2, \ldots, n)\}$$

of minimal factorizations (of the n-cycle into transpositions).

$\rightsquigarrow$ Question: $\#\mathfrak{M}_n = n^{n-2}$ (Dénes, 1959)

For example (multiply from left to right):

$$(1, 2, 3) = (1, 3)(2, 3) = (2, 3)(1, 2) = (1, 2)(1, 3),$$

$\#\mathfrak{M}_3 = 3.$

$\rightsquigarrow$ Question:

Minimal factorizations

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$\leadsto$ Question: for n large, what does a typical minimal factorization look like?

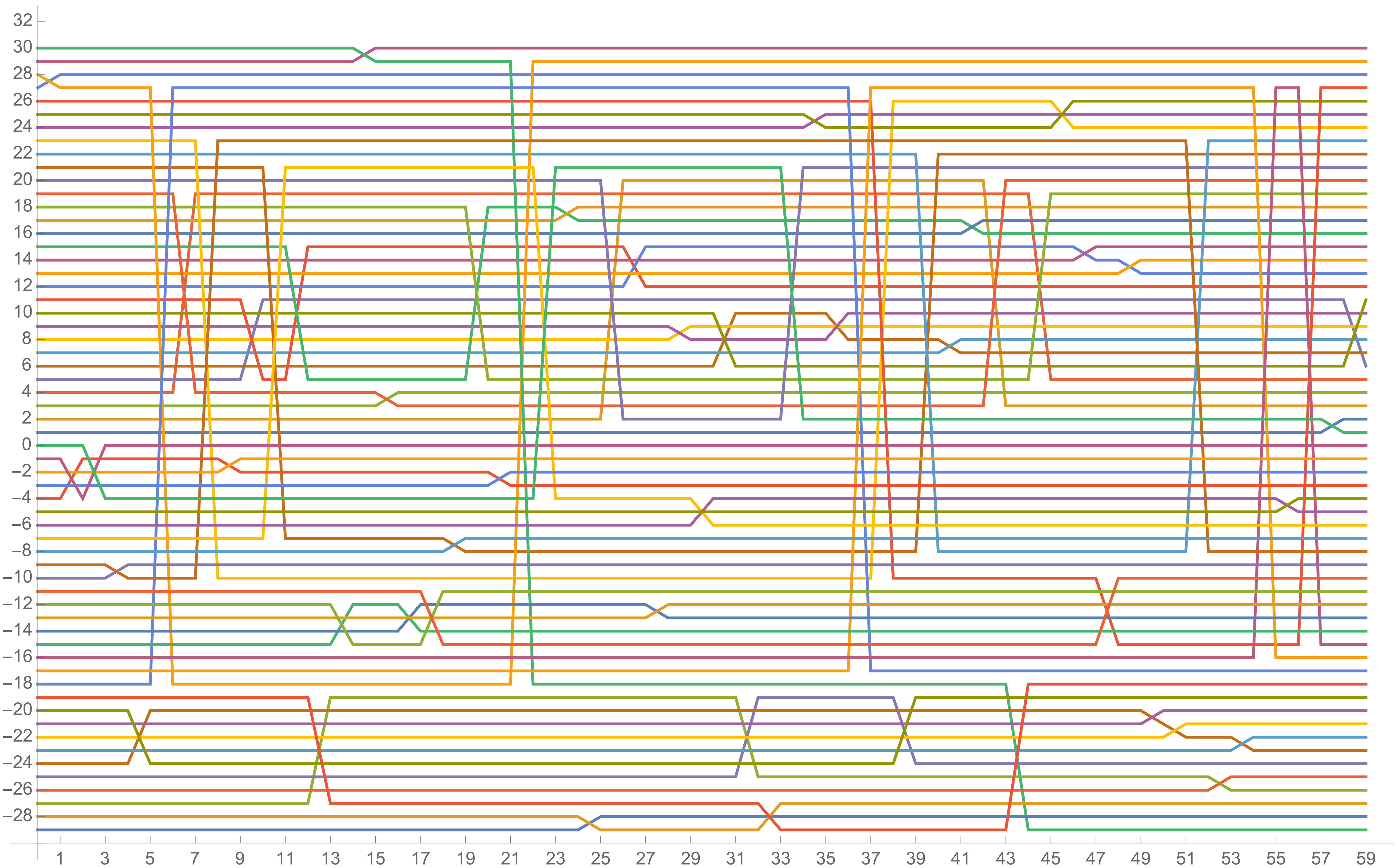

Figure: Graphical representation of the trajectories of a random minimal factorization of the 60-cycle.

Minimal factorizations: motivations

Motivations:

- combinatorics of minimal factorizations (Moszkowski, Goulden & Pepper, Goulden & Yong, Stanley, Biane);

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- rich probabilistic structure (additive coalescent, Aldous-Pitman fragmentation of the CRT,Thévenin).

What space for minimal factorizations?

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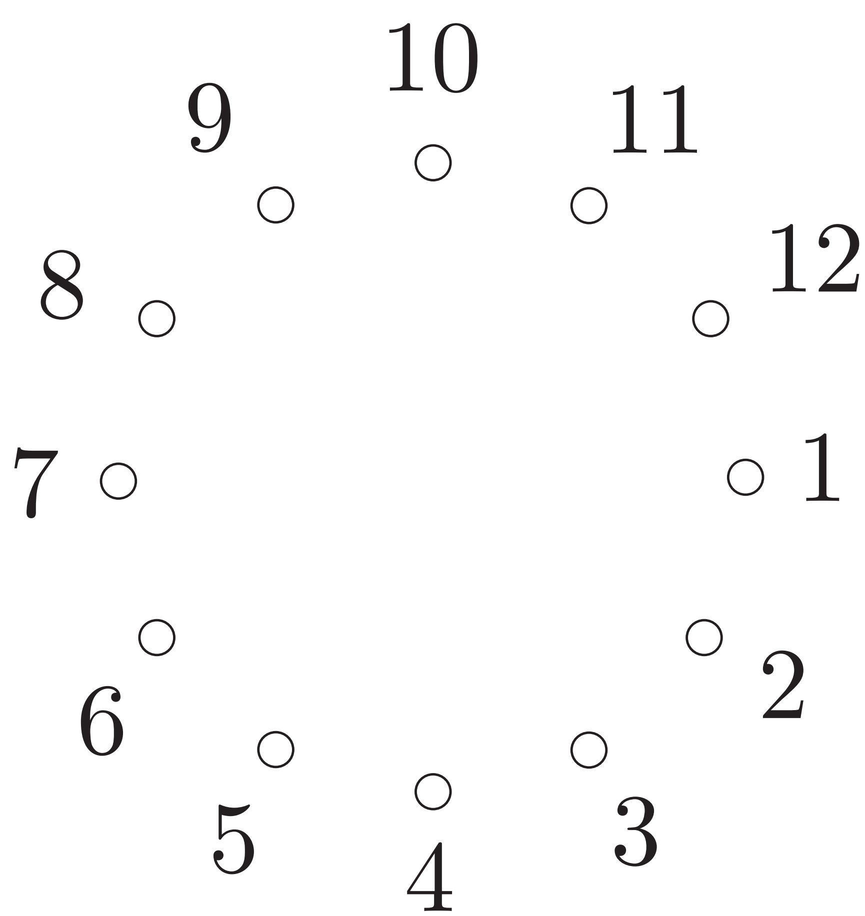
compact subsets of the unit disk.

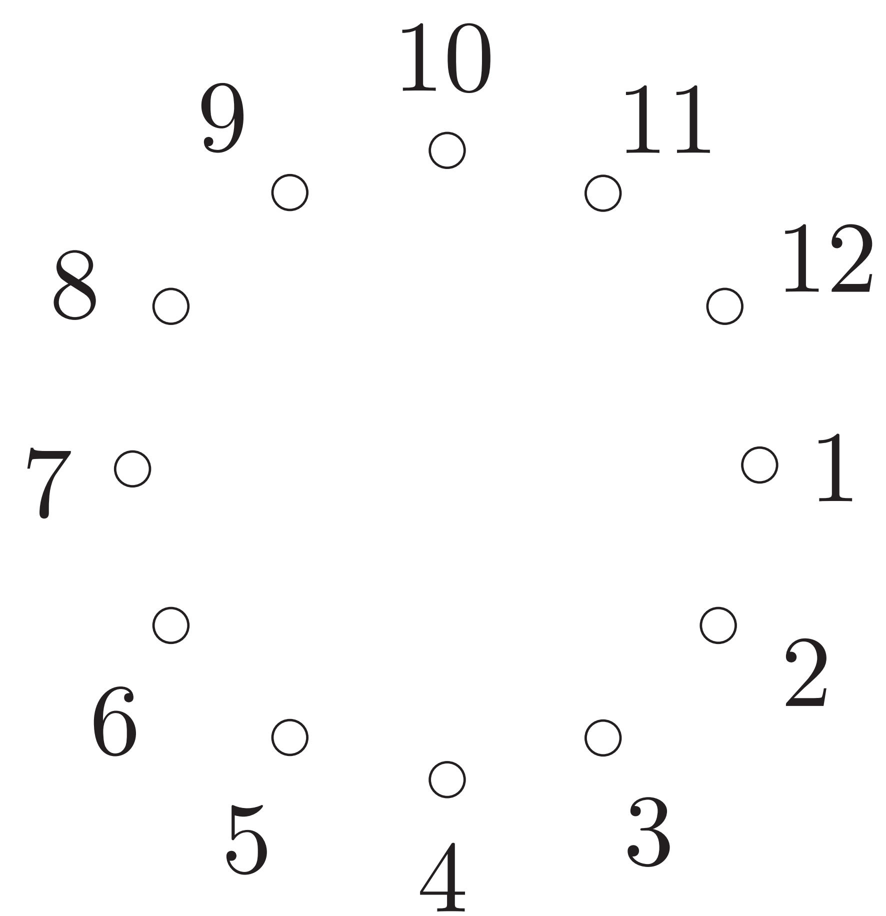
If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
- $\mathcal{P}_k$ is the compact subset associated to the cycles of $\tau_1\tau_2\cdots\tau_k$.

$\leadsto$ Example ($n = 12$). Take

$$\big((1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11)\big)$$





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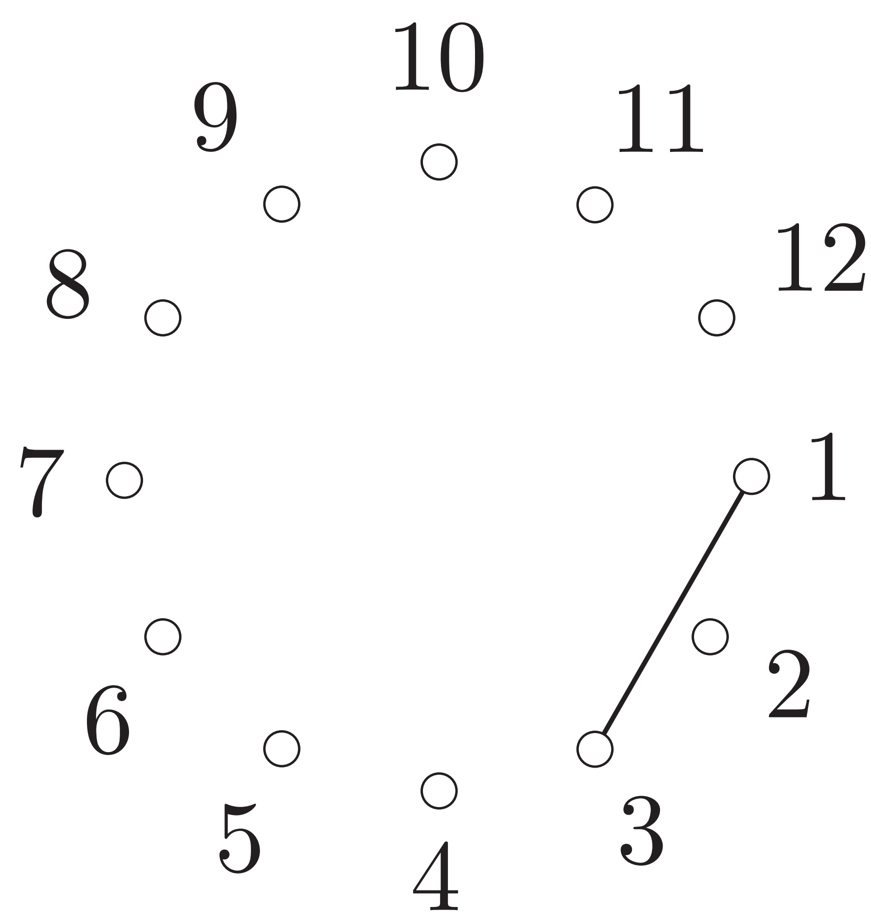
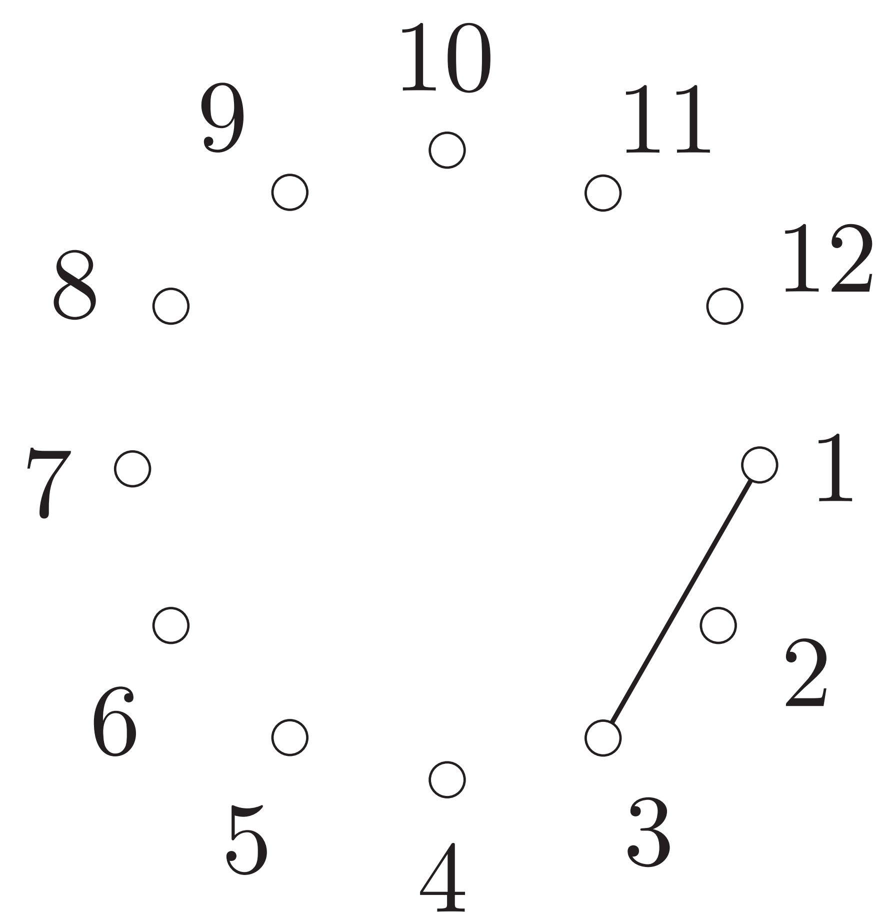
Example $(n = 12)$. For $k = 1$:

$$\Big(\underbrace{(1,3)}_{\text{product}=(1,3)}, (6,12), (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11$$



