Recent Progress in Ramsey Theory

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Outline

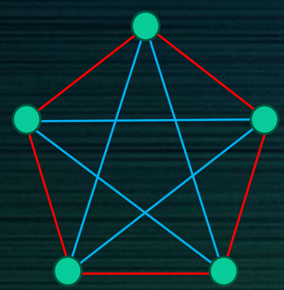
- Classical Ramsey Theory
- Random Graphs
- Pseudorandom Graphs
- Ramsey Numbers
- Random Blocks
- $r(4,t)$
- Erdős-Rogers Functions
- Open Problems

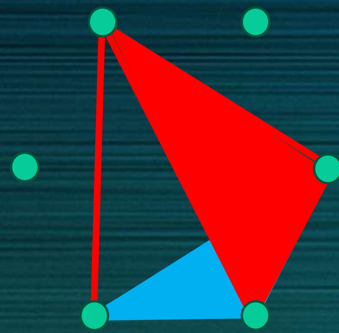
Classical Ramsey Theory

- In any sufficiently large "structure", a relatively large "uniform" substructure exists. "Perfect disorder is mathematically impossible."
- For integers $s,t \geqslant 2$, let $r(s,t)$ denote the minimum n such that every red-blue edge-coloring of K_n contains a red K_s or a blue K_t .
- These are the classical Ramsey numbers.

Classical Ramsey Theory

- Example: $r(2,t) = t$
- Example: $r(3,3) = 6$





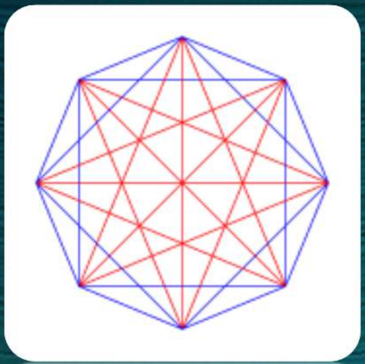
- The only other classical Ramsey numbers $r(3,t)$ known are

$$r(3,4) = 9 \quad r(3,5) = 14 \quad r(3,6) = 18$$
$$r(3,7) = 23 \quad r(3,8) = 36 \quad r(3,9) = 39$$

Classical Ramsey Theory

- The only other known classical Ramsey numbers $r(3,t)$ for $t \geqslant 4$ are:

$$r(3,4) = 9 \quad r(3,5) = 14 \quad r(3,6) = 18$$
$$r(3,7) = 23 \quad r(3,8) = 36 \quad r(3,9) = 39$$



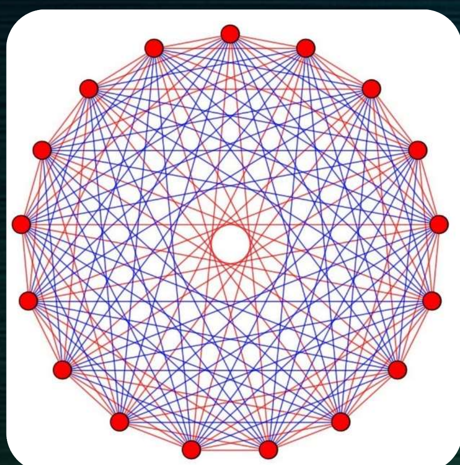
- The only known classical Ramsey numbers $r(4,t)$ for $t \geqslant 4$ are:

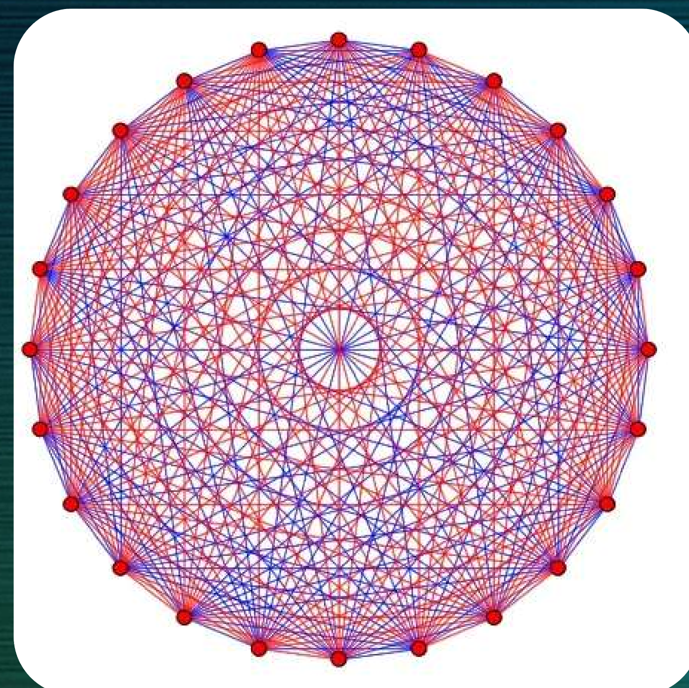
$$r(4,4) = 18 \qquad r(4,5) = 25$$

Classical Ramsey Theory

$r(4,5) = 25$

$r(4,4) = 18$





Classical Ramsey Theory

- Erdős-Szekeres Theorem (1935)

  For $s, t \geqslant 2$,

$$r(s,t) \leqslant r(s-1,t) + r(s,t-1) \leqslant \binom{s+t-2}{s-1} < t^{s-1}.$$

Classical Ramsey Theory

- Theorem (Shearer 1982, Ajtai-Komlós-Szemerédi 1983, Li-Rousseau-Zang 1991)

  For $s > 2$, as $t \to \infty$:

$$r(s,t) \quad \lesssim \quad \frac{t^{s-1}}{(\log t)^{s-2}}.$$

Classical Ramsey Theory

- Maximum sets of points in the plane with no k in convex position.
- Permutations of n letters with no monotone subsequence of length k.
- Unit distance graphs.
- Sets with no three-term arithmetic progressions.
- Embeddings of metric spaces with low distortion.

