Graph discrepancy

István Tomon

Umeå University

joint work with Eero Räty and Benny Sudakov

What is **discrepancy**?

What is **discrepancy**?

The **maximum deviation** from the **expected value**.

What is **discrepancy**?

The **maximum deviation** from the **expected value**.

Example: **discrepancy** of n points $P \subset [0,1]^2$ is

$$\text{disc}(P) = \max_{\substack{R \subset [0,1] \\ R \text{ rectangle}}} \left| |R \cap P| - n \cdot \text{area}(R) \right| .$$

What is **discrepancy**?

The **maximum deviation** from the **expected value**.

Example: **discrepancy** of n points $P \subset [0,1]^2$ is

$$\mathrm{disc}(P) = \max_{\substack{R \subset [0,1] \\ R \text{ rectangle}}} \left| |R \cap P| - n \cdot \mathrm{area}(R) \right| .$$

Problem. What is $f(n) = \min_{|P|=n} \mathrm{disc}(P)$?

What is **discrepancy**?

The **maximum deviation** from the **expected value**.

Example: **discrepancy** of n points $P \subset [0,1]^2$ is

$$\mathrm{disc}(P) = \max_{\substack{R \subset [0,1] \\ R \text{ rectangle}}} \left| |R \cap P| - n \cdot \mathrm{area}(R) \right| .$$

Problem. What is $f(n) = \min_{|P|=n} \mathrm{disc}(P)$?

- If P is **random**, then $\mathrm{disc}(P) \approx \sqrt{n}$.

What is **discrepancy**?

The **maximum deviation** from the **expected value**.

Example: **discrepancy** of n points $P \subset [0,1]^2$ is

$$\operatorname{disc}(P) = \max_{\substack{R \subset [0,1] \\ R \text{ rectangle}}} \left| |R \cap P| - n \cdot \operatorname{area}(R) \right| .$$

Problem. What is $f(n) = \min_{|P|=n} \operatorname{disc}(P)$?

- If P is **random**, then $\operatorname{disc}(P) \approx \sqrt{n}$.
- $f(n) = \Theta(\log n)$.

G is a graph on n vertices of density $p = e(G)/\binom{n}{2}$.

G is a graph on n vertices of density $p = e(G)/\binom{n}{2}$.

Discrepancy of G:

$$\mathrm{disc}(G) = \max_{U \subset V(G)} \left| e(G[U]) - p\binom{|U|}{2} \right|.$$

G is a graph on n vertices of density $p = e(G)/\binom{n}{2}$.

Discrepancy of G:

$$\mathrm{disc}(G) = \max_{U \subset V(G)} \left| e(G[U]) - p\binom{|U|}{2} \right|.$$

Problem. What is $f_p(n) = \min_G \mathrm{disc}(G)$?

G is a graph on n vertices of density $p = e(G)/\binom{n}{2}$.

Discrepancy of G:

$$\mathrm{disc}(G) = \max_{U \subset V(G)} \left| e(G[U]) - p\binom{|U|}{2} \right|.$$

Problem. What is $f_p(n) = \min_G \mathrm{disc}(G)$?

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then $\mathbf{G}(n, p)$ has discrepancy $\approx p^{1/2} n^{3/2}$.

G is a graph on n vertices of density $p = e(G)/\binom{n}{2}$.

Discrepancy of G:

$$\mathrm{disc}(G) = \max_{U \subset V(G)} \left| e(G[U]) - p\binom{|U|}{2} \right|.$$

Problem. What is $f_p(n) = \min_G \mathrm{disc}(G)$?

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then $\mathbf{G}(n,p)$ has discrepancy $\approx p^{1/2}n^{3/2}$.

Theorem (Erdős, Goldbach, Pach, Spencer 1988)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then $\mathrm{disc}(G) = \Omega(p^{1/2}n^{3/2})$.

Positive discrepancy:

$$\mathrm{disc}^+(G) = \max_{U \subset V(G)} e(G[U]) - p\binom{|U|}{2}.$$

Positive discrepancy:

$$\operatorname{disc}^+(G) = \max_{U \subset V(G)} e(G[U]) - p\binom{|U|}{2}.$$

Negative discrepancy:

$$\operatorname{disc}^-(G) = \max_{U \subset V(G)} p\binom{|U|}{2} - e(G[U]).$$

Positive discrepancy:

$$\mathrm{disc}^+(G) = \max_{U \subset V(G)} e(G[U]) - p\binom{|U|}{2}.$$

Negative discrepancy:

$$\mathrm{disc}^-(G) = \max_{U \subset V(G)} p\binom{|U|}{2} - e(G[U]).$$

Problem. What is $f_p^+(n) = \min_G \mathrm{disc}^+(G)$ and similarly $f_p^-(n)$?