$\mathcal{F}_1$



$\mathcal{P}_1$

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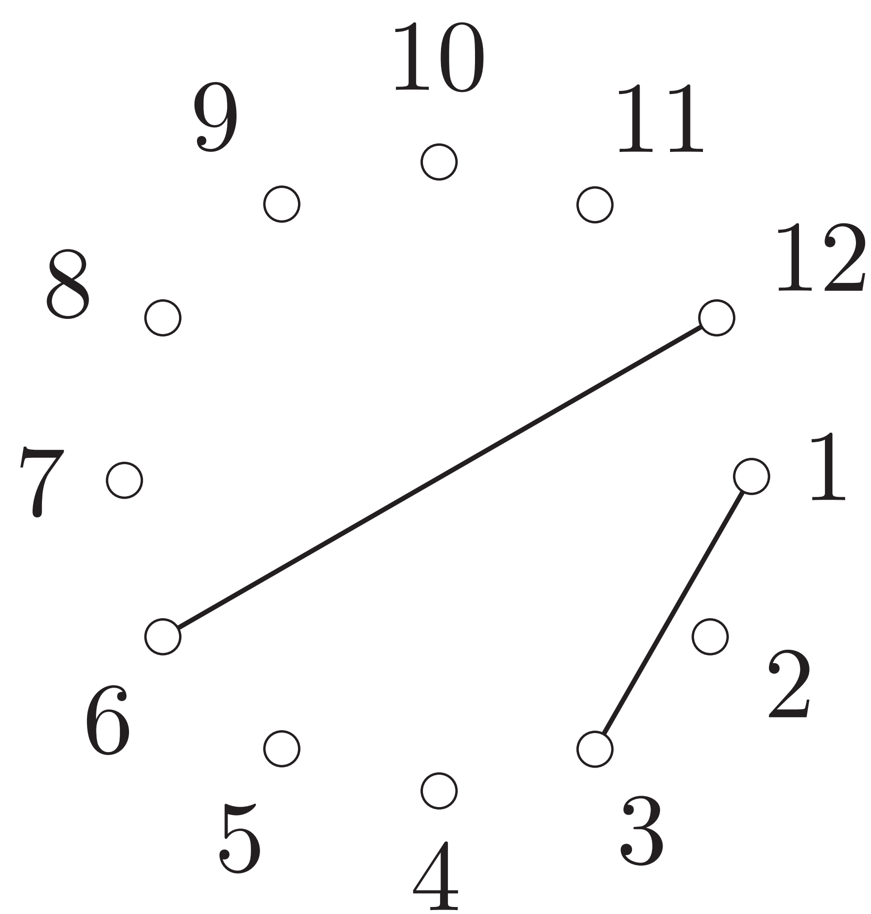
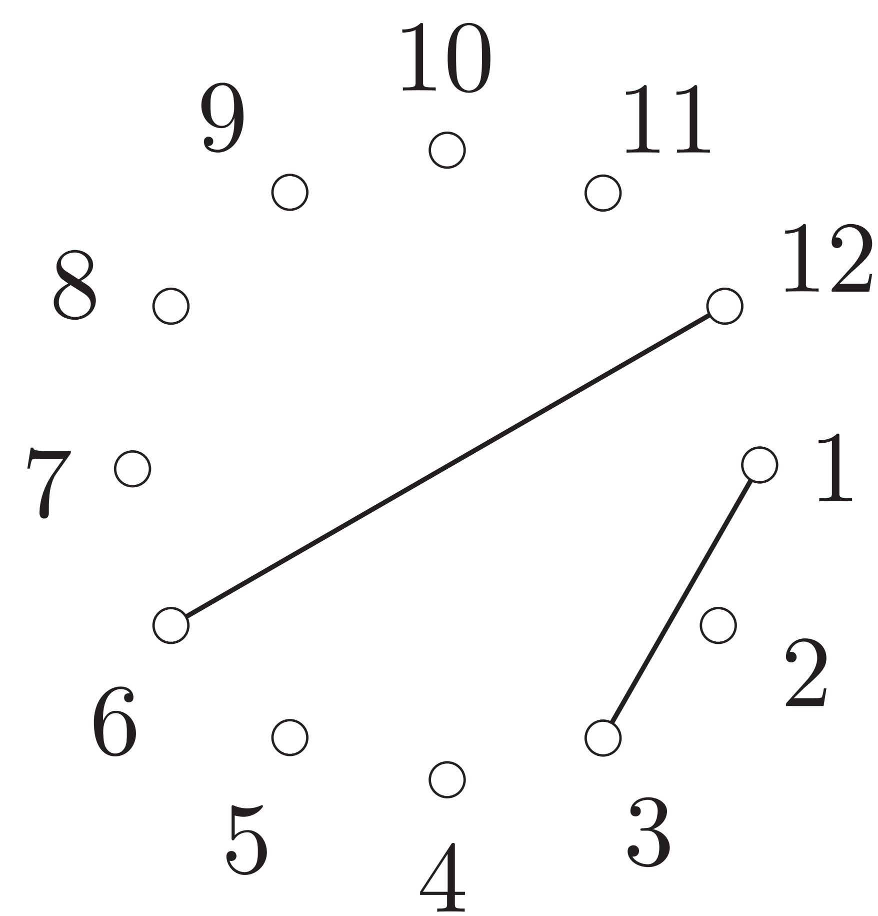
$\rightsquigarrow$ Example ($n = 12$). For $k = 2$:

$$\Big(\underbrace{(1,3),(6,12)}_{\text{product}=(1,3)(6,12)}, (1,5), (7,12), (9,10), (11,12), (2,3), (4,5), (1,6), (8,11), (9,11) \Big)$$



$\mathcal{F}_2$



$\mathcal{P}_2$

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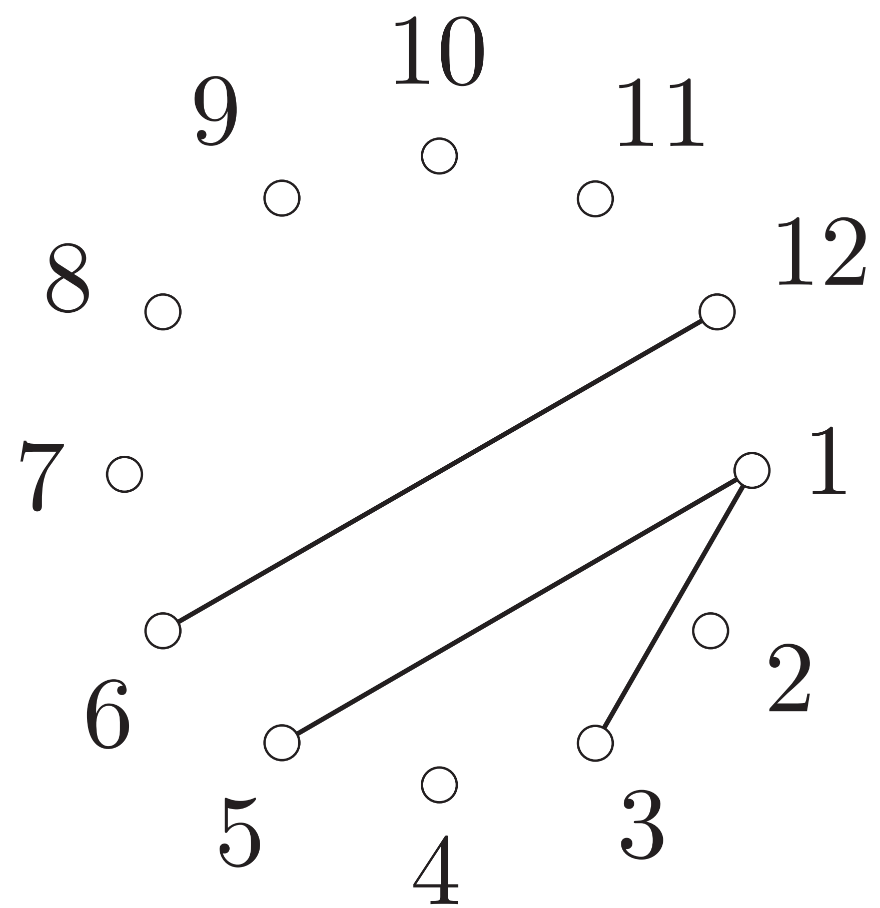
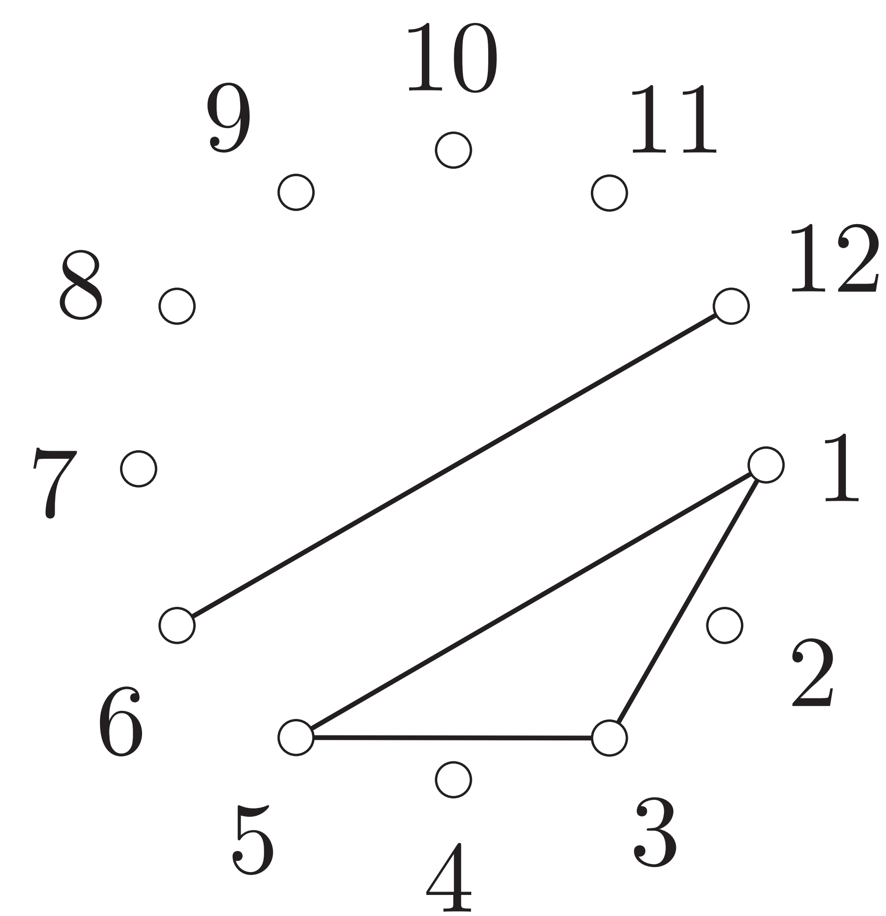
Example ($n = 12$). For $k = 3$:

$$\Big(\underbrace{(1,3),(6,12),(1,5)}_{\text{product}=(1,3,5)(6,12)}, (7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11) \Big)$$



$\mathcal{F}_3$



$\mathcal{P}_3$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
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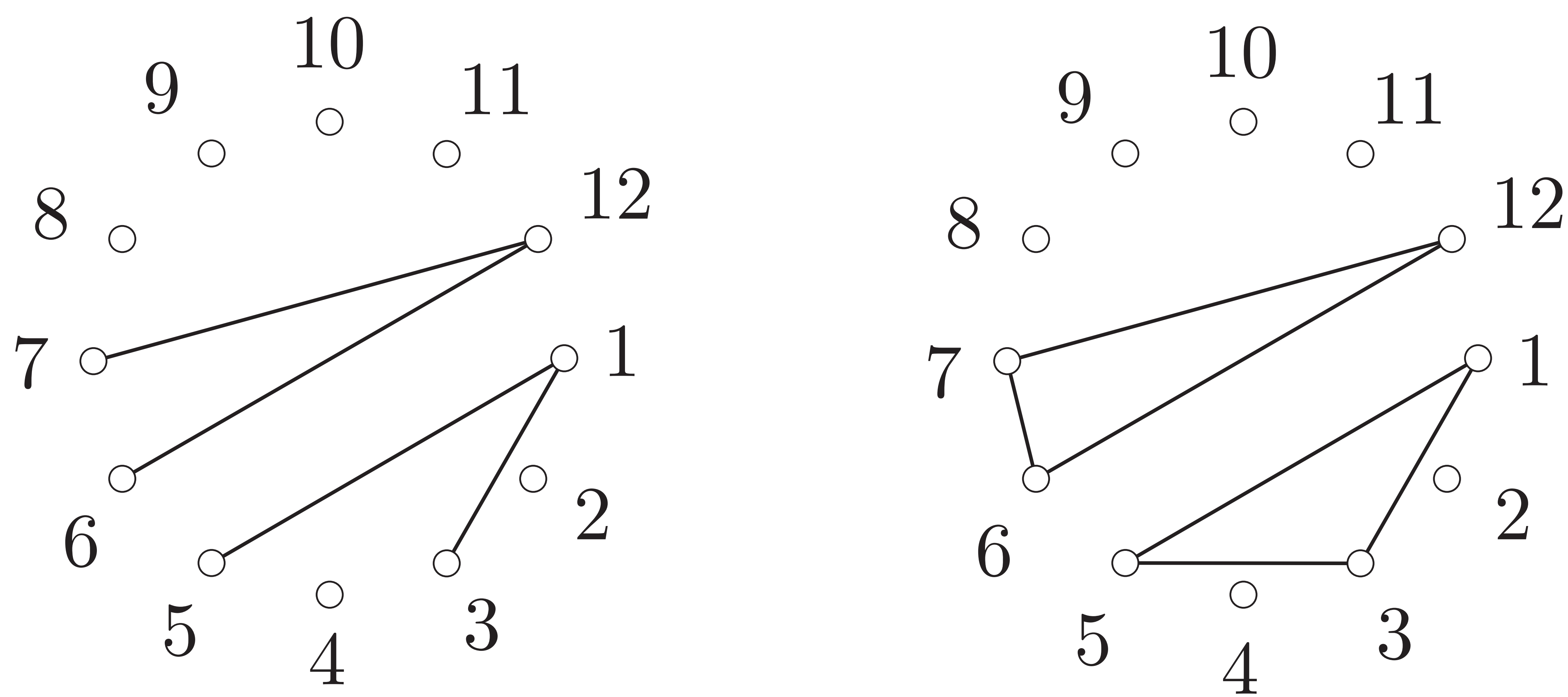
Example ($n = 12$). For $k = 4$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12)}_{\text{product}=(1,3,5)(6,7,12)},(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11)\Big)$$