Classical Ramsey Theory

- Sets of points in the square with no triangles of small area.
- Grid points with no three on a line.
- Roth's Theorem and arithmetic progressions of primes.
- Random graphs, percolation and cellular automata.
- Orchard planting problem.

Classical Ramsey Theory

- Theorem (Kim 1995, Bohman-Keevash 2013, Fiz Pontiveros, Griffiths, Morris 2021)

  As $t \to \infty$:

$$r(3,t) \gtrsim \frac{t^2}{4(\log t)}.$$

Classical Ramsey Theory

For $r(4,n)$, the best lower bound known is $c(n \log n)^{5/2}$ due to Spencer, [33] again by using the Lovász local lemma. The best upper bound known is $c'n^3/\log^2 n$, proved by Ajtai, Komlós and Szemerédi[27]. So there is a nontrivial gap still remaining, as repeatedly pointed out in many problems papers [34] of Erdős.

Problem [19] ($250)
Prove or disprove that

$$r(4,n) > \frac{n^3}{\log^c n} \tag{2.12}$$

for some c, provided n is sufficiently large.

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[31]P. Erdős, On the construction of certain graphs, *J. Comb. Theory* **17** (1966), 149-153

[33]J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Math.* **20** (1977/78), 69–76.

[34]P. Erdős, Problems and results on graphs and hypergraphs: similarities and differences, *Mathematics of Ramsey theory, Algorithms Combin., 5*, (J. Nešetřil and V. Rödl, eds.), 12–28, Springer, Berlin, 1990.

[36]P. Erdős, Some new problems and results in graph theory and other branches of combinatorial mathematics, *Combinatorics and graph theory (Calcutta, 1980), Lecture Notes in Math., 885*, 9–17, Springer, Berlin-New York, 1981.

Classical Ramsey Theory

- Theorem (Mattheus-V, Ann. Math. 2024)

  As $t \to \infty$:

$$r(4,t) \gtrsim \frac{t^3}{(64\log t)^4}.$$

- The proof of this theorem illustrates the philosophy that good Ramsey graphs "hide" inside pseudorandom graphs.

Media

Quanta Magazine:

- https://www.quantamagazine.org/mathematicians-discover-new-way-to-predict-structure-in-graphs-20230622/

UC San Diego Today

- https://today.ucsd.edu/story/ramsey-problems

Carnegie Mellon MCS News

- https://www.cmu.edu/math/news-events/articles/20230712_random-algorithms-math-conference.html

SIAM News

- https://sinews.siam.org/Details-Page/off-diagonal-ramsey-numbers-from-pseudorandom-graphs

Belgian HLN Newspaper

- https://www.hln.be/binnenland/jonge-vlaming-zorgt-voor-doorbraak-in-wiskunde-sam-mattheus-29-kraakt-aartsmoeilijke-code~a8b931d8/

The Brussels Times Newspaper:

- https://www.brusselstimes.com/brussels-2/610175/young-vub-researcher-solves-decades-old-maths-problem

Media

Quanta magazine

UC TODAY

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Mathematics

Breakthrough in fiendishly hard puzzle

The key to a successful party is a good mix of people

fight off an alien invasion. Now mathematicians have made the first major advance in nearly a century

The key to a successful party is a good mix of people

Random graphs

- Erdős and Rényi defined the Bernouilli or mean field model of random graphs G_n where the edges of the complete n-vertex graph are sampled independently with probability $1/2$.