Positive discrepancy:

$$\text{disc}^+(G) = \max_{U \subset V(G)} e(G[U]) - p\binom{|U|}{2}.$$

Negative discrepancy:

$$\text{disc}^-(G) = \max_{U \subset V(G)} p\binom{|U|}{2} - e(G[U]).$$

Problem. What is $f_p^+(n) = \min_G \text{disc}^+(G)$ and similarly $f_p^-(n)$?

Examples:

- $\text{disc}^+(\mathbf{G}(n,p)) = \Theta(p^{1/2}n^{3/2}) = \text{disc}^-(\mathbf{G}(n,p))$.

Positive discrepancy:

$$\mathrm{disc}^+(G) = \max_{U \subset V(G)} e(G[U]) - p\binom{|U|}{2}.$$

Negative discrepancy:

$$\mathrm{disc}^-(G) = \max_{U \subset V(G)} p\binom{|U|}{2} - e(G[U]).$$

Problem. What is $f_p^+(n) = \min_G \mathrm{disc}^+(G)$ and similarly $f_p^-(n)$?

Examples:

- $\mathrm{disc}^+(\mathbf{G}(n,p)) = \Theta(p^{1/2}n^{3/2}) = \mathrm{disc}^-(\mathbf{G}(n,p))$.
- $\mathrm{disc}^+(\mathbf{K_{n,n}}) = \Theta(n)$ and $\mathrm{disc}^-(\mathbf{K_{n,n}}) = \Theta(n^2)$.

Theorem (Bollobás, Scott 2006)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then

- $\mathrm{disc}^+(G) \cdot \mathrm{disc}^-(G) = \Omega(pn^3)$.

Theorem (Bollobás, Scott 2006)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then

- $\mathrm{disc}^+(G) \cdot \mathrm{disc}^-(G) = \Omega(pn^3)$.
- $\mathrm{disc}^+(G) = \Omega(n)$ and $\mathrm{disc}^-(G) = \Omega(n)$,

Theorem (Bollobás, Scott 2006)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then

- $\mathrm{disc}^{+}(G) \cdot \mathrm{disc}^{-}(G) = \Omega(pn^3)$.
- $\mathrm{disc}^{+}(G) = \Omega(n)$ and $\mathrm{disc}^{-}(G) = \Omega(n)$,
  Turán graphs have $\mathrm{disc}^{+} = \Theta(n)$.

Theorem (Bollobás, Scott 2006)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then

- $\text{disc}^{+}(G) \cdot \text{disc}^{-}(G) = \Omega(pn^3)$.
- $\text{disc}^{+}(G) = \Omega(n)$ and $\text{disc}^{-}(G) = \Omega(n)$,
  Turán graphs have $\text{disc}^{+} = \Theta(n)$.

$f^{+}_{1/2}(n) = \Theta(n)$

Theorem (Bollobás, Scott 2006)

If $p \in [\frac{1}{n}, \frac{1}{2}]$, then

- $\mathrm{disc}^{+}(G) \cdot \mathrm{disc}^{-}(G) = \Omega(pn^3)$.
- $\mathrm{disc}^{+}(G) = \Omega(n)$ and $\mathrm{disc}^{-}(G) = \Omega(n)$,
  Turán graphs have $\mathrm{disc}^{+} = \Theta(n)$.

$f^{+}_{1/2}(n) = \Theta(n)$

Conjecture (Verstraete). If $\frac{1}{n} \leq p \leq \frac{1}{2} - \varepsilon$, then

$$\mathrm{disc}^{+}(G) = \Omega(p^{1/2} n^{3/2}).$$

MaxCut in a graph is the maximum number of edges in a **cut**.

MaxCut in a graph is the maximum number of edges in a **cut**.

Minimum bisection is the min. size of a cut into 2 equal parts.

MaxCut in a graph is the maximum number of edges in a **cut**.

Minimum bisection is the min. size of a cut into 2 equal parts.

Lemma

If G is regular, then

- $\textbf{MaxCut} = \frac{e(G)}{2} + \Theta(\mathrm{disc}^-(G))$,

MaxCut in a graph is the maximum number of edges in a **cut**.

Minimum bisection is the min. size of a cut into 2 equal parts.