$\mathcal{F}_4$ $\qquad\qquad$ $\mathcal{P}_4$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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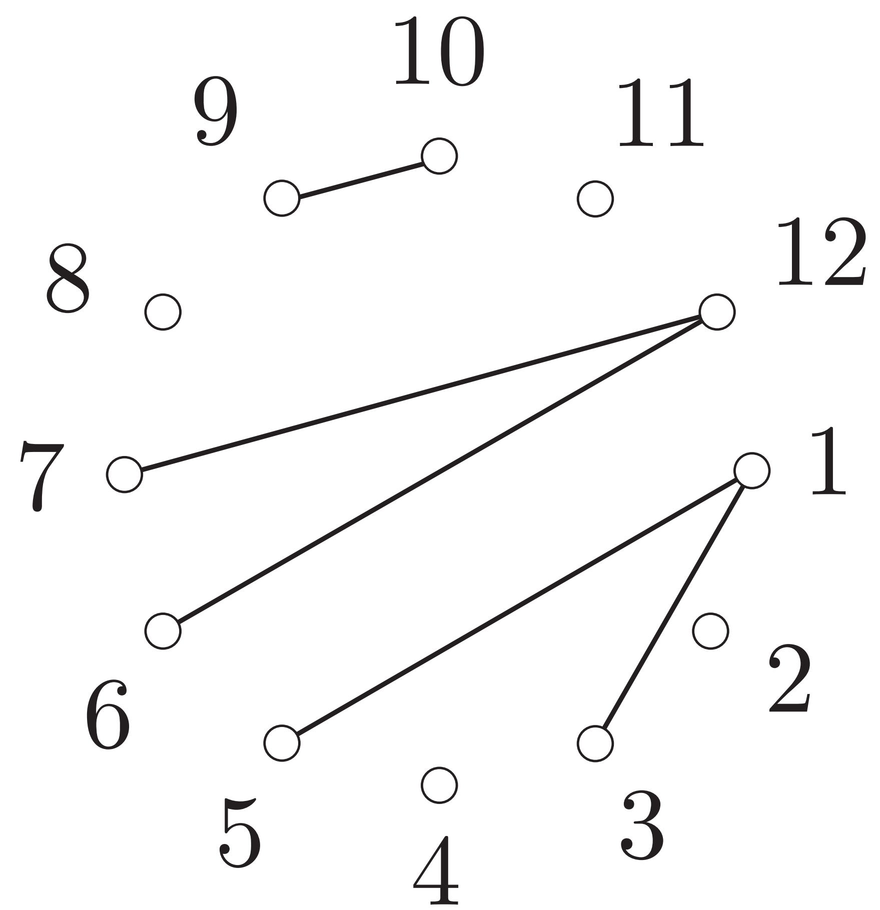
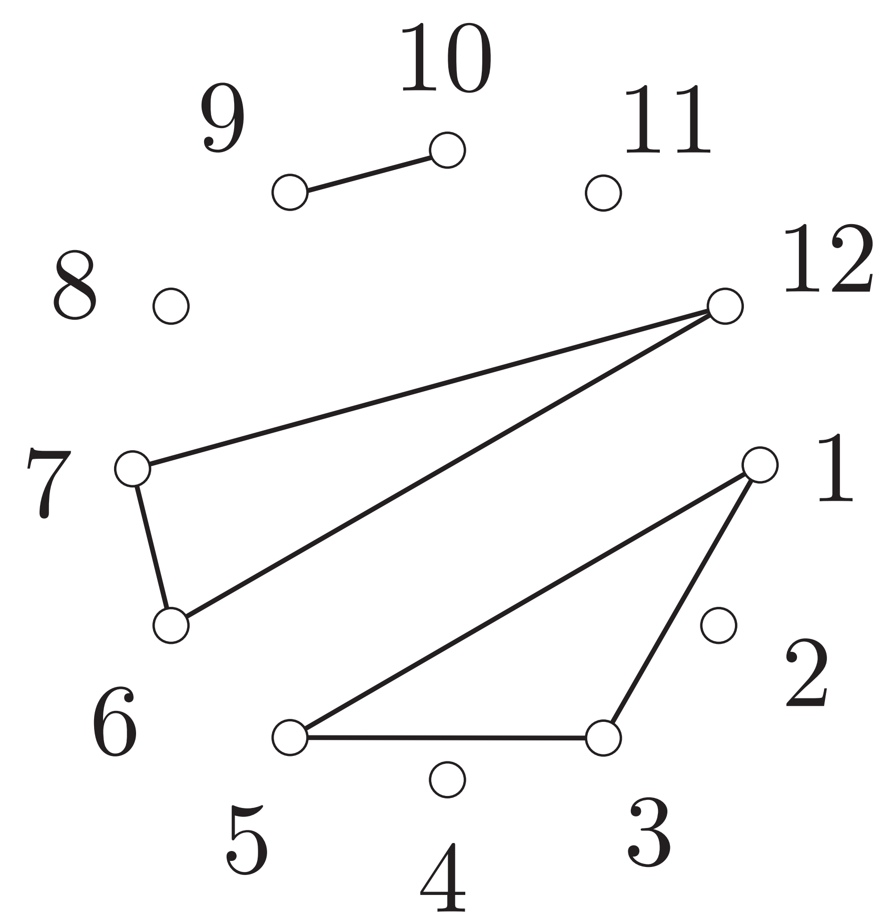
⇝ Example $(n = 12)$. For $k = 5$:

$$\Big(\underbrace{(1,3), (6,12), (1,5), (7,12), (9,10)}_{\text{product}=(1,3,5)(6,7,12)(9,10)}, (11,12), (2,3), (4,5), (1,6), (8,11), (9,11) \Big)$$



$\mathcal{F}_5$



$\mathcal{P}_5$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

- $\mathcal{F}_k$ is the compact subset obtained by drawing the chords τ_i, $1 \leqslant i \leqslant k$.
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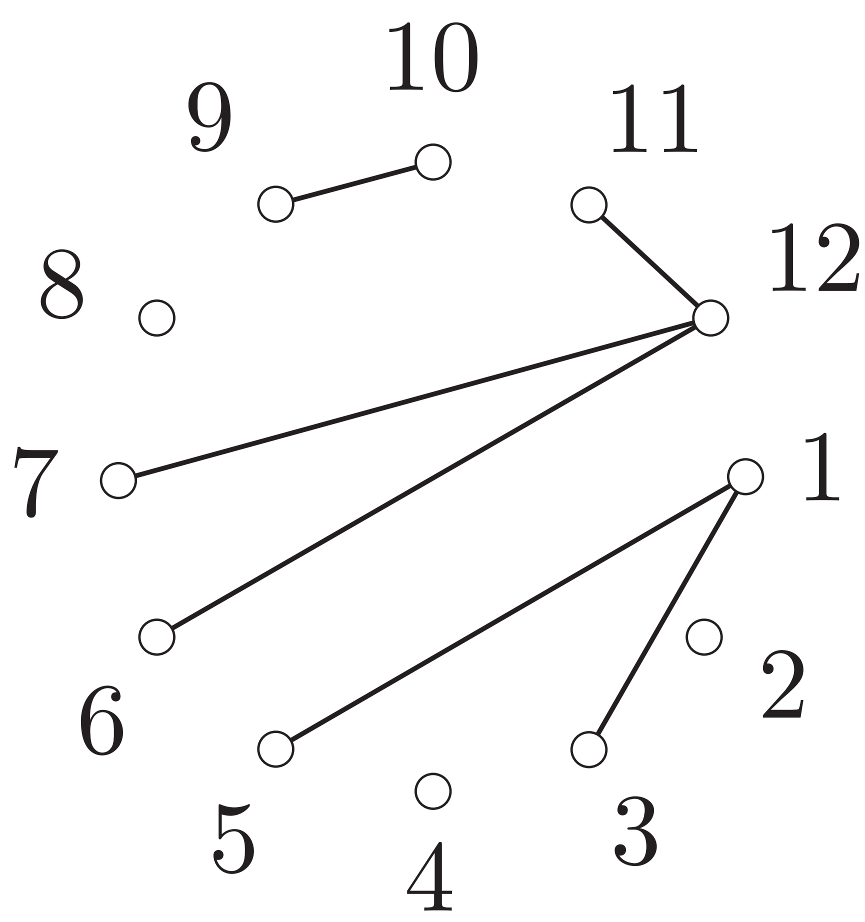
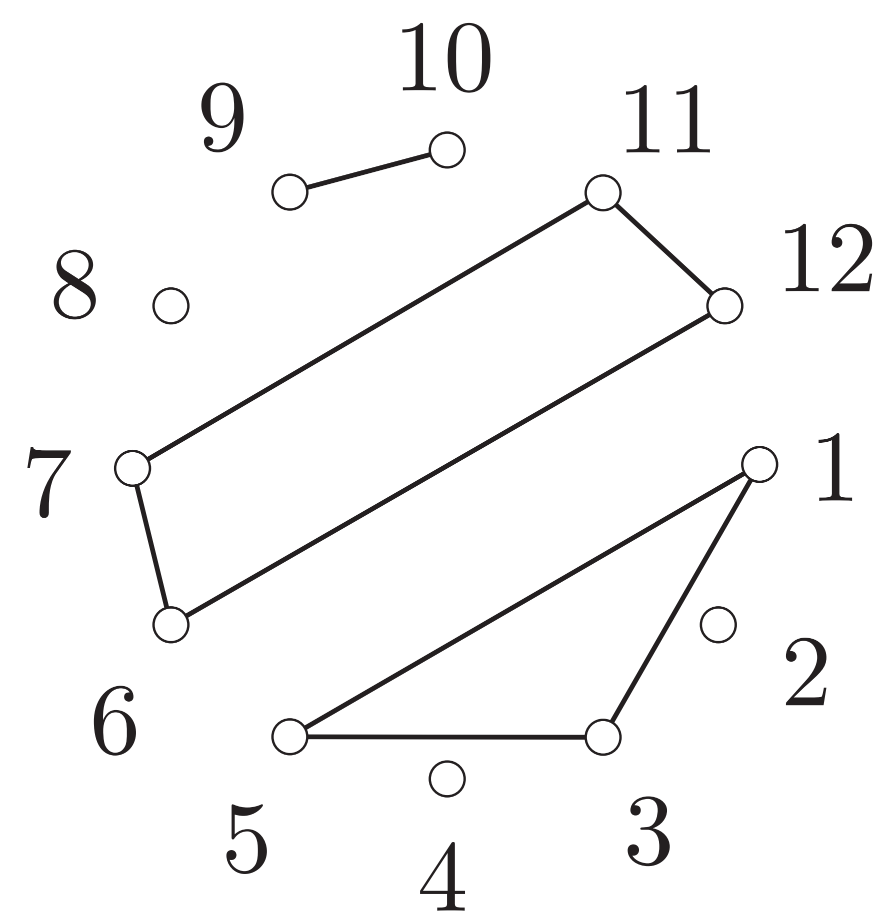
Example $(n = 12)$. For $k = 6$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12)}_{\text{product}=(1,3,5)(6,7,11,12)(9,10)},(2,3),(4,5),(1,6),(8,11),(9,11)\Big)$$



$\mathcal{F}_6$



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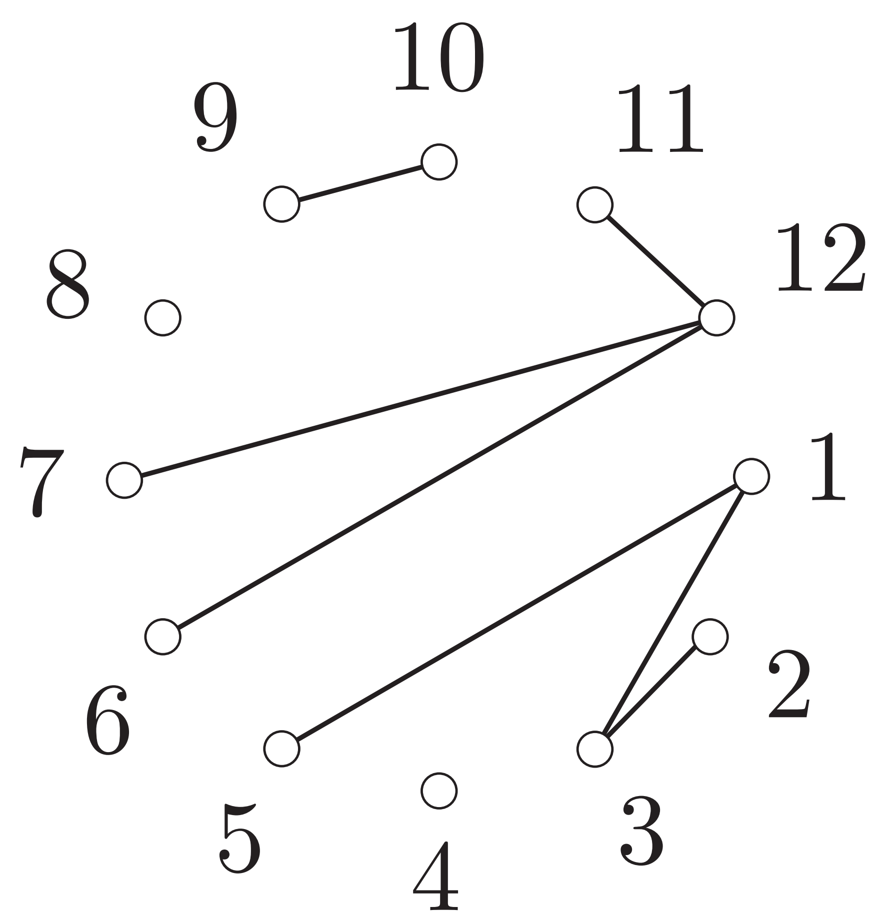
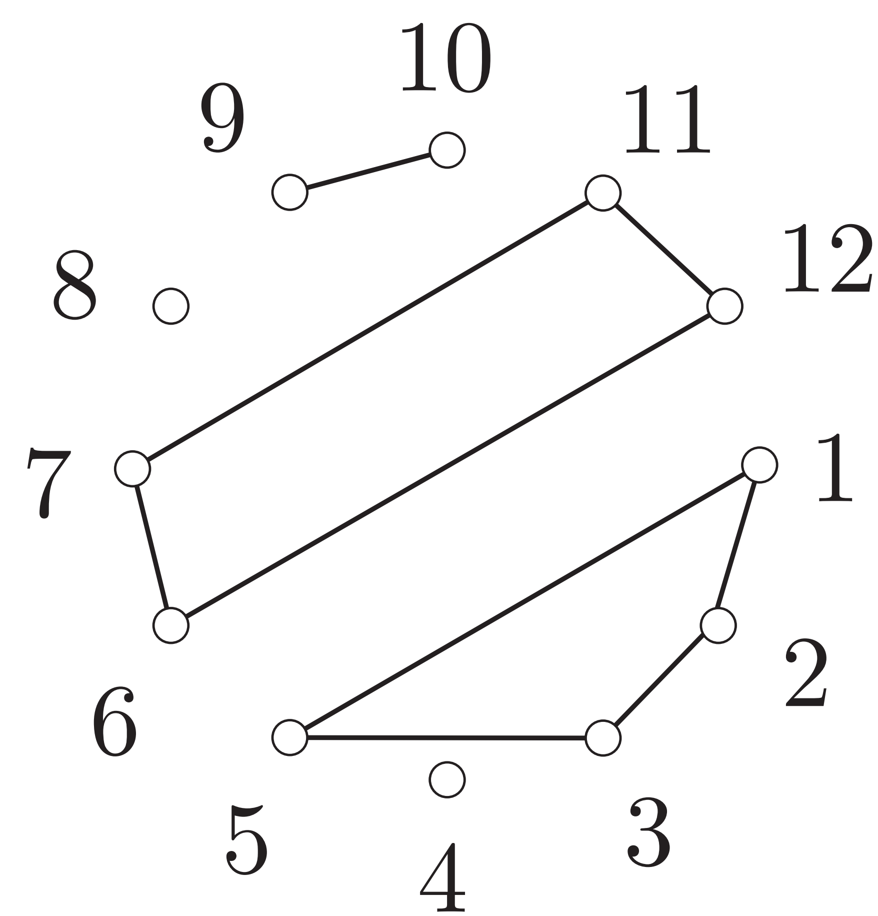
Example ($n = 12$). For $k = 7$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3)}_{\text{product}=(1,2,3,5)(6,7,11,12)(9,10)}, (4,5),(1,6),(8,11),(9,11) \Big)$$