- An event A_n occurs a.a.s (asymptotically almost surely) if

$$\Pr(A_n) \to 1 \quad \text{as} \quad n \to \infty.$$

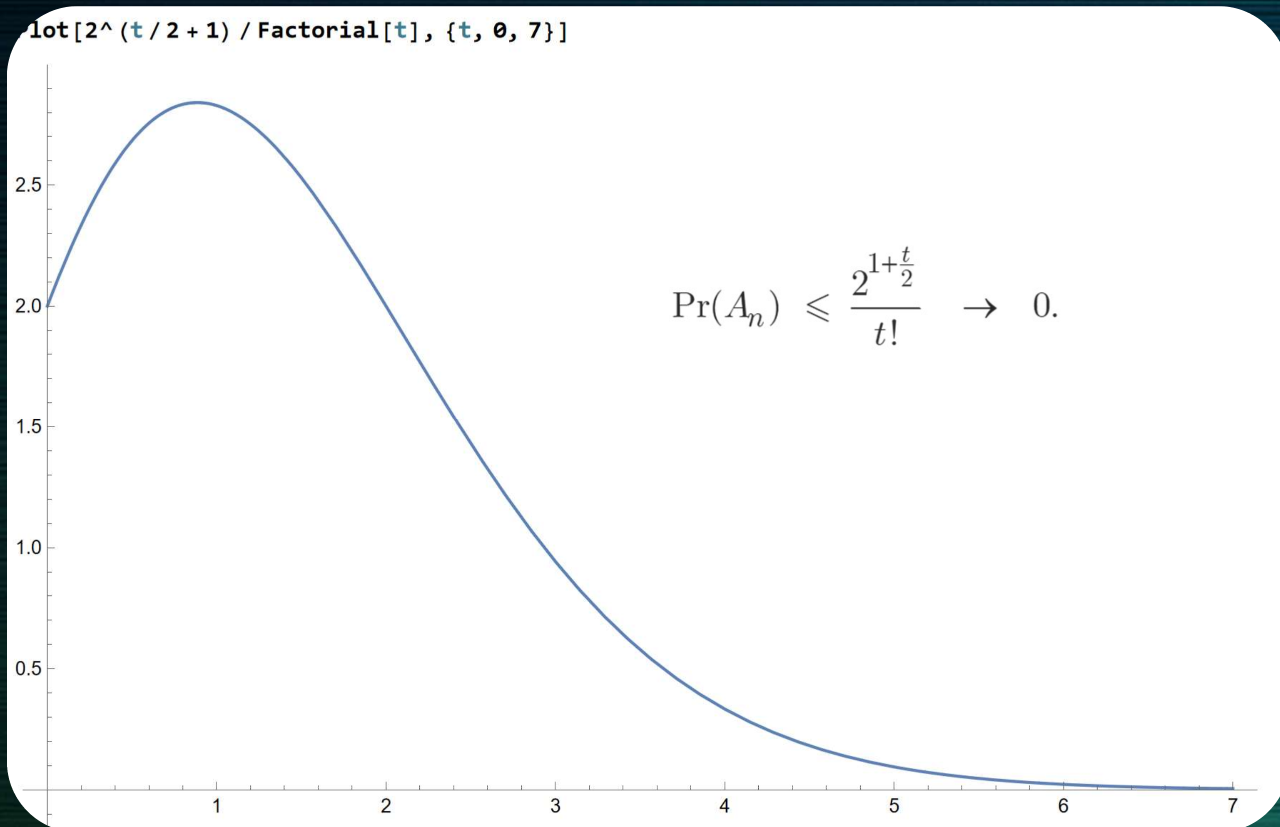
Random graphs

- Let A_n be the event that $G = G_n$ has no clique or independent set of size t.

- If $n > \sqrt{2}^t$

$$
\begin{aligned}
\Pr(A_n) &\leqslant 2 \cdot 2^{-\binom{t}{2}} \cdot \binom{n}{t} \\
&\leqslant 2^{1-\frac{1}{2}t(t-1)} \cdot \frac{n^t}{t!} \\
&\leqslant 2^{1-\frac{1}{2}t(t-1)} \cdot \frac{\sqrt{2}^t}{t!} < \frac{2^{1+\frac{t}{2}}}{t!} \to 0.
\end{aligned}
$$

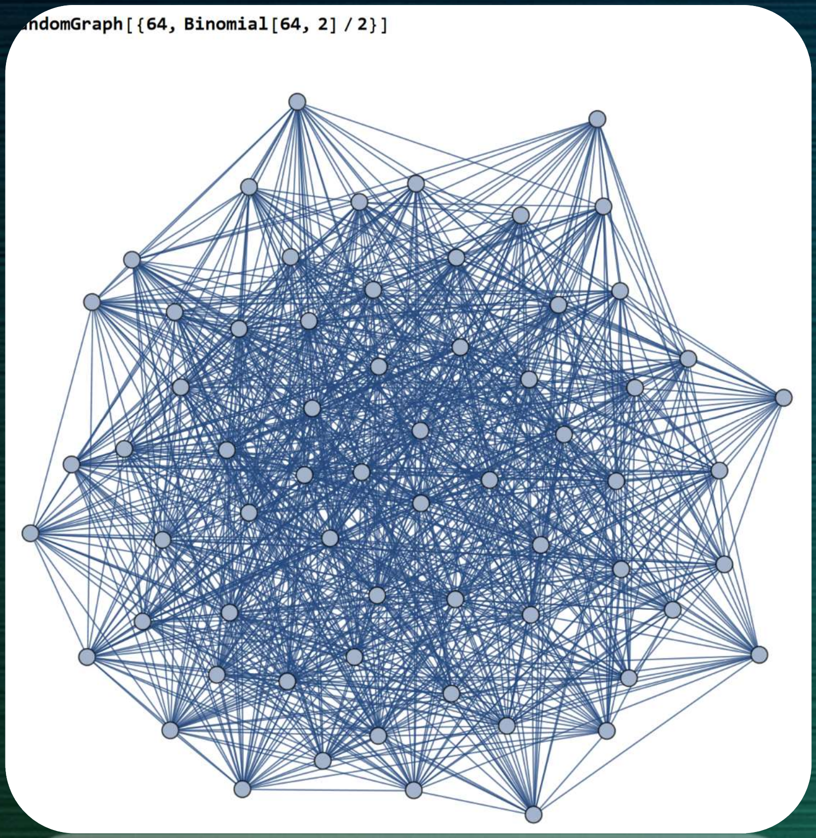
Random graphs



$$\Pr(A_n) \leqslant \frac{2^{1+\frac{t}{2}}}{t!} \rightarrow 0.$$

Random graphs

- Therefore there is a graph with $n > \sqrt{2}^t$ vertices and no clique or independent set of size t.



Random graphs

- We can do better (deletion method): delete one vertex from each clique or independent set of size t so that the average number of vertices left is at least

$$n - 2 \cdot 2^{-\binom{t}{2}} \cdot \binom{n}{t}$$

- Maximize this over n to get an N-vertex graph with no clique or independent set of size t, where