Lemma

If G is regular, then

- **MaxCut** $= \frac{e(G)}{2} + \Theta(\text{disc}^{-}(G))$,
- **Minimum bisection** $= \frac{e(G)}{2} - \Theta(\text{disc}^{+}(G))$.

MaxCut in a graph is the maximum number of edges in a **cut**.

Minimum bisection is the min. size of a cut into 2 equal parts.

Lemma

If G is regular, then

- **MaxCut** $= \frac{e(G)}{2} + \Theta(\mathrm{disc}^-(G))$,
- **Minimum bisection** $= \frac{e(G)}{2} - \Theta(\mathrm{disc}^+(G))$.

Theorem (Alon 1993)

If G is d-regular and $d = O(n^{1/9})$, then

$$\textbf{Minimum bisection} < \frac{dn}{4} - cd^{1/2}n.$$

MaxCut in a graph is the maximum number of edges in a **cut**.

Minimum bisection is the min. size of a cut into 2 equal parts.

Lemma

If G is regular, then

- **MaxCut** $= \frac{e(G)}{2} + \Theta(\text{disc}^-(G))$,
- **Minimum bisection** $= \frac{e(G)}{2} - \Theta(\text{disc}^+(G))$.

Theorem (Alon 1993)

If G is d-regular and $d = O(n^{1/9})$, then

$$\textbf{Minimum bisection} < \frac{dn}{4} - cd^{1/2}n.$$

Equivalently, $\text{disc}^+(G) = \Omega(d^{1/2}n) = \Omega(p^{1/2}n^{3/2})$.

Theorem (Räty, Sudakov, T. 2023+)

The minimum **positive discrepancy** among n vertex graphs of average degree d is

Theorem (Räty, Sudakov, T. 2023+)

The minimum **positive discrepancy** among n vertex graphs of average degree d is

- $\Theta(d^{1/2}n)$ if $d \in [1, n^{2/3}]$

Theorem (Räty, Sudakov, T. 2023+)

The minimum **positive discrepancy** among n vertex graphs of average degree d is

- $\Theta(d^{1/2}n)$ if $d \in [1, n^{2/3}]$
- $\Theta(n^2/d)$ if $d \in [n^{2/3}, n^{3/4}]$

Theorem (Räty, Sudakov, T. 2023+)

The minimum **positive discrepancy** among n vertex graphs of average degree d is

- $\Theta(d^{1/2}n)$ if $d \in [1, n^{2/3}]$
- $\Theta(n^2/d)$ if $d \in [n^{2/3}, n^{3/4}]$
- at least $\Omega(n^2/d)$ if $d \in [n^{3/4}, n^{4/5}]$

Theorem (Räty, Sudakov, T. 2023+)

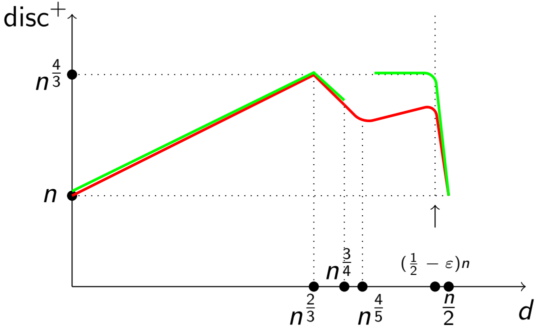
The minimum **positive discrepancy** among n vertex graphs of average degree d is

- $\Theta(d^{1/2}n)$ if $d \in [1, n^{2/3}]$
- $\Theta(n^2/d)$ if $d \in [n^{2/3}, n^{3/4}]$
- at least $\Omega(n^2/d)$ if $d \in [n^{3/4}, n^{4/5}]$
- between $\tilde{\Omega}(d^{1/4}n)$ and $O(n^{4/3})$ if $d \in [n^{4/5}, (\frac{1}{2} - \varepsilon)n]$

Theorem (Räty, Sudakov, T. 2023+)

The minimum **positive discrepancy** among n vertex graphs of average degree d is

- $\Theta(d^{1/2}n)$ if $d \in [1, n^{2/3}]$
- $\Theta(n^2/d)$ if $d \in [n^{2/3}, n^{3/4}]$
- at least $\Omega(n^2/d)$ if $d \in [n^{3/4}, n^{4/5}]$
- between $\tilde{\Omega}(d^{1/4}n)$ and $O(n^{4/3})$ if $d \in [n^{4/5}, (\frac{1}{2} - \varepsilon)n]$



Lemma

If G is d-regular with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\mathrm{disc}^+(G) \leq \frac{\lambda_2}{2} n + d.$$

Proof. Let A be the adjacency matrix, $v_1, \ldots, v_n$ an orthonormal basis.