$\mathcal{F}_7$



$\mathcal{P}_7$

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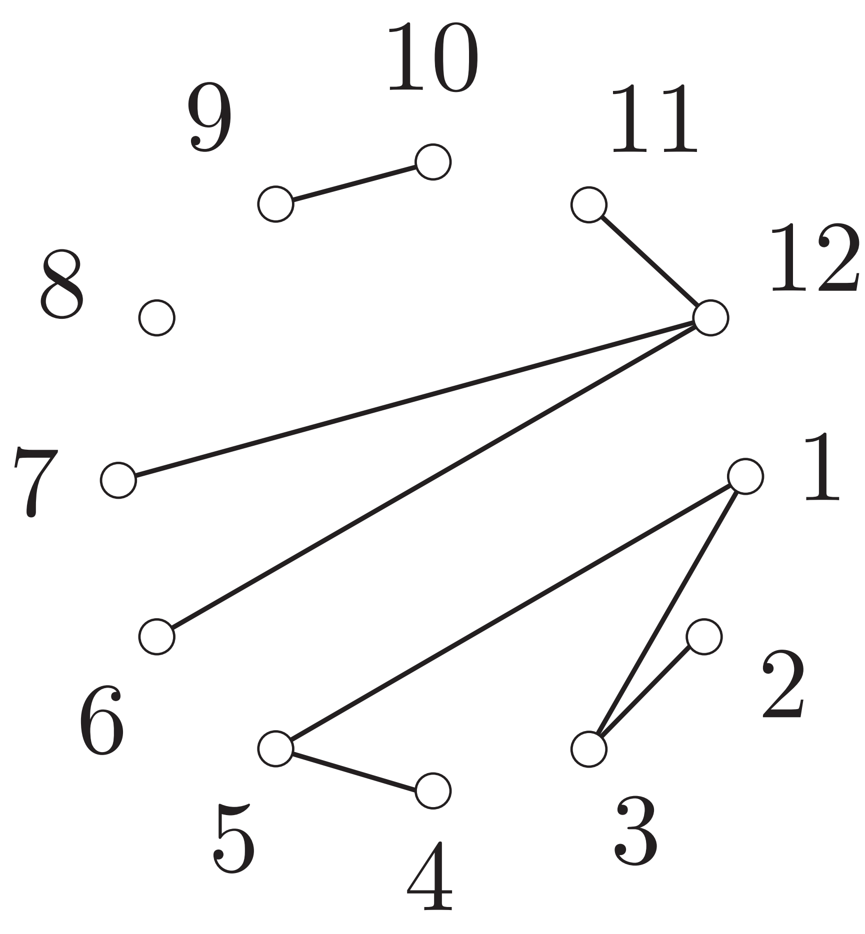
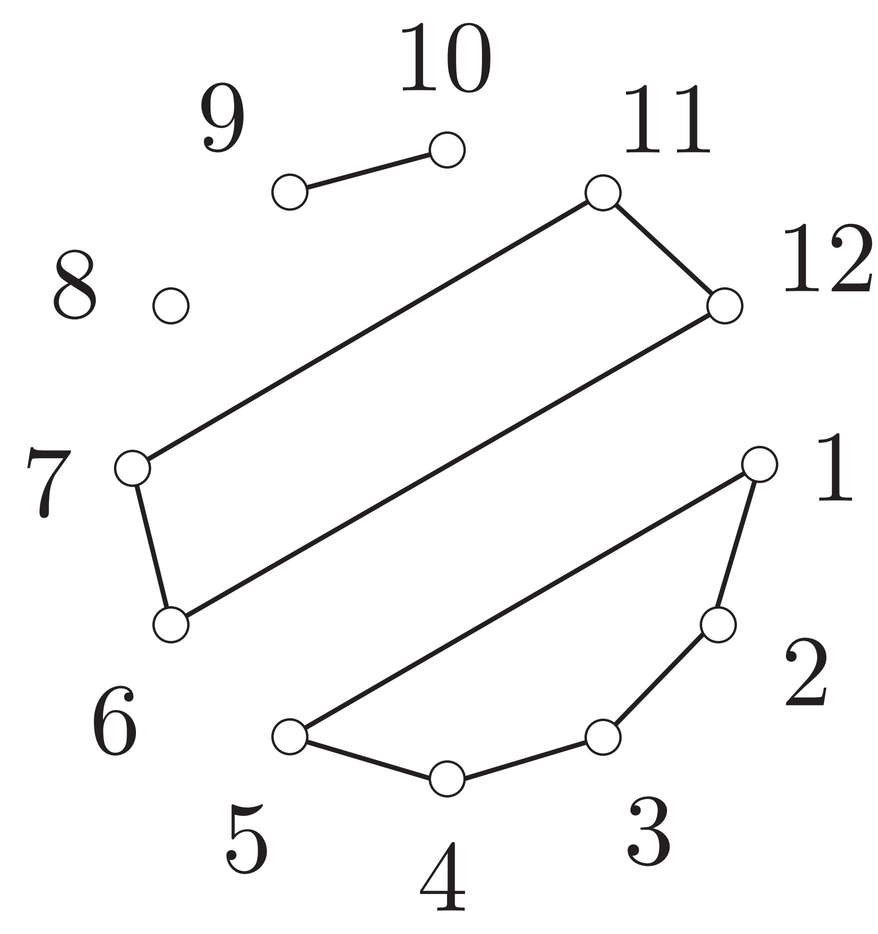
Example $(n = 12)$. For $k = 8$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5)}_{\text{product}=(1,2,3,4,5)(6,7,11,12)(9,10)},(1,6),(8,11),(9,11)\Big)$$



$\mathcal{F}_8$



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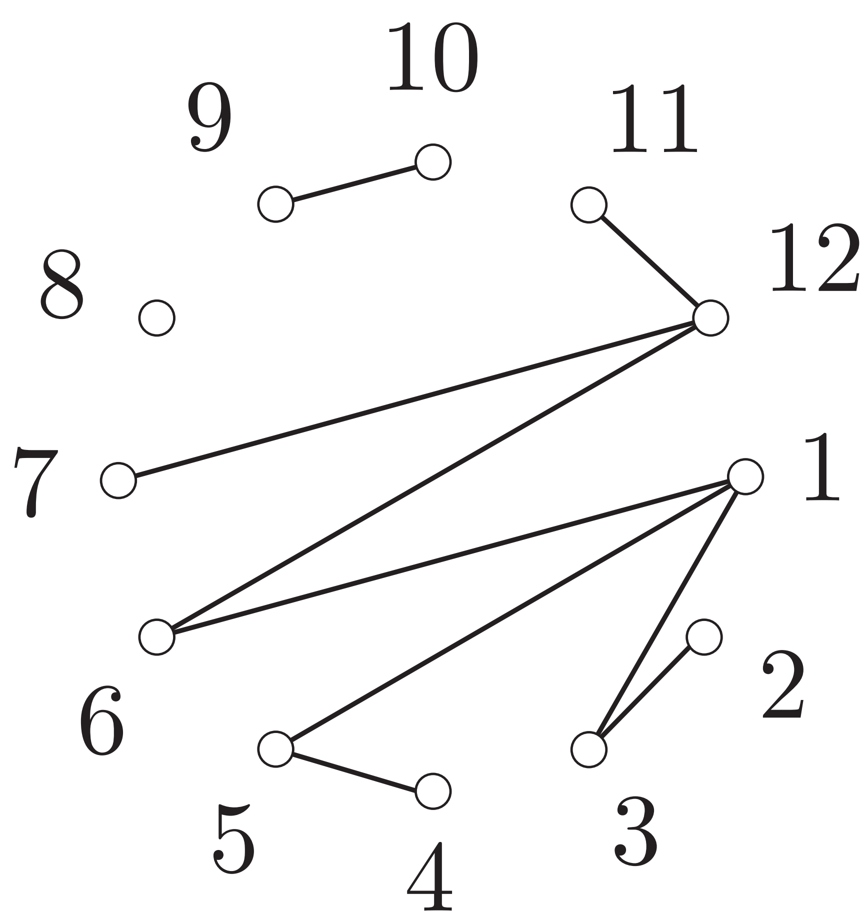
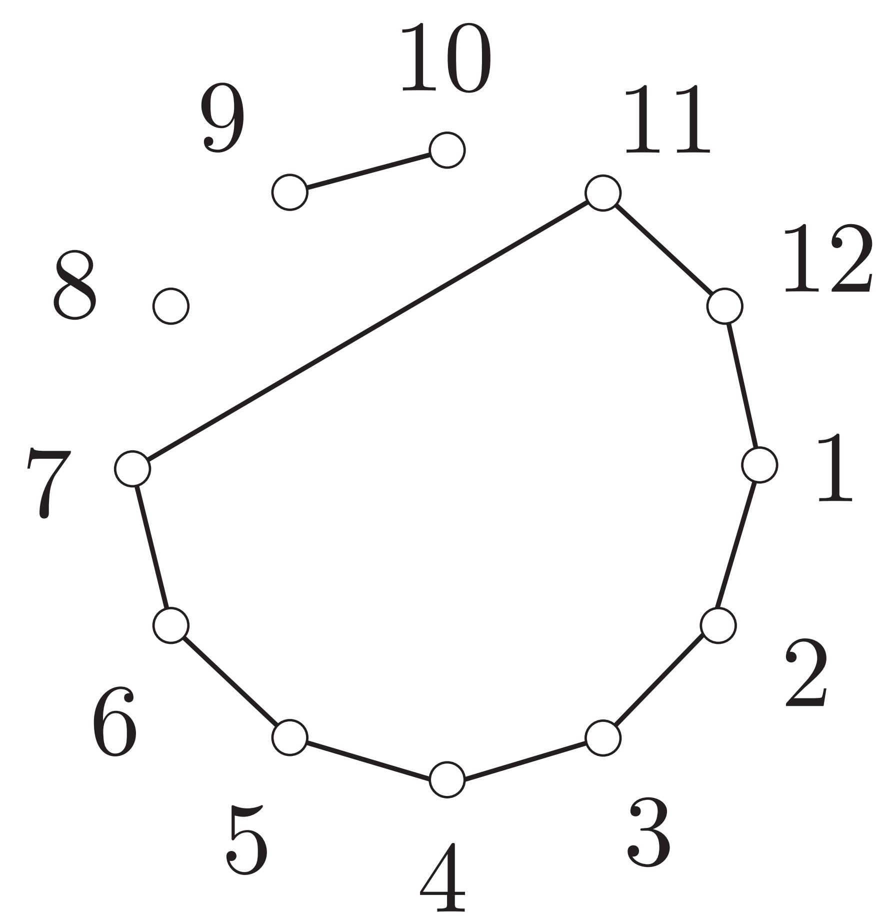
Example ($n = 12$). For $k = 9$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6)}_{\text{product}=(1,2,3,4,5,6,7,11,12)(9,10)},(8,11),(9,11) \Big)$$