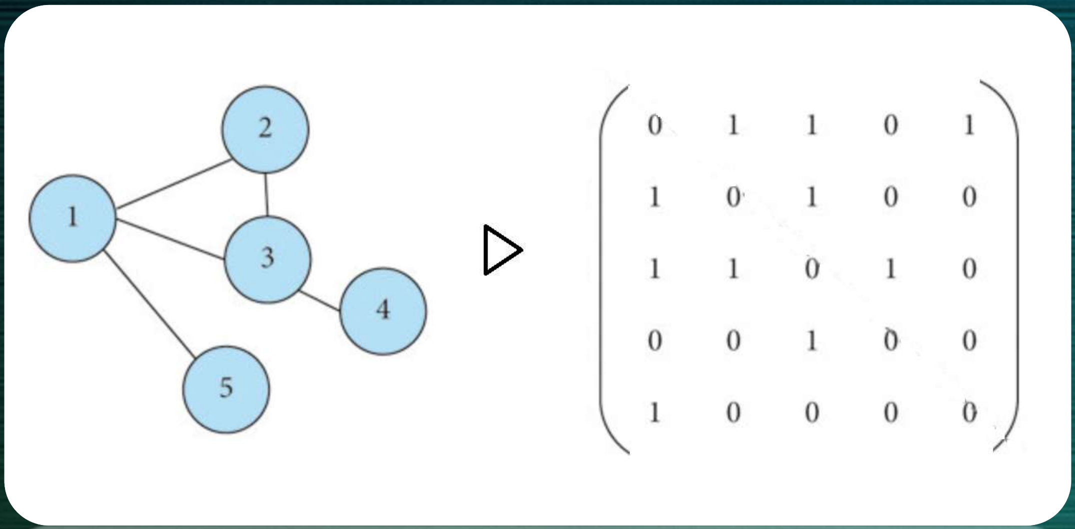
$$N > \frac{t}{e} \cdot \sqrt{2}^{t}$$

Pseudorandom graphs

- Any graph G has an adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{if } ij \notin E(G) \end{cases}$$

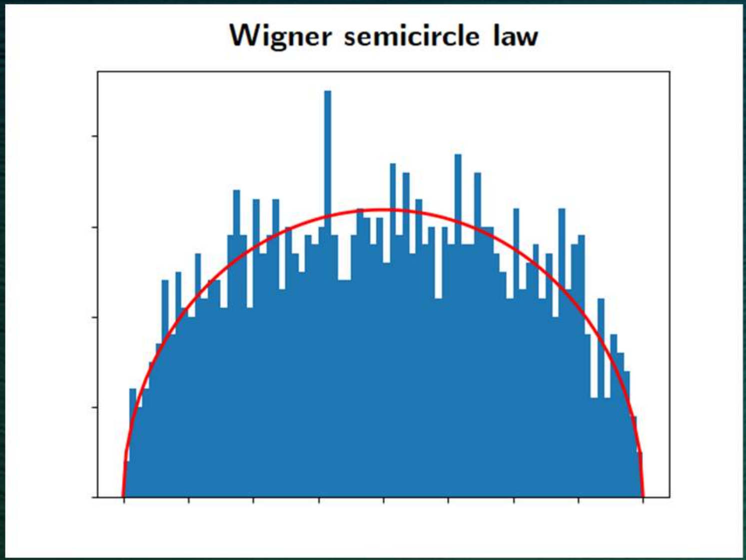
- For example,



Pseudorandom graphs

- The adjacency matrix has eigenvalues that are real, since it is symmetric.

- In the case of random graphs, Wigner's semicircle law shows they look like:



Pseudorandom graphs

- If the eigenvalues are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and G is d-regular and

$$\lambda = \max\{|\lambda_i| : i > 1\}$$

then G is λ-pseudorandom.

- Alon-Boppana Theorem

$$\lambda \;>\; \left(2 - \frac{1}{\lfloor \mathrm{diam}(G)/2 \rfloor}\right) \cdot \sqrt{d-1}.$$

- The infinite d-ary tree is the universal cover of d-regular graphs.

Pseudorandom graphs

- If X is any set of in a λ-pseudorandom graph, then

$$2e(X) = \langle Ax, x \rangle$$

- If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of eigenvectors let

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

Pseudorandom graphs

- Then

$$2e(X) = \langle Ax, x \rangle = \sum_{i=1}^{n} \lambda_i x_i^2$$

- Recalling $\lambda_1 = d$

$$\left| 2e(X) - dx_1^2 \right| = \left| \sum_{i=2}^{n} \lambda_i x_i^2 \right|$$

Pseudorandom graphs

- Since the first eigenvector is constant,

$$x_1 = \langle x, e_1 \rangle = \frac{|X|}{\sqrt{n}}$$

- Finally,

$$\left|2e(X) - \tfrac{d}{n}|X|^2\right| \leqslant \lambda \left|\sum_{i=1}^{n} x_i^2\right| \leqslant \lambda |X|.$$

Pseudorandom graphs

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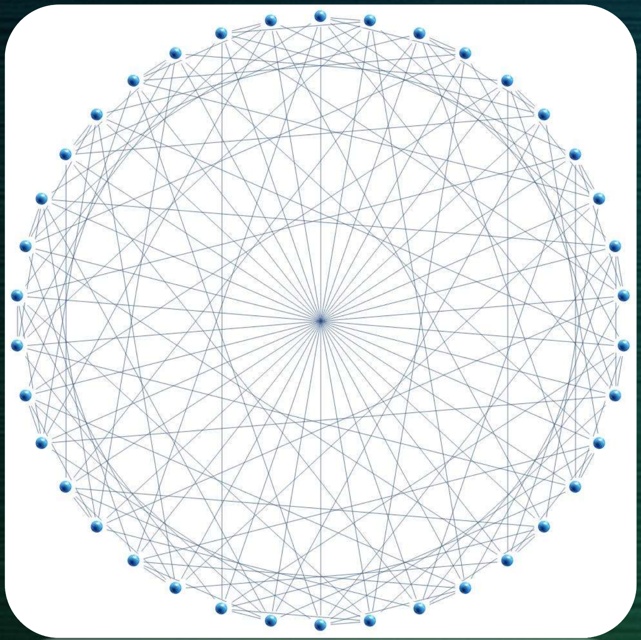