Lemma

If G is d-regular with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\mathrm{disc}^+(G) \leq \frac{\lambda_2}{2} n + d.$$

Proof. Let A be the adjacency matrix, $v_1, \ldots, v_n$ an orthonormal basis.
If $U \subset V(G)$ and x is the characteristic vector, then

$$e(G[U]) - p\binom{|U|}{2} = \frac{1}{2}\left(x^T A x - p x^T (J - I) x\right).$$

Lemma

If G is d-regular with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\text{disc}^+(G) \leq \frac{\lambda_2}{2} n + d.$$

Proof. Let A be the adjacency matrix, $v_1, \ldots, v_n$ an orthonormal basis.

If $U \subset V(G)$ and x is the characteristic vector, then

$$e(G[U]) - p\binom{|U|}{2} = \frac{1}{2}\left(x^T A x - p x^T (J - I) x\right).$$

Writing $x = \sum a_i v_i$, we have

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i a_i^2 - p n a_1^2 + p||x||_2^2\right)$$

Lemma

If G is d-regular with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\text{disc}^+(G) \leq \frac{\lambda_2}{2} n + d.$$

Proof. Let A be the adjacency matrix, $v_1, \ldots, v_n$ an orthonormal basis.
If $U \subset V(G)$ and x is the characteristic vector, then

$$e(G[U]) - p\binom{|U|}{2} = \frac{1}{2}\left(x^T A x - p x^T (J - I) x\right).$$

Writing $x = \sum a_i v_i$, we have

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i a_i^2 - p n a_1^2 + p||x||_2^2\right)$$

$$\leq \frac{1}{2}\lambda_2 \sum_{i=2}^{n} a_i^2 + \frac{1}{2} p||x||_2^2$$

Lemma

If G is d-regular with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\mathrm{disc}^+(G) \leq \frac{\lambda_2}{2} n + d.$$

Proof. Let A be the adjacency matrix, $v_1, \ldots, v_n$ an orthonormal basis.

If $U \subset V(G)$ and x is the characteristic vector, then

$$e(G[U]) - p\binom{|U|}{2} = \frac{1}{2}\left(x^T A x - p x^T (J - I) x\right).$$

Writing $x = \sum a_i v_i$, we have

$$= \frac{1}{2}\left(\sum_{i=1}^{n} \lambda_i a_i^2 - p n a_1^2 + p||x||_2^2\right)$$

$$\leq \frac{1}{2}\lambda_2 \sum_{i=2}^{n} a_i^2 + \frac{1}{2} p ||x||_2^2 \leq \frac{\lambda_2}{2} n + d.$$

Constructions:

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\text{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{4/3} \approx nd^{1/3}$.

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{4/3} \approx nd^{1/3}$.

Alon-Boppana bound. If G has diameter ≥ 4, then $\lambda_2 \geq d^{1/2}$.

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\text{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

This gives a graph with $\text{disc}^+(G) \approx n^{4/3} \approx nd^{1/3}$.

Alon-Boppana bound. If G has diameter ≥ 4, then $\lambda_2 \geq d^{1/2}$.

Theorem (Balla, Räty, Sudakov, T. 2023+)

The minimum of λ_2 among n-vertex d-regular graphs is

- $\Theta(d^{1/2})$ if $d \in [1, n^{2/3}]$

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

This gives a graph with $\mathrm{disc}^+(G) \approx n^{4/3} \approx nd^{1/3}$.

Alon-Boppana bound. If G has diameter ≥ 4, then $\lambda_2 \geq d^{1/2}$.

Theorem (Balla, Räty, Sudakov, T. 2023+)

The minimum of λ_2 among n-vertex d-regular graphs is

- $\Theta(d^{1/2})$ if $d \in [1, n^{2/3}]$
- $\Theta(n/d)$ if $d \in [n^{2/3}, n^{3/4}]$

Constructions:

(i) $\exists$ *strongly-regular graph* with $d \approx n^{3/4}$ and $\lambda_2 \approx n^{1/4}$.

This gives a graph with $\text{disc}^+(G) \approx n^{5/4} \approx n^2/d$.

(ii) $\exists$ *strongly-regular graph* with $d \approx \frac{1}{3}n$ and $\lambda_2 \approx n^{1/3}$.

This gives a graph with $\text{disc}^+(G) \approx n^{4/3} \approx nd^{1/3}$.