$\mathcal{F}_9$



$\mathcal{P}_9$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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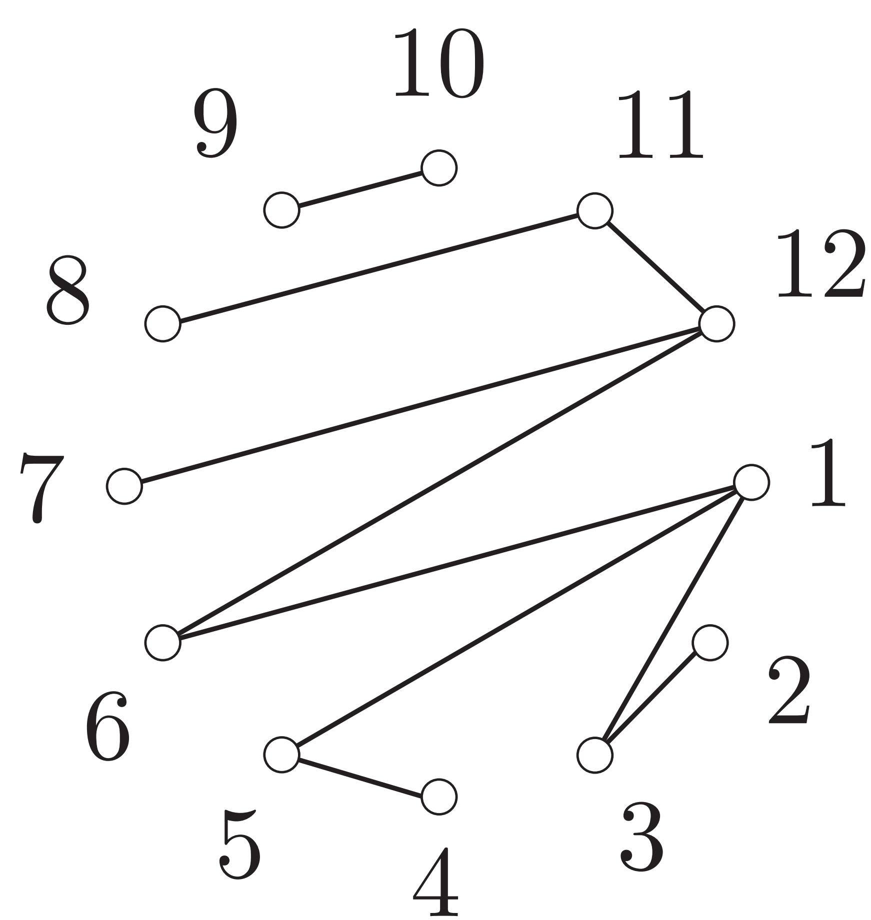
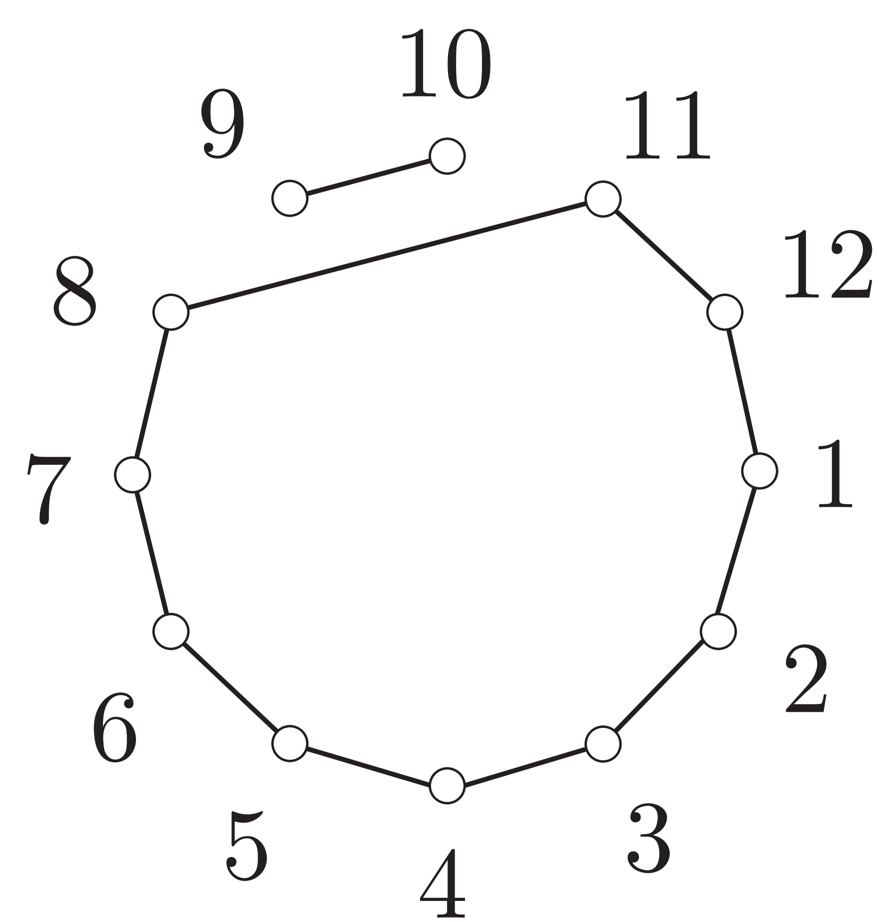
Example $(n = 12)$. For $k = 10$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11)}_{\text{product}=(1,2,3,4,5,6,7,8,11,12)(9,10)},(9,11)\Big)$$



$\mathcal{F}_{10}$



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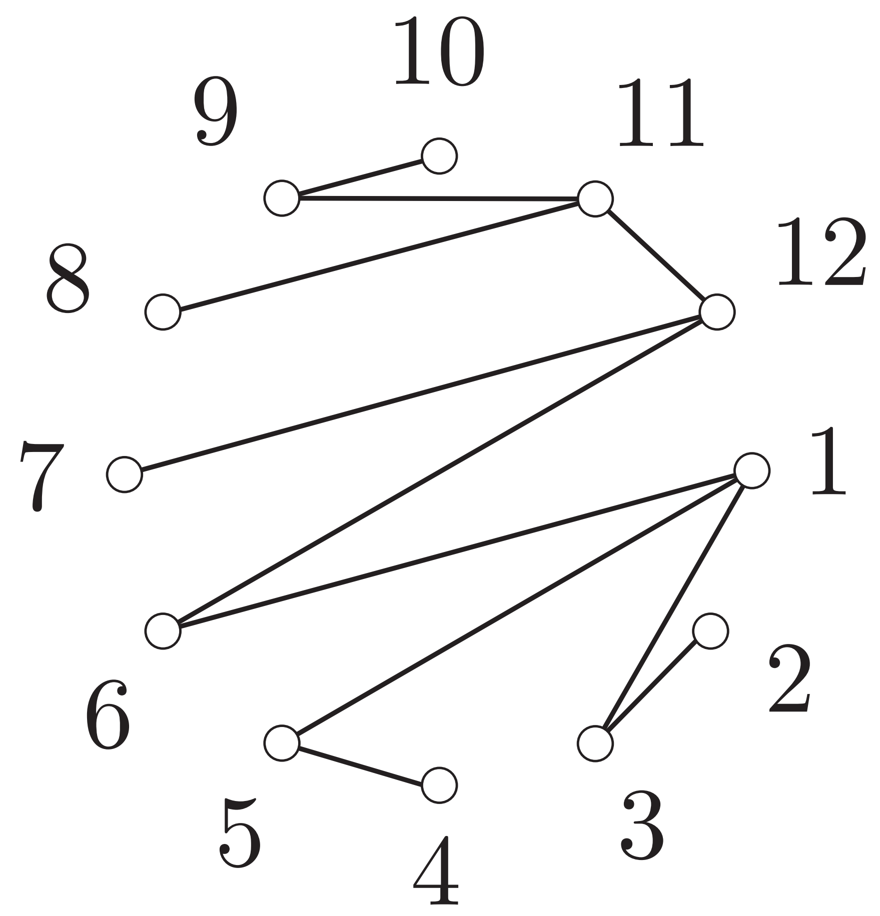
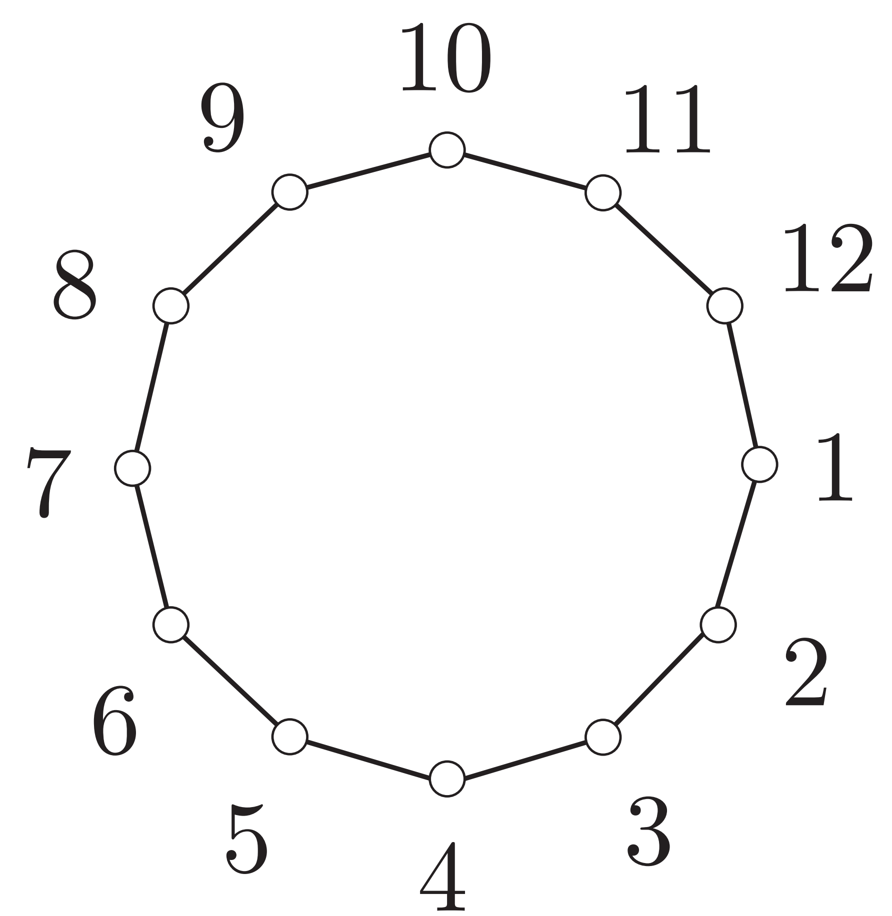
Example ($n = 12$). For $k = 11$:

$$\Big(\underbrace{(1,3),(6,12),(1,5),(7,12),(9,10),(11,12),(2,3),(4,5),(1,6),(8,11),(9,11)}_{\text{product}=(1,2,3,4,5,6,7,8,9,10,11,12)} \Big)$$



$\mathcal{F}_{11}$



$\mathcal{P}_{11}$

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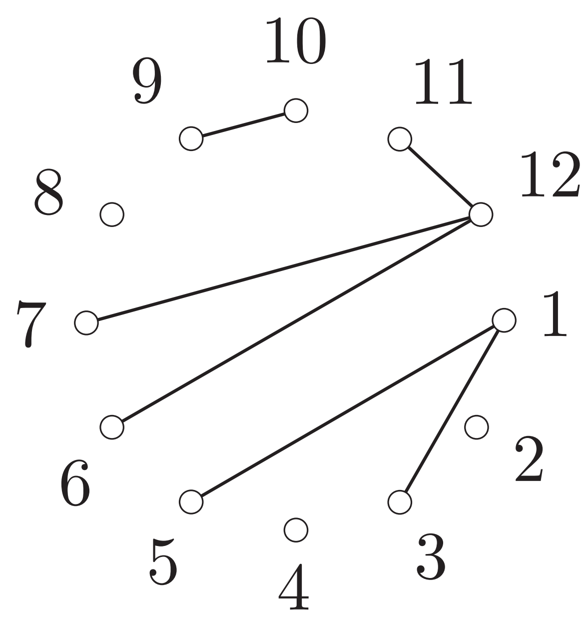
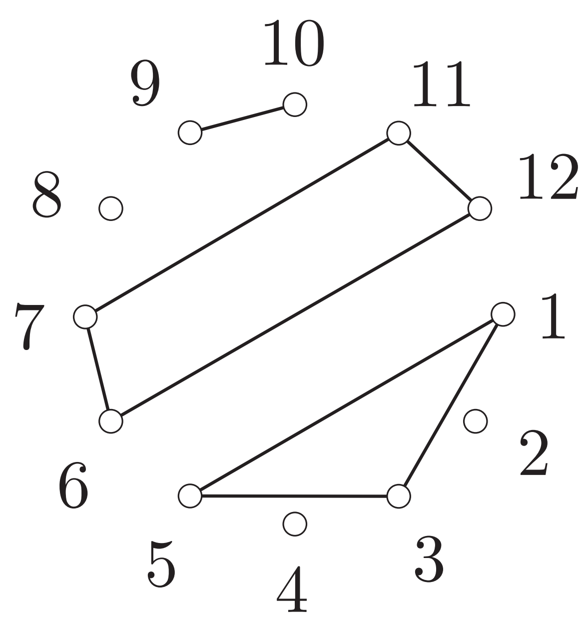


$\mathcal{F}_6$



$\mathcal{P}_6$

$\rightsquigarrow$ Remarks:

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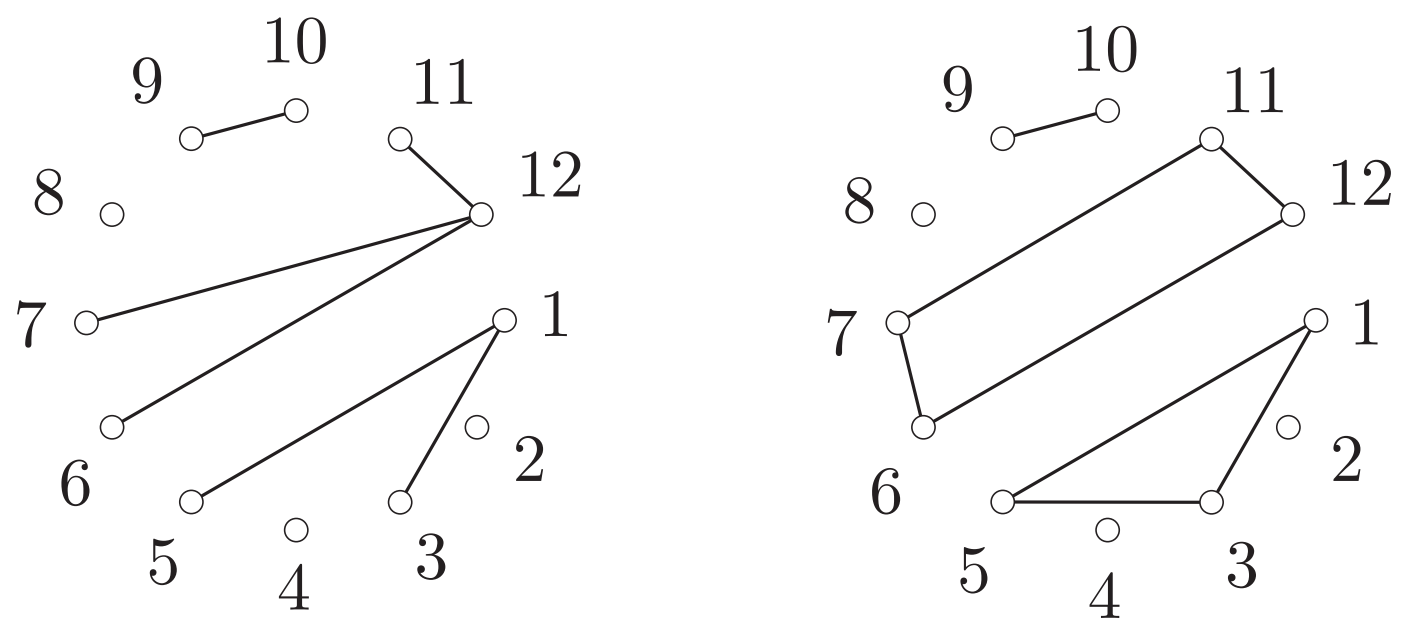

$\mathcal{F}_6$ $\mathcal{P}_6$

⤳ Remarks:

- The chords are non-crossing

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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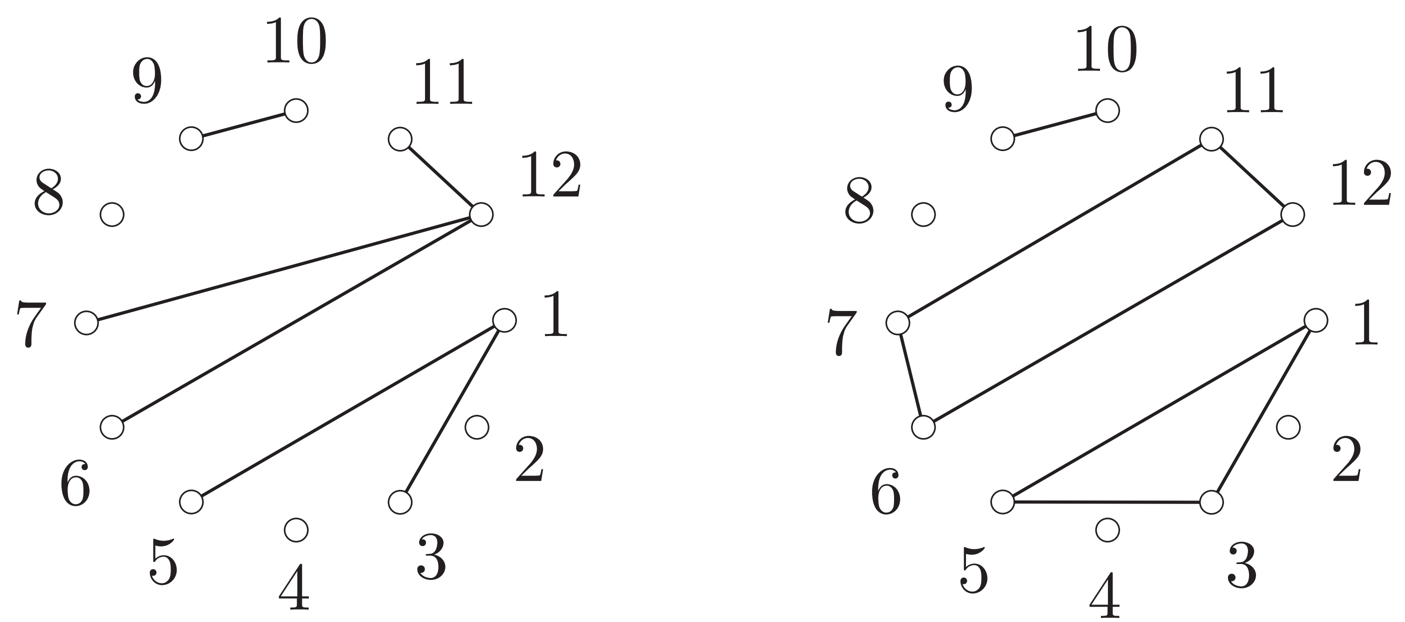

$\mathcal{F}_6$ $\mathcal{P}_6$

Remarks:

- The chords are non-crossing
- the cycles of $\mathcal{P}_k$ are the connected components of $\mathcal{F}_k$

If $(\tau_1, \ldots, \tau_{n-1})$ is a minimal factorization of length n and $1 \leqslant k \leqslant n$:

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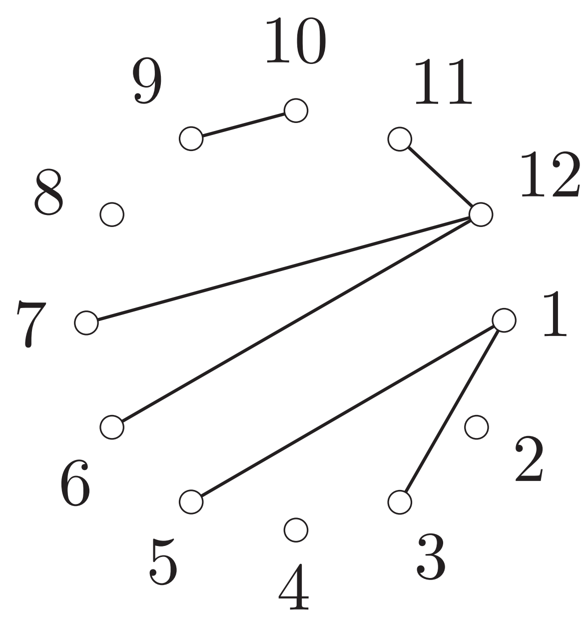
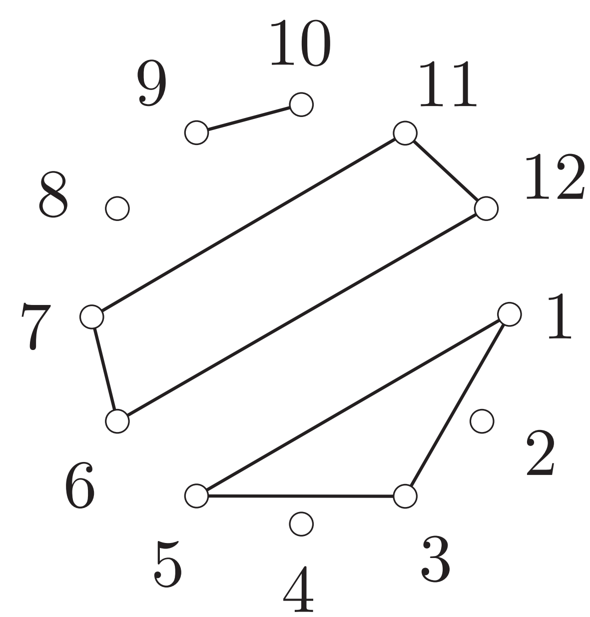


$\mathcal{F}_6$



$\mathcal{P}_6$

Remarks:

- The chords are non-crossing
- the cycles of $\mathcal{P}_k$ are the connected components of $\mathcal{F}_k$
- $\mathcal{P}_k$ has $n - k$ blocks.

Let $(\tau_1^n, \ldots, \tau_{n-1}^n)$ be a uniform minimal factorization of the n-cycle.

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The following film represents

$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right)$$

with $\boxed{K_n = \lfloor cn \rfloor}$ for fixed n, as c varies.

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The following film represents

$$\left(\mathcal{F}^n_{K_n}, \mathcal{P}^n_{K_n}\right)$$

with $\boxed{K_n = \lfloor c\sqrt{n} \rfloor}$ for fixed n, as c varies.

Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

(i)

(ii)

(iii)

(iv)

Theorem (Féray, K.).

Let $(t_1^{(n)}, \ldots, t_{n-1}^{(n)})$ be a uniform minimal factorization of length n and $1 \leqslant K_n \leqslant n-1$ with $K_n \to \infty$.

(i) If $K_n = o(\sqrt{n})$:

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(iii) If $\frac{K_n}{\sqrt{n}} \to \infty$ and $\frac{n-K_n}{\sqrt{n}} \to \infty$:

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Theorem (Féray, K.).

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Theorem (Féray, K.).

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$\leadsto$ $\mathbf{L}_0$ is the unit circle.

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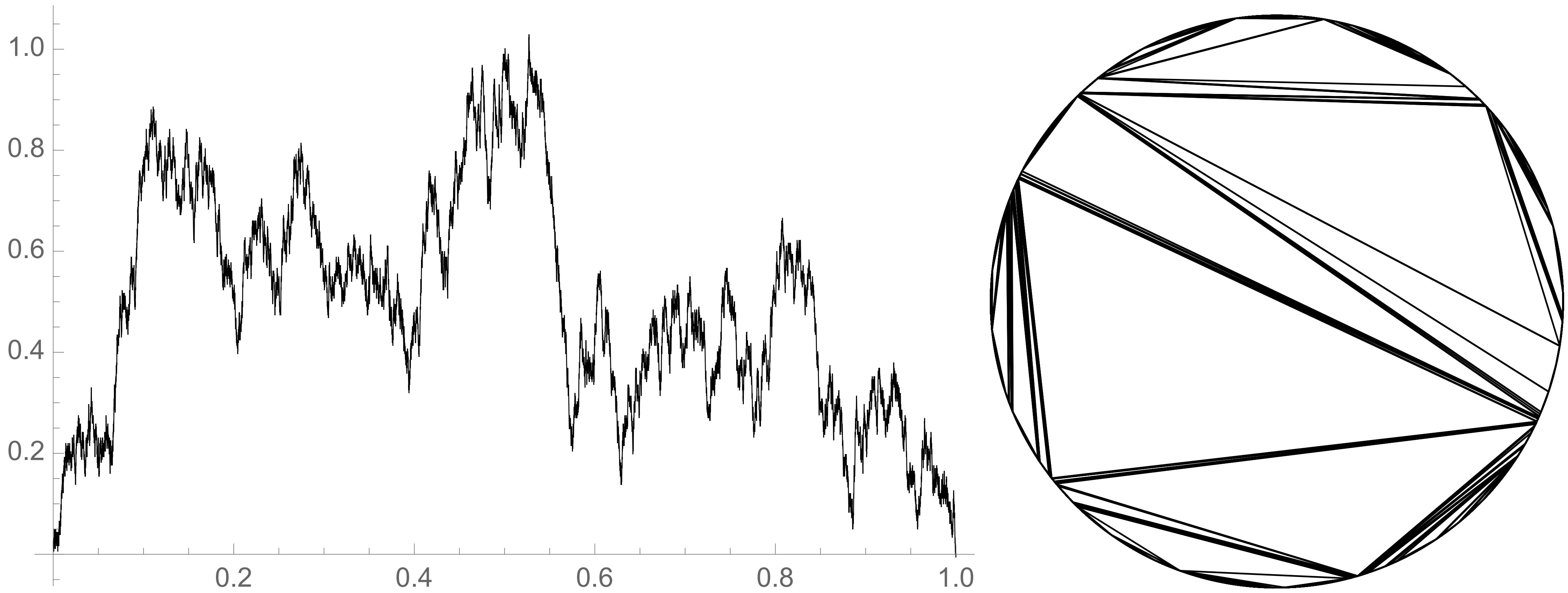

Figure: A Brownian excursion (left) coding $\mathbf{L}_\infty$ (right).

$\rightsquigarrow$ For $0 < c < \infty$, $\mathbf{L}_c$ is a *lamination* (compact subset of the unit disk made of noncrossing chords), coded by an excursion of an explicit spectrally positive Lévy process.

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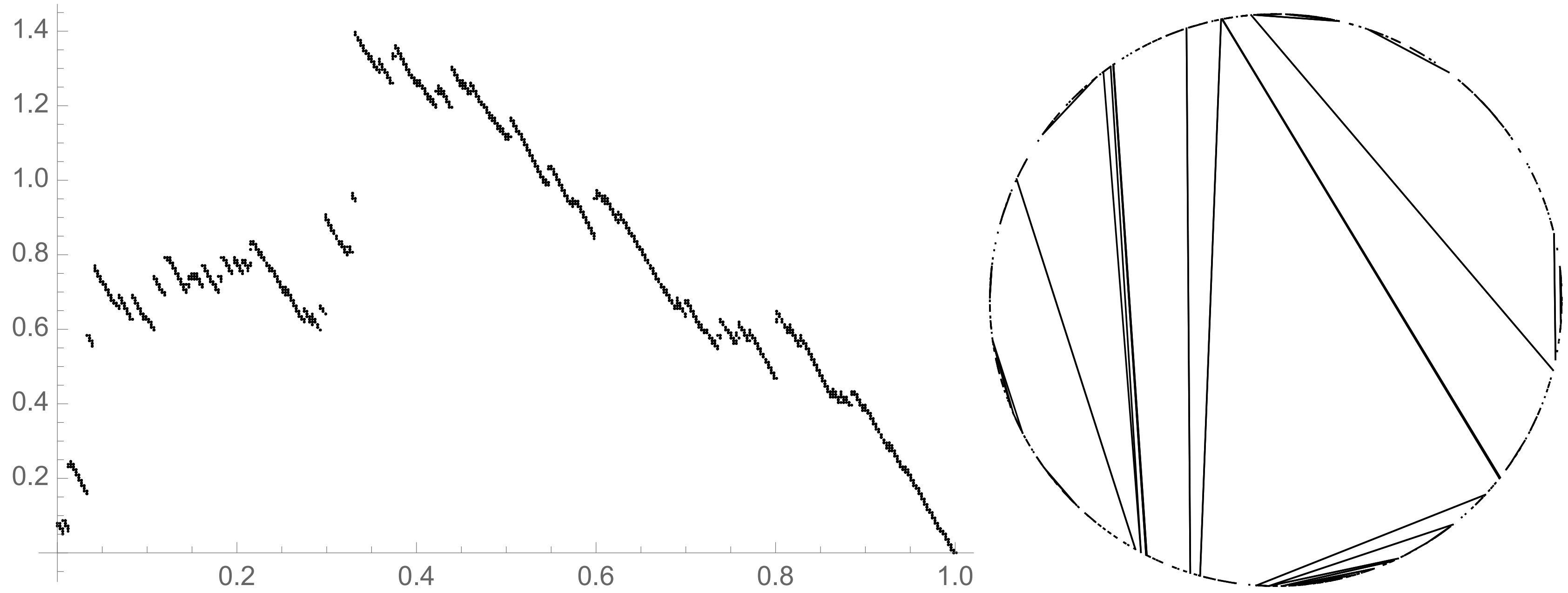

Figure: An excursion of a spectrally positive Lévy process (left) coding $\mathbf{L}_5$ (right).