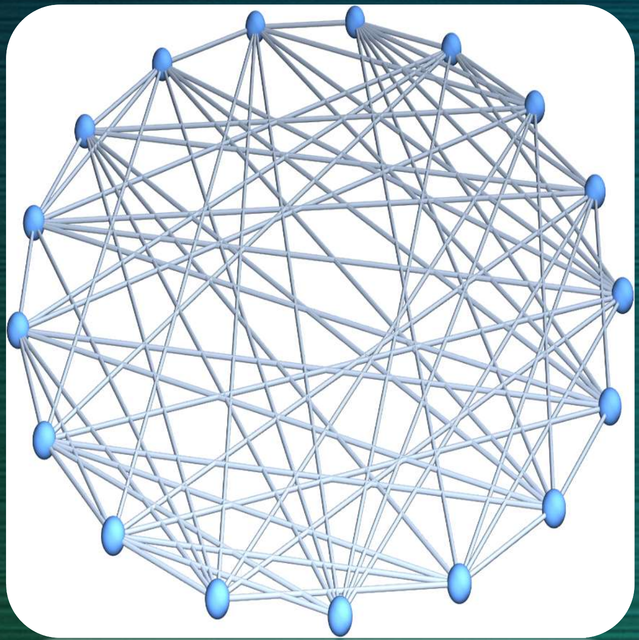

Pseudorandom graphs

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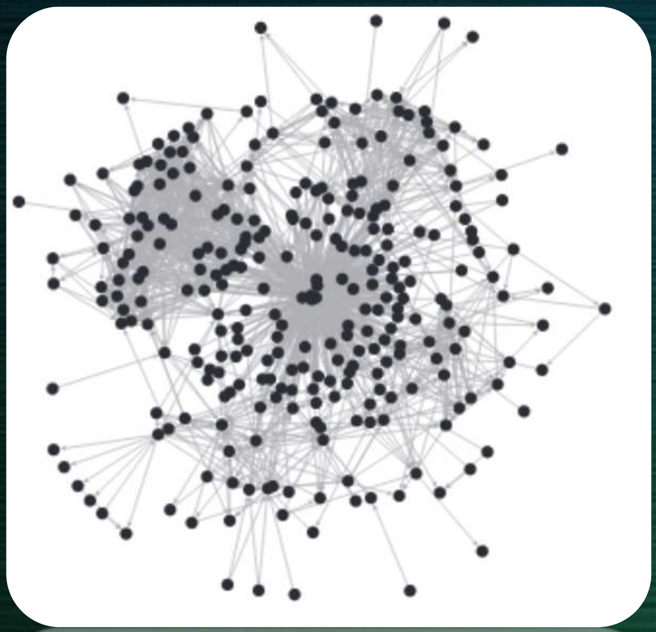
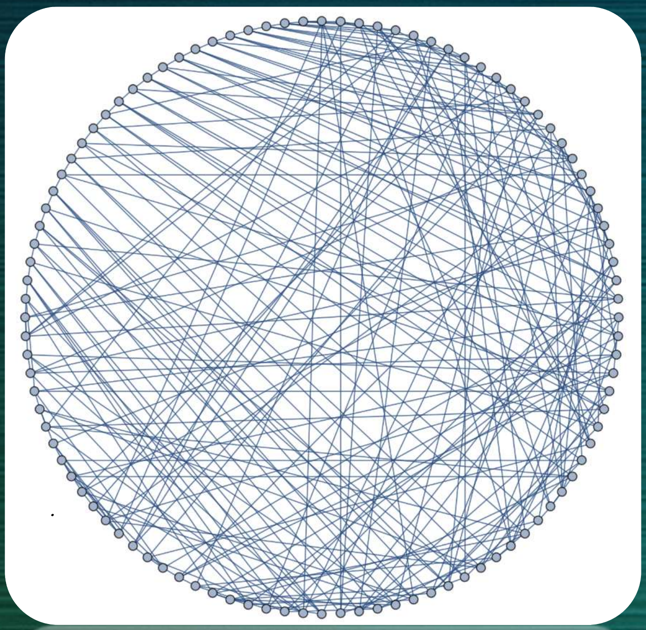




Pseudorandom graphs

- Let n be a prime congruent to 1 mod 4.
- The vertex set of the Paley graph P_n is $\{0,1,2,\ldots,n\}$.
- The edges are pairs $\{i,j\}$ such that $|i-j|$ is a quadratic residue mod n.
- Every edge is in $(n-5)/4$ triangles and every pair of non-adjacent vertices has $(n-1)/4$ common neighbors.