Alon-Boppana bound. If G has diameter ≥ 4, then $\lambda_2 \geq d^{1/2}$.

Theorem (Balla, Räty, Sudakov, T. 2023+)

The minimum of λ_2 among n-vertex d-regular graphs is

- $\Theta(d^{1/2})$ if $d \in [1, n^{2/3}]$
- $\Theta(n/d)$ if $d \in [n^{2/3}, n^{3/4}]$
- between $\Omega(d^{1/3})$ and $O(n^{1/3})$ if $d \in [n^{3/4}, (\frac{1}{2} - \varepsilon)n]$.

Theorem (baby version)

If G is d-regular and $d \ll n^{2/3}$, then

$$\mathrm{disc}^+(G) = \Omega(d^{1/2} n).$$

Theorem (baby version)

If G is d-regular and $d \ll n^{2/3}$, then

$$\mathrm{disc}^+(G) = \Omega(d^{1/2} n).$$

Proof. For each $v \in V(G)$, **assign the vector** $x_v \in \mathbb{R}^{V(G)}$ defined

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, \\ 0 & \text{if } v \not\sim w. \end{cases}.$$

Theorem (baby version)

If G is d-regular and $d \ll n^{2/3}$, then

$$\mathrm{disc}^{+}(G) = \Omega(d^{1/2}n).$$

Proof. For each $v \in V(G)$, **assign the vector** $x_v \in \mathbb{R}^{V(G)}$ defined

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, \\ 0 & \text{if } v \not\sim w. \end{cases}.$$

Let H be a **random half-space** in $\mathbb{R}^{V(G)}$ through the origin, and let U be the set of vertices v such that $x_v \in H$.

Theorem (baby version)

If G is d-regular and $d \ll n^{2/3}$, then

$$\mathrm{disc}^+(G) = \Omega(d^{1/2} n).$$

Proof. For each $v \in V(G)$, **assign the vector** $x_v \in \mathbb{R}^{V(G)}$ defined

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, \\ 0 & \text{if } v \not\sim w. \end{cases}.$$

Let H be a **random half-space** in $\mathbb{R}^{V(G)}$ through the origin, and let U be the set of vertices v such that $x_v \in H$.

Observation: For $u, v \in V(G)$,

$$\mathbb{P}(u, v \in U) = \frac{1}{4} + \Theta(\langle x_u, x_v \rangle).$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

Claim.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right] = \Omega(d^{1/2}n).$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

Claim.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right] = \Omega(d^{1/2}n).$$

Proof.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right]$$

$$\approx \sum_{u \sim v} \langle x_u, x_v \rangle - \frac{d}{n} \sum_{u,v} \langle x_u, x_v \rangle$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

Claim.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right] = \Omega(d^{1/2}n).$$

Proof.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right]$$

$$\approx \sum_{u\sim v}\langle x_u, x_v\rangle - \frac{d}{n}\sum_{u,v}\langle x_u, x_v\rangle$$

$$> \sum_{u\sim v}\frac{1}{\sqrt{d}} - \frac{d}{n}\sum_{u,v}\sum_{w\sim u,v}\frac{1}{d}$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

Claim.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right] = \Omega(d^{1/2}n).$$

Proof.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right]$$

$$\approx \sum_{u\sim v}\langle x_u, x_v\rangle - \frac{d}{n}\sum_{u,v}\langle x_u, x_v\rangle$$

$$> \sum_{u\sim v}\frac{1}{\sqrt{d}} - \frac{d}{n}\sum_{u,v}\sum_{w\sim u,v}\frac{1}{d} \approx \sqrt{d}n - d^2.$$

$$x_v(w) = \begin{cases} 1 & \text{if } v = w, \\ \frac{1}{\sqrt{d}} & \text{if } v \sim w, . \\ 0 & \text{if } v \not\sim w. \end{cases}$$

Claim.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right] = \Omega(d^{1/2}n).$$

Proof.

$$\mathbb{E}\left[e(G[U]) - \frac{d}{n-1}\binom{|U|}{2}\right]$$

$$\approx \sum_{u \sim v} \langle x_u, x_v \rangle - \frac{d}{n} \sum_{u,v} \langle x_u, x_v \rangle$$

$$> \sum_{u \sim v} \frac{1}{\sqrt{d}} - \frac{d}{n} \sum_{u,v} \sum_{w \sim u,v} \frac{1}{d} \approx \sqrt{d}n - d^2.$$

$RHS \gtrapprox \sqrt{d}n$ if $d \ll n^{2/3}$. □