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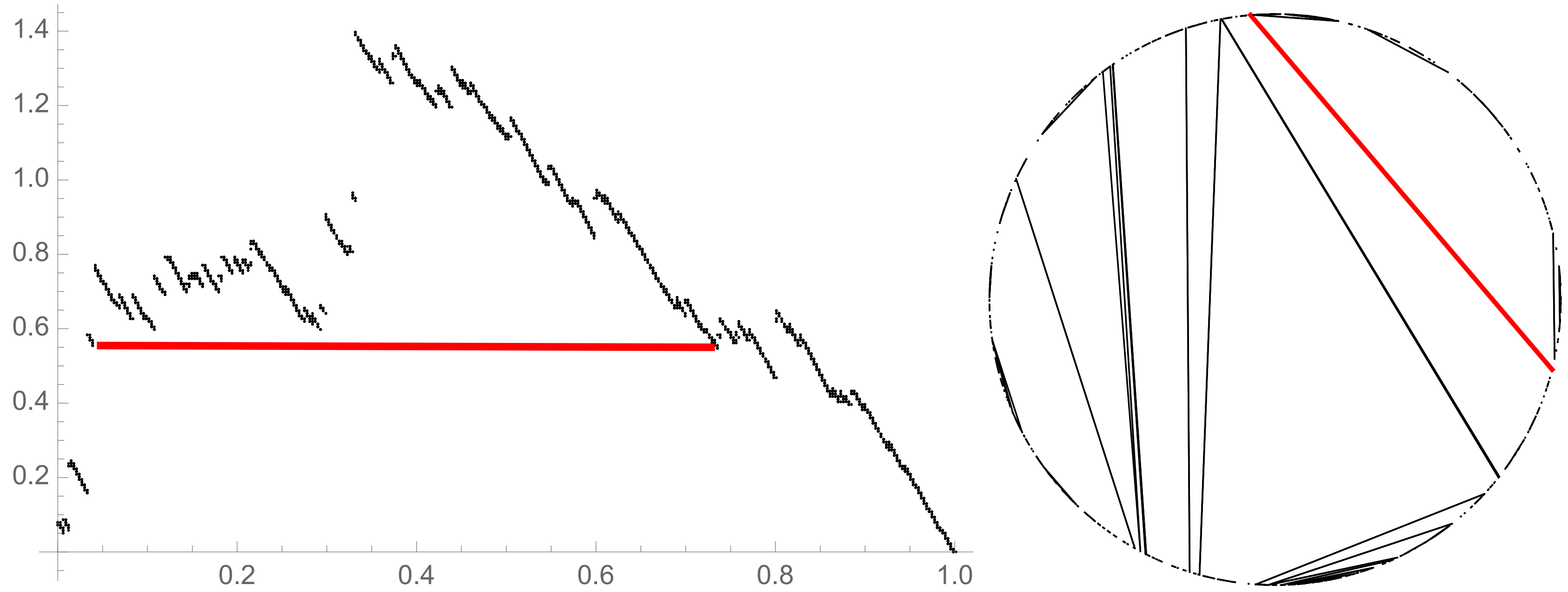

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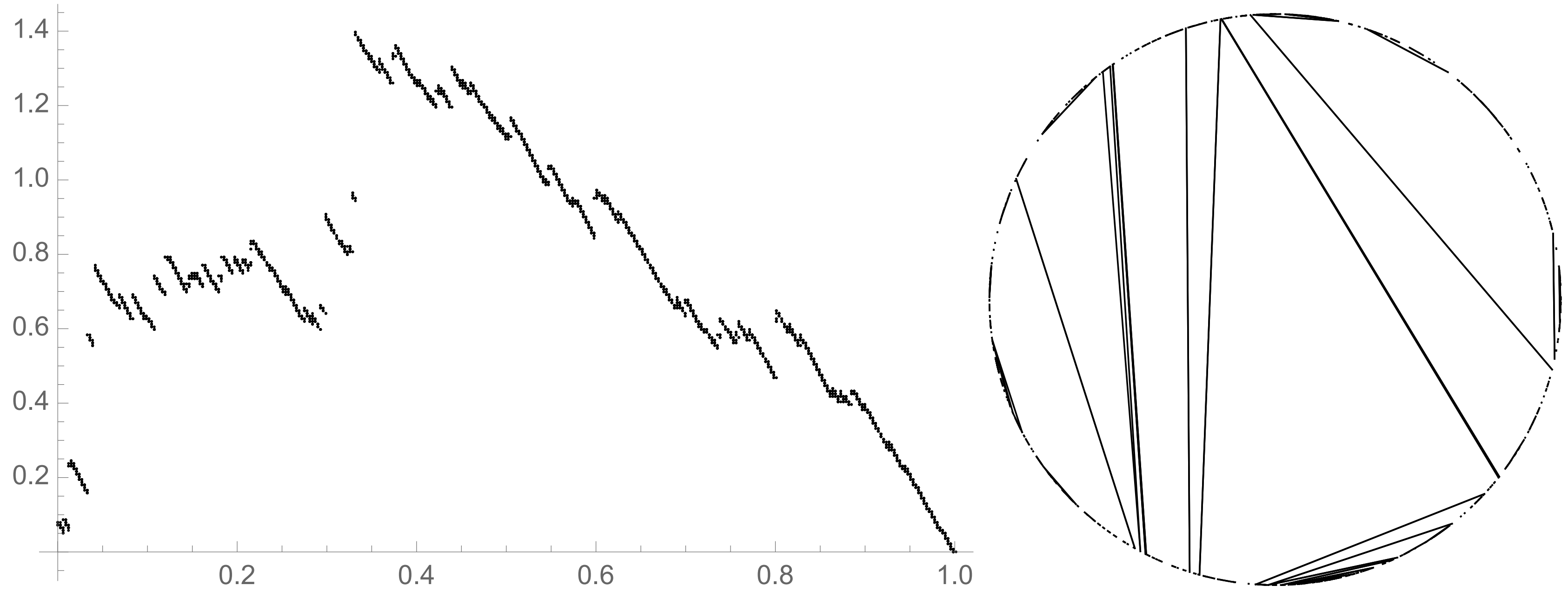

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↝ The Laplace exponent of the Lévy process is

$$\Phi_c(\lambda) = c^2\left(1 - \sqrt{1 + \frac{2\lambda}{c}}\right) + \lambda c.$$

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We have $\ell_0 = 0$ a.s. and ℓ_∞ is the "length" of the maximal chord of the Brownian triangulation (we have the same limit for $K_n = \ln(n)\sqrt{n}$ or $K_n = n - 1$!).

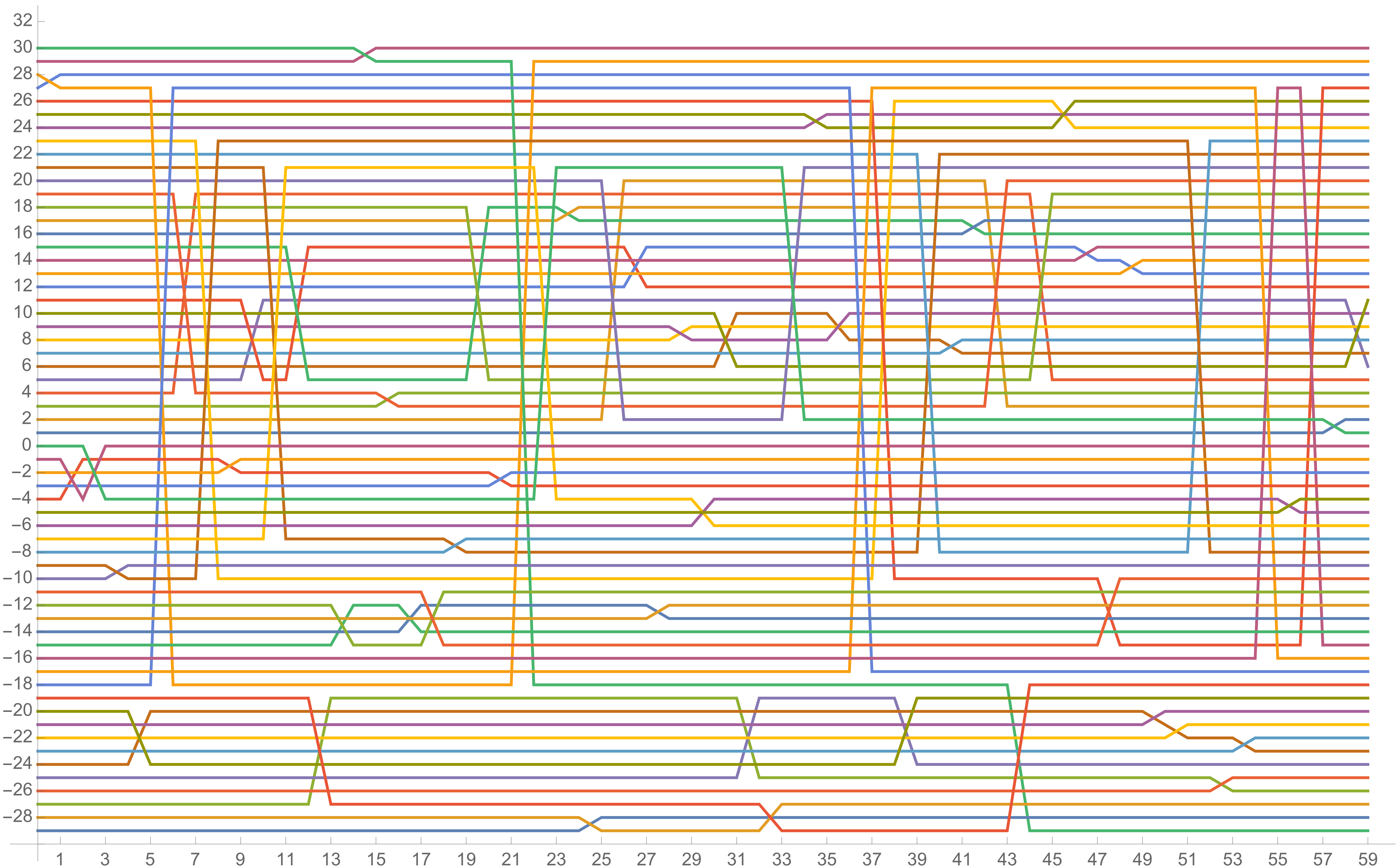

Figure: Graphical representation of the trajectories of a random minimal factorization of the 60-cycle.

I. TRIANGULATIONS & DISSECTIONS

II. MINIMAL FACTORIZATIONS

III. MAIN STEPS OF THE PROOF

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Fix $1 \leqslant k \leqslant n-1$ and let P be a non-crossing partition with n vertices and $n-k$ blocks. Then

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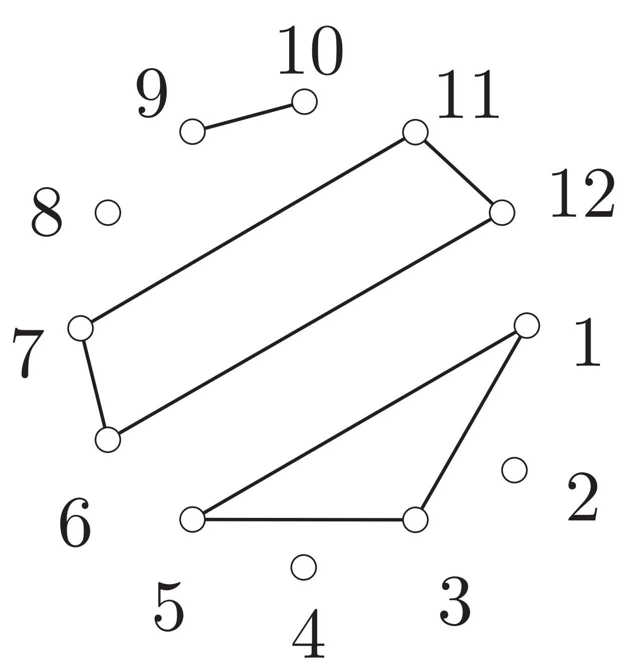

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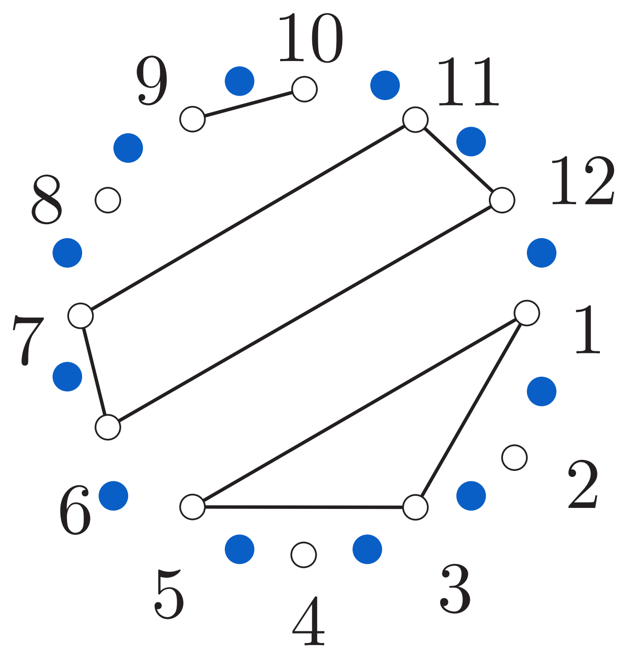

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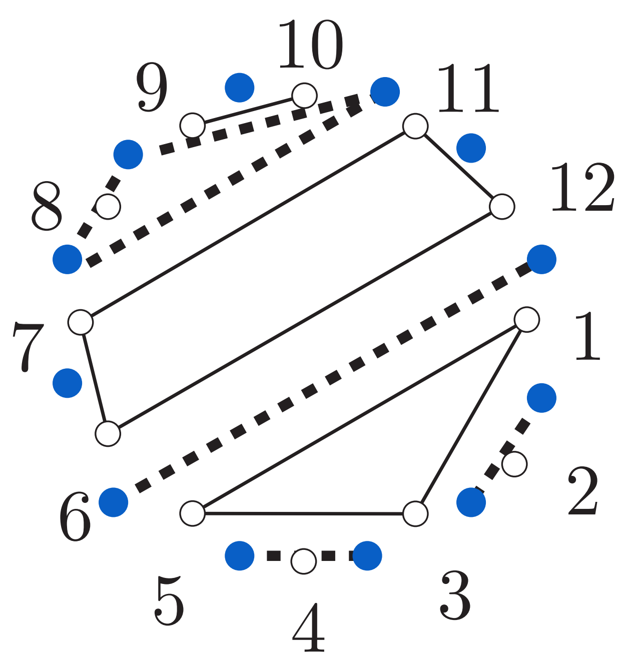

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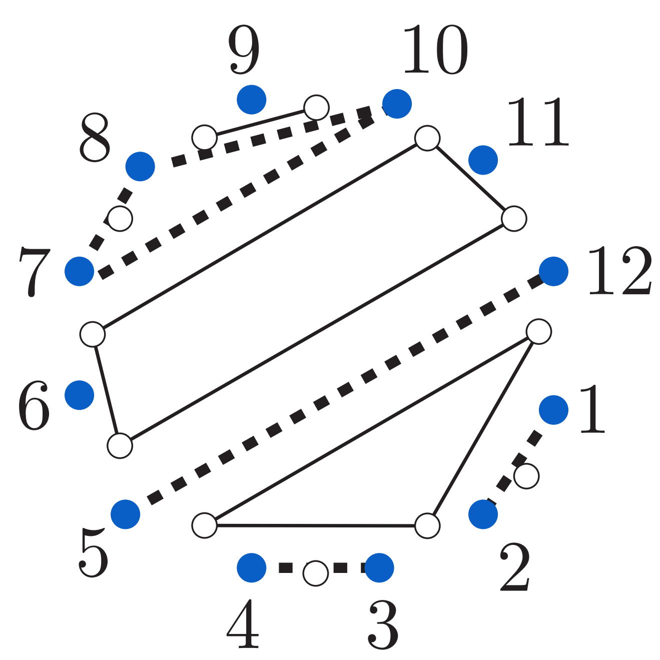

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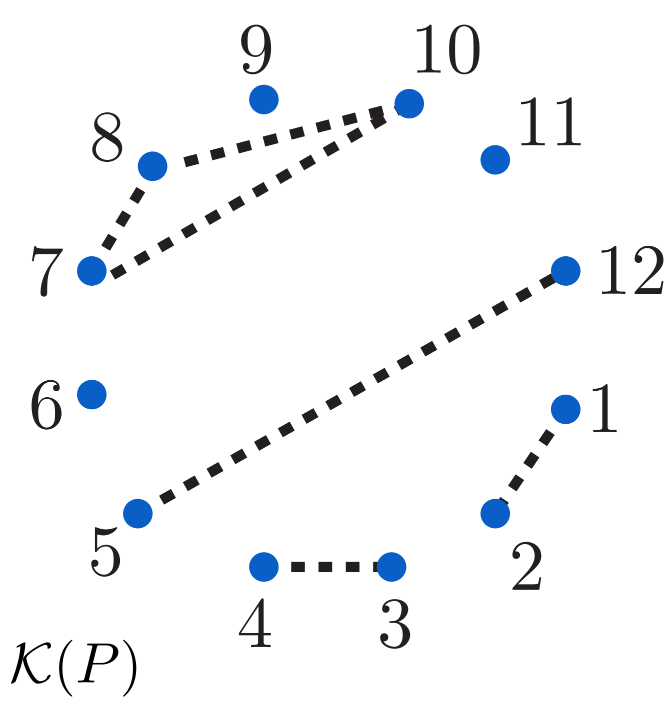