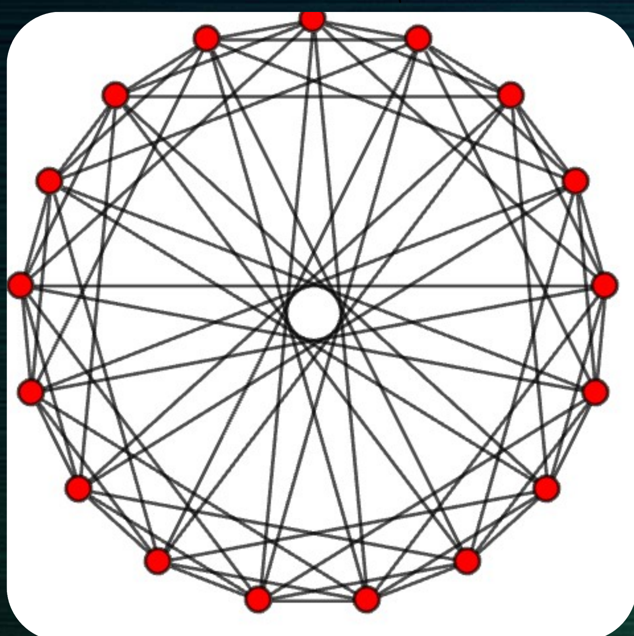
Pseudorandom graphs

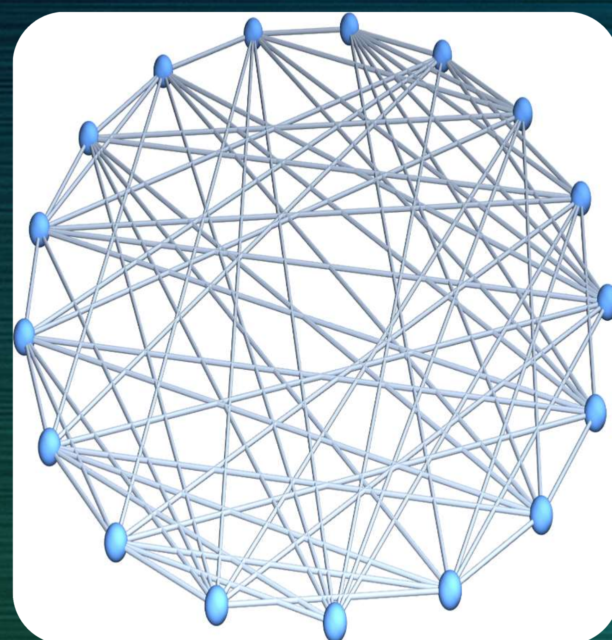
- The vertex set of the Paley graph P_{17} is $\{0,1,2,\ldots,17\}$.
- The edges are pairs $\{i,j\}$ such that $|i-j|$ is a quadratic residue mod 17.
- Paley graph P_{17} is 8-regular and eigenvalues are

$$8,\ \underbrace{\frac{1+\sqrt{17}}{2},\ \ldots,\ \frac{1+\sqrt{17}}{2}}_{8\ \text{times}},\ \underbrace{\frac{1-\sqrt{17}}{2},\ \ldots,\ \frac{1-\sqrt{17}}{2}}_{8\ \text{times}}$$

Pseudorandom graphs

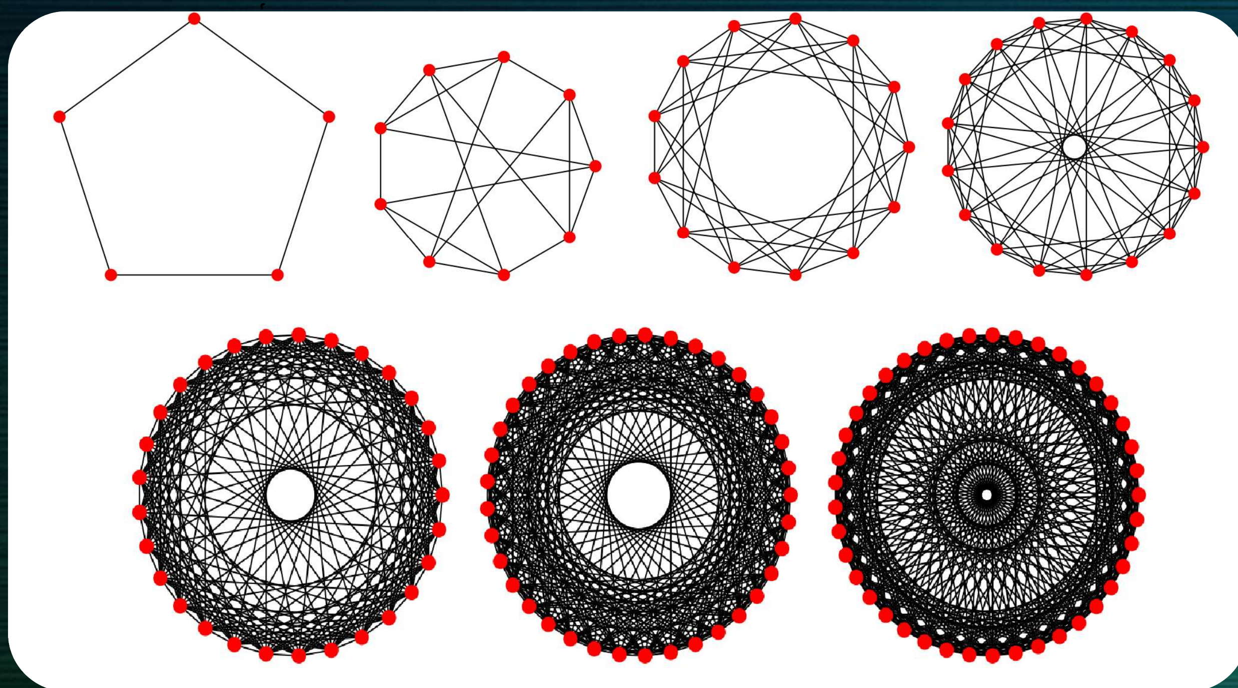
- Paley graph P_{17} is 8-regular





Pseudorandom graphs

- Paley graphs



Pseudorandom graphs

- In a d-regular n-vertex graph, the number of independent sets of size $n/(d+1)$ is at least $(d+1)^t$.
- Alon and Rödl (2004) showed that one can do better in pseudorandom graphs.

Pseudorandom graphs

- Theorem (Mubayi-V, JEMS 2023)

The number of independent sets of size $t = n(\log n)^2/d$ *is at most*

$$\left(\frac{4e^2\lambda}{(\log n)^2}\right)^t$$

Pseudorandom graphs

- When $n = 2^k - 1$ the number of independent sets of size $t = kn/d$ in P_n is at most

$$\left(\frac{4e^2\lambda}{k}\right)^t = \left(\frac{2e^2(1+\sqrt{n})}{k}\right)^t$$

- Randomly sample vertices with probability

$$p = \frac{k}{2e^2(1+\sqrt{n})}$$

Pseudorandom graphs

- Let X be the number of sampled vertices minus one vertex from each independent set of size $t = kn/d < 2k+1$. Then

$$\mathrm{E}(X) = pn - 1$$

- This gives a graph with no independent set of size t and no clique of size t, where the number of vertices is

$$pn - 1 = \frac{kn}{2e^2(1+\sqrt{n})} - 1 > \frac{k}{2e^2}(\sqrt{n} - 2) > \frac{t}{4e^2} 2^{\frac{t}{2}}$$

Ramsey Numbers

- In a triangle-free graph whose adjacency matrix has eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$

$$tr(A^3) = \sum_{i=1}^{n} \lambda_i^3 = 0.$$

- If the graph is λ-pseudorandom, then

$$d^3 \geqslant (n-1)\lambda^3.$$

Ramsey Numbers

- The Alon-Boppana Theorem shows

$$\lambda \gtrsim 2\sqrt{d-1}.$$

- We conclude an optimal pseudorandom triangle-free graph has

$$d = \Omega(n^{\frac{2}{3}}) \quad \text{and} \quad \lambda = O(n^{\frac{1}{3}}).$$

- The first examples were constructed by Alon (1991) and later by Kopparty.

Ramsey Numbers

- Consider the Cayley sum graph with vertex set $\mathbb{F}_q^3$ and generators

$$S = \{(xy, xy^2, xy^3) : x \in A, y \in \mathbb{F}_q^*\}$$

  where A is a sum-free set closed under additive inverse.

- Eigenvalues are controlled by character sums / Gauss sums.

Ramsey Numbers

- Similarly, an optimal pseudorandom K_s-free graph has

$$d = \Omega(n^{1-\frac{1}{2s-3}}) \quad \text{and} \quad \lambda = O(d^{\frac{1}{2}}).$$

- Theorem (Mubayi-V, 2023)

If an optimal pseudorandom K_s-free graph exists, then

$$r(s,t) = \Omega\left(\frac{t^{s-1}}{\log^{2s-4} t}\right).$$

Random Blocks

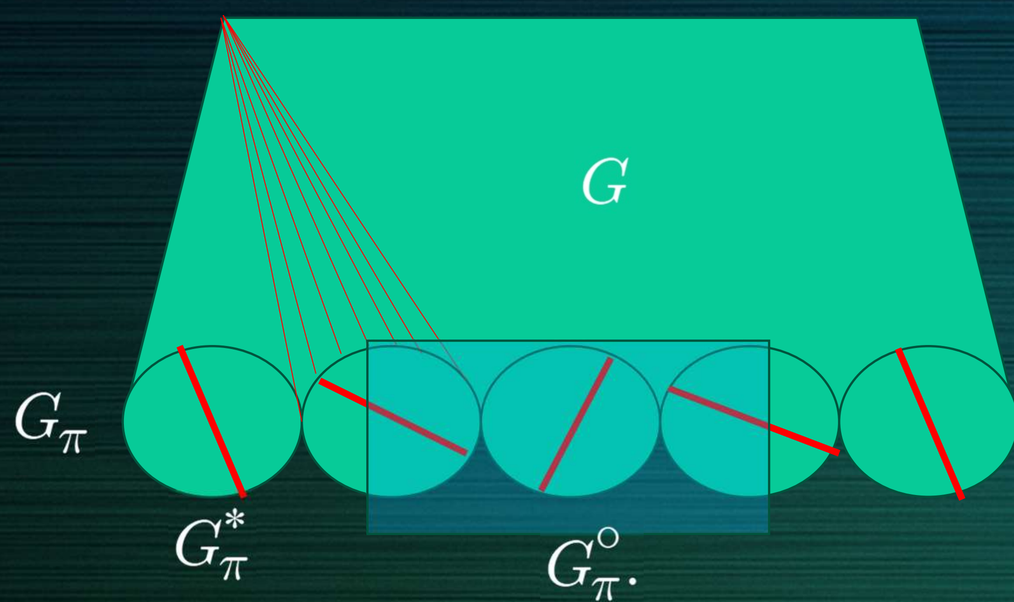
- Let G be a bipartite graph with parts A and B of sizes n and m.
- The projection G_π of G onto is the graph consisting of edges between vertices of A at distance two.
- This graph is a union of m designated cliques, one for each vertex in B.
- If the graph is a point-line incidence graph, then the designated cliques are edge-disjoint.

Random Blocks

- In each designated clique of G_π, independently take a random complete bipartite graph to obtain a random graph G_π^* .
- We refer to this as a random block construction.

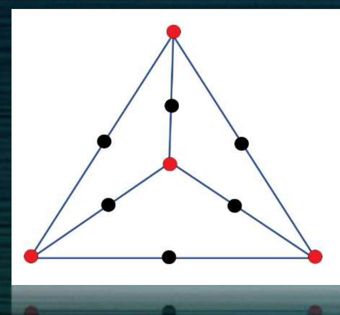
Random Blocks

$$G \;\rightarrow\; G_\pi \;\rightarrow\; G_\pi^* \;\rightarrow\; G_\pi^\circ.$$



Random Blocks

- Suppose G does not contain a 1-subdivision of K_4 :



- Then every K_4 in G_π contains a triangle in one of the designated cliques.
- Therefore G_π^* is a random K_s-free graph.

$r(4,t) = \tilde{\Theta}(t^3)$

$$G \to G_\pi \to G_\pi^* \to G_\pi^\circ.$$

bipartite incidence graph
Hermitian unital
no subdivision of K_4
O'Nan/Paasche configuration

projection
strongly regular graph
pseudorandom
designated clique structure

random blocks
martingales
pseudorandom

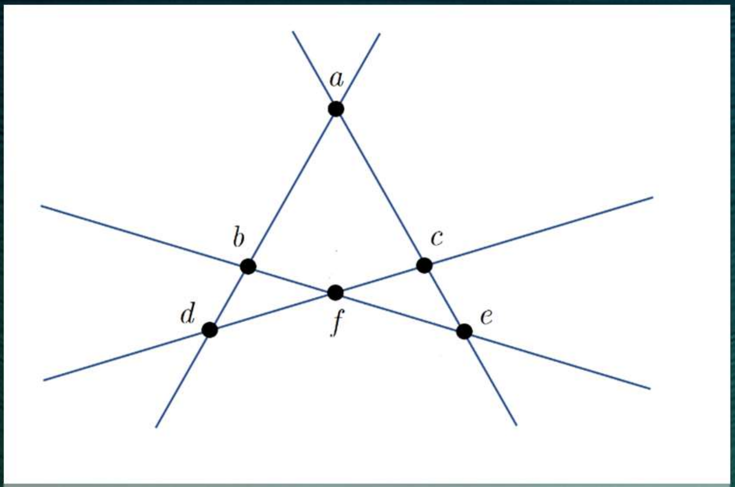
counting independent sets
random sampling
Ramsey graph