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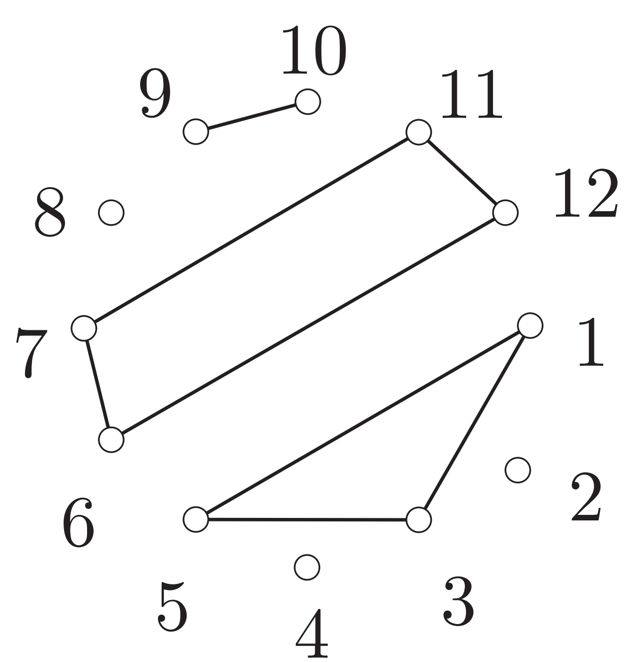

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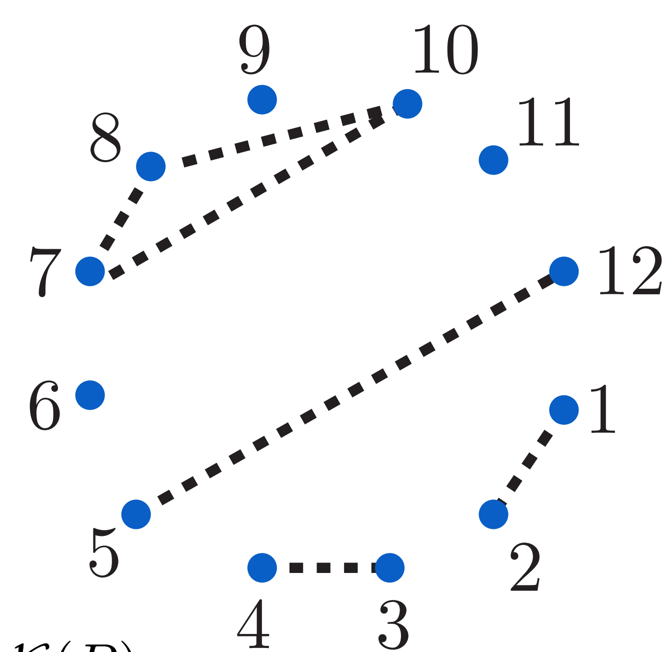

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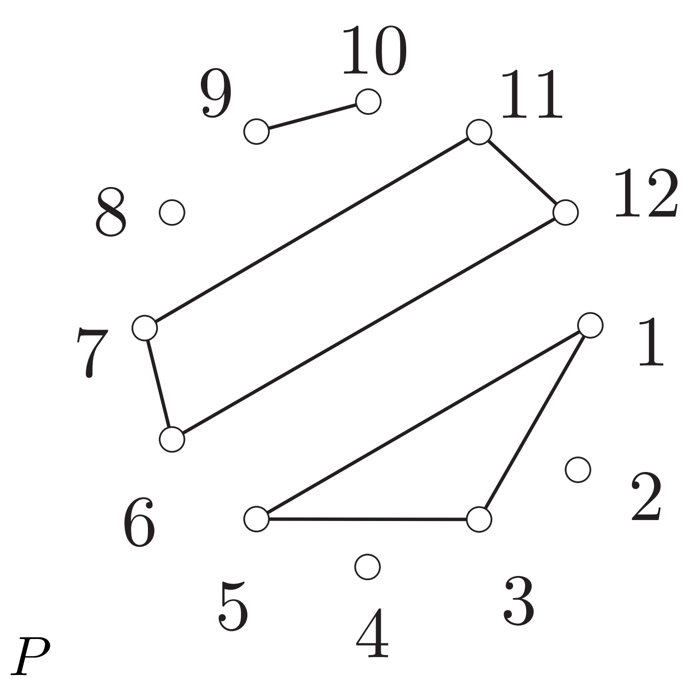

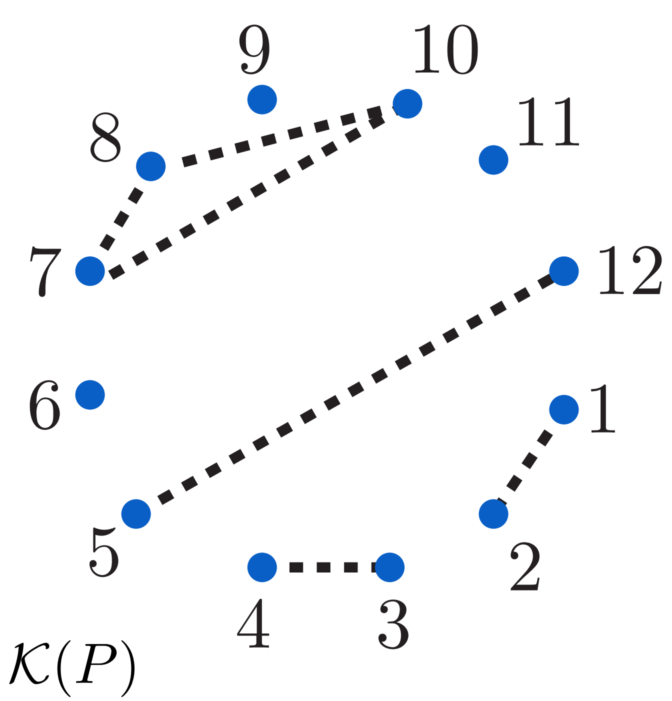

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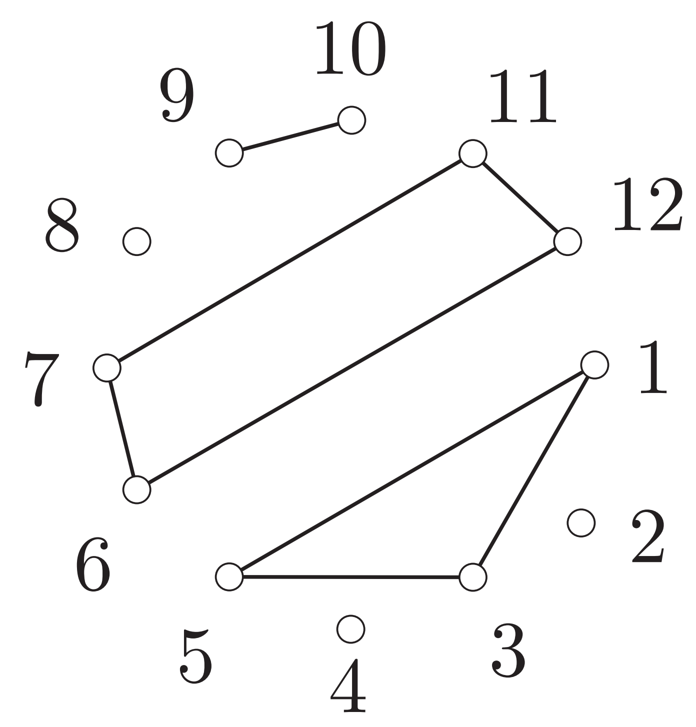

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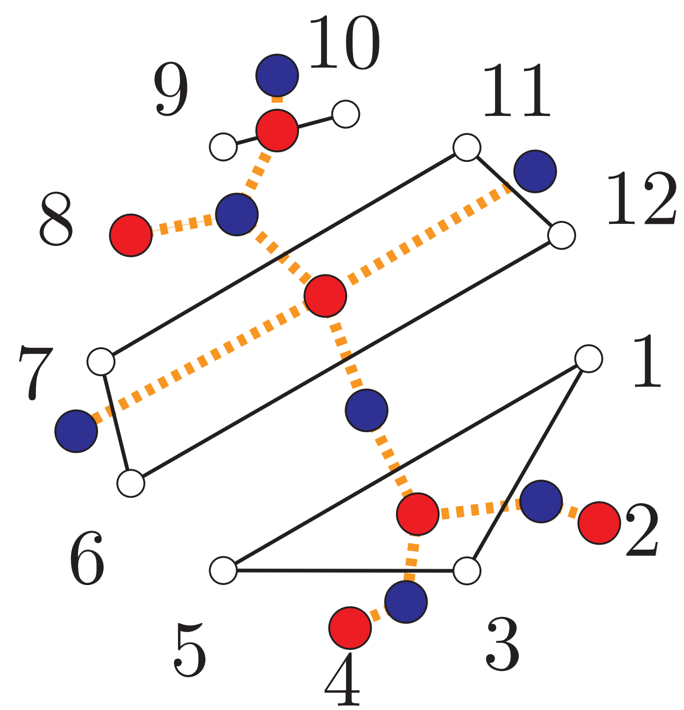

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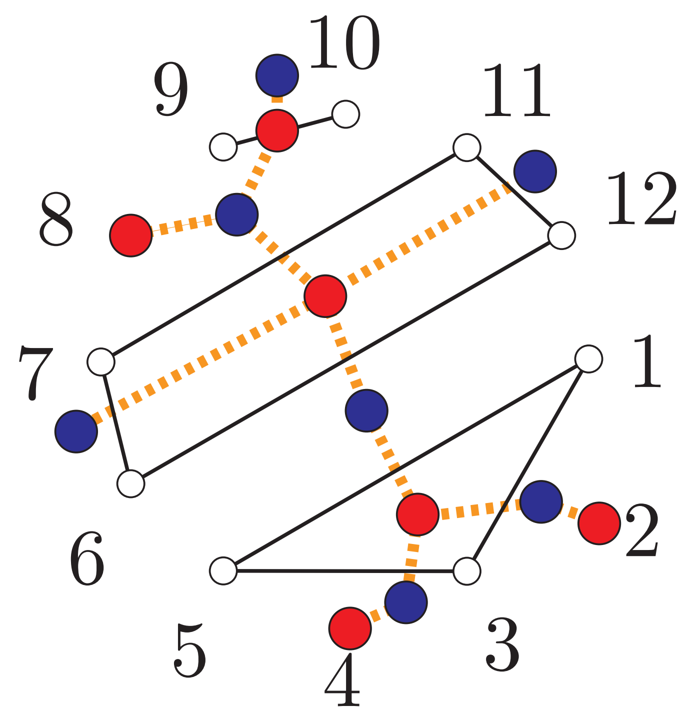

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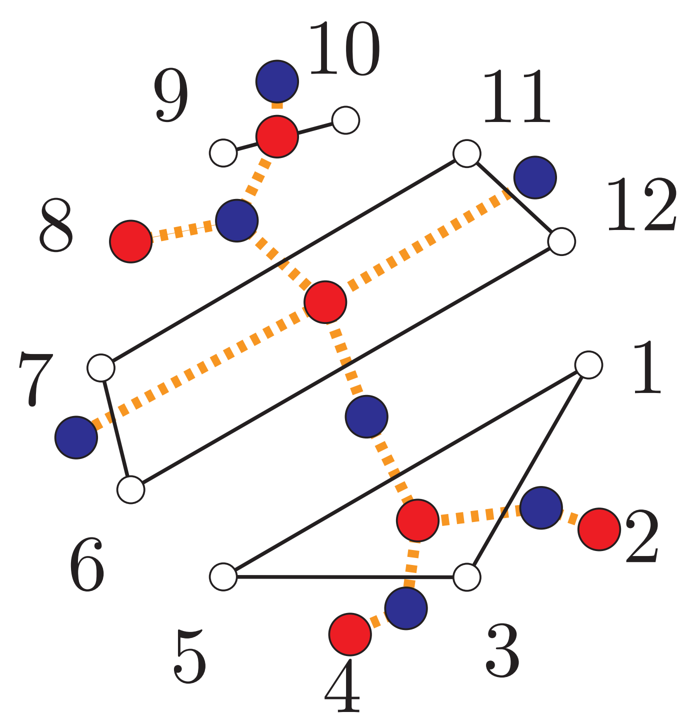

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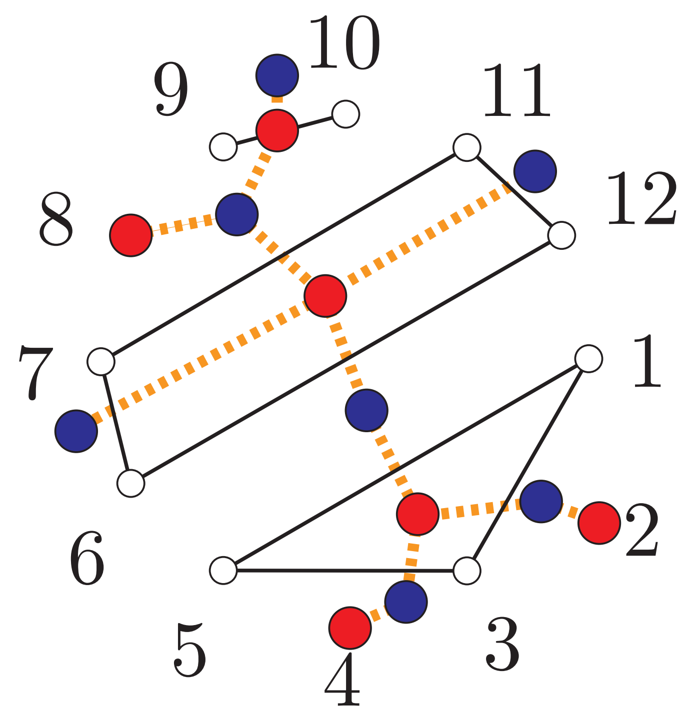

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We develop a new machinery to study limits of such random trees.

Concluding remarks

$\leadsto$ We obtain the convergence of the one-dimensional marginal distributions of the process $\left(\mathcal{F}^n_{\lfloor c\sqrt{n} \rfloor}\right)_{c \geqslant 0}$ by studying bi-conditioned bi-type BGW trees.

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$\rightsquigarrow$ Factorization models have a very rich combinatorics (Hurwitz numbers, maps, genomics), but few probabilistic results.