$r(4,t) = \tilde{\Theta}(t^3)$

- Starting point: the bipartite incidence graph G of a Hermitian unital in $\mathrm{PG}(2,q^2)$.
- Take all points (x,y,z) satisfying $x^{q+1} + y^{q+1} + z^{q+1} = 0$.
- Then G is an m by n bipartite graph with $m = q^3 + 1$, $n = q^2(q^2 - q + 1)$, and every vertex in the part of size m has degree q^2, and every vertex in the part of size n has degree $q + 1$.

$r(4,t) = \tilde{\Theta}(t^3)$

- The key is that the Hermitian unital does not contain four lines in general position (the O'Nan/Paasche configuration), as proved by O'Nan (1972).

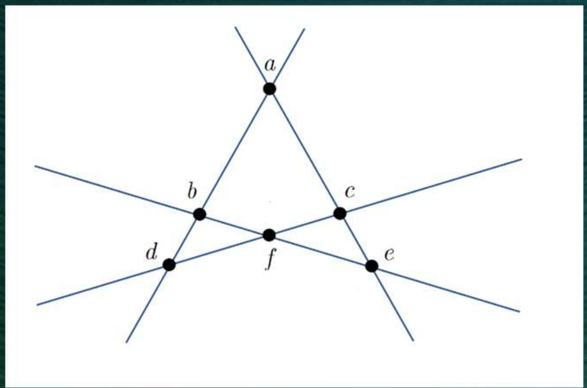


$r(4,t) = \tilde{\Theta}(t^3)$

- As $\{a,b,d\},\{a,c,d\},\{b,e,f\},\{c,d,f\}$ are collinear triples, we may choose generators so

$$d = a + b \qquad e = a + c \qquad f = a + b + c.$$

- Let A be the matrix whose rows are a,b and c and let B be the matrix whose rows are a^q,b^q and c^q. Then A and B are non-singular and so is AB.



$r(4, t) = \tilde{\Theta}(t^3)$

- However

$$\det(AB) = \det \begin{pmatrix} \sigma(a,a) & \sigma(a,b) & \sigma(a,c) \\ \sigma(b,a) & \sigma(b,b) & \sigma(b,c) \\ \sigma(c,a) & \sigma(c,b) & \sigma(c,c) \end{pmatrix}$$

$$= \sigma(a,b)\sigma(b,c)\sigma(c,a) - \sigma(b,a)\sigma(a,c)\sigma(c,b) = 0$$

Erdős-Rogers Functions

- Given graphs F and G, let $f_{F,G}(n)$ denote the maximum number of vertices in an F-free subgraph of every G-free n-vertex graph. (Erdős-Rogers 1962)

- Erdős-Rogers, Bollobás-Hind, Dudek-Rödl, Dudek-Retter-Rödl, Krivelevich, Alon-Krivelevich, Wolfovitz, Gowers-Janzer, Janzer-Sudakov, ...

- Theorem (Wolfovits)

$$f_{K_3,K_4}(n) \leqslant n^{\frac{1}{2}}(\log n)^{120}$$

Erdős-Rogers Functions

- It is easy to see

$$f_{K_s,K_{s+1}}(n) \ > \ \sqrt{n} - 1.$$

- In fact (Shearer)

$$f_{K_s,K_{s+1}}(n) \ > \ \sqrt{\frac{n \log n}{\log \log n}}.$$

note.jnt

Erdős-Rogers Functions

- Theorem (Mubayi-V, 2024)

Let $s > 3$. Then for any K_4-free graph F containing a k-cycle,

$$bn^{\frac{1}{3} + \frac{1}{3k}} < f_{F,K_4}(n) < cn^{\frac{1}{2}} \log n.$$

Erdős-Rogers Functions

- Theorem (Mubayi-V, 2024)

  For any triangle-free graph F,

$$f_{F,K_3}(n) = n^{\frac{1}{2}+o(1)}.$$

note.jnt

Open Problem I

- Conjecture (V)

  $For\ s > 2,$

$$f_{K_s,K_{s+1}}(n) \;=\; \Theta(n^{\frac{1}{2}}\log n).$$

- Problem

  Suppose the neighborhood of every vertex in an n-vertex graph G of maximum degree d induces a bipartite graph. Does G contain an induced triangle-free subgraph of size

$$\omega(d)\cdot\frac{n\log d}{d}?$$

Open Problem I

- Theorem (Ajtai, Komlós, Pintz, Spencer, Szemerédi 1982, V-Wilson, 2024)

  Every locally sparse n-vertex triple system of maximum degree d has an independent set of size at least

$$\frac{n\sqrt{\log d}}{4\sqrt{d}}.$$

Open Problem II

- For some C > 0, is the number of independent sets of size

$$t = \left\lceil \frac{Cn \log n}{d} \right\rceil$$

in a λ-pseudorandom graph at most

$$\left(\frac{C\lambda}{\log n} \right)^t ?$$

Open Problem III

- In PG$(2, q^3)$ is there a set of roughly q^4 points and q^6 lines of size q each such that no five lines are in general position?