Correlated stochastic block models: graph matching and community recovery

Based on joint works with Julia Gaudio and Anirudh Sridhar

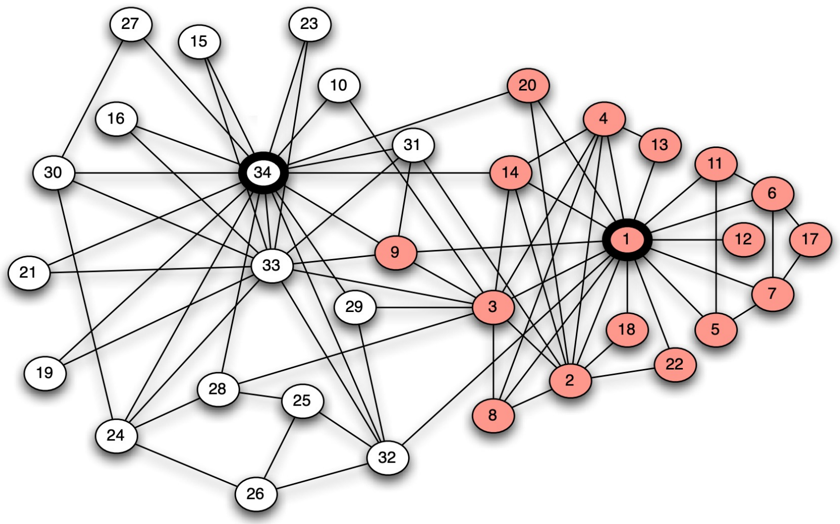
Miklós Z. Rácz

Northwestern University

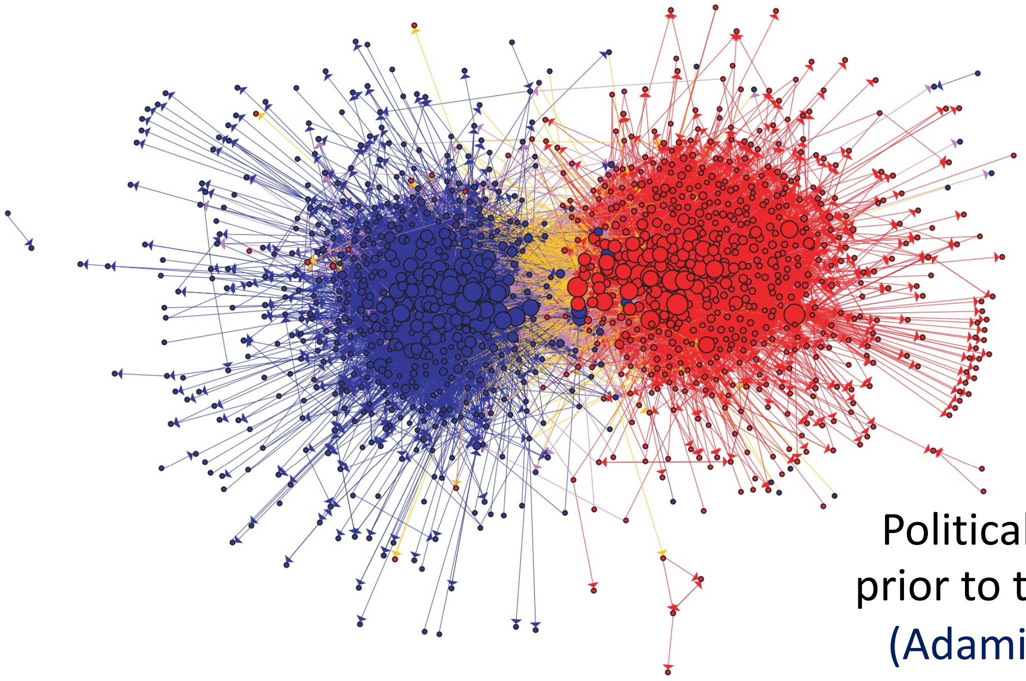
Oxford Discrete Mathematics and Probability Seminar

March 7, 2023

Recovering communities in networks

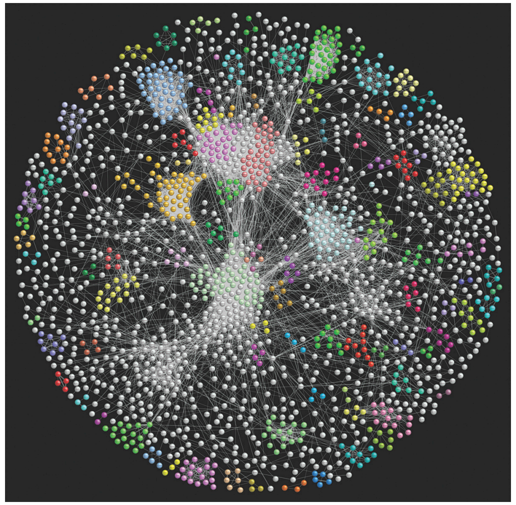


Zachary's karate club (1970-72; 1977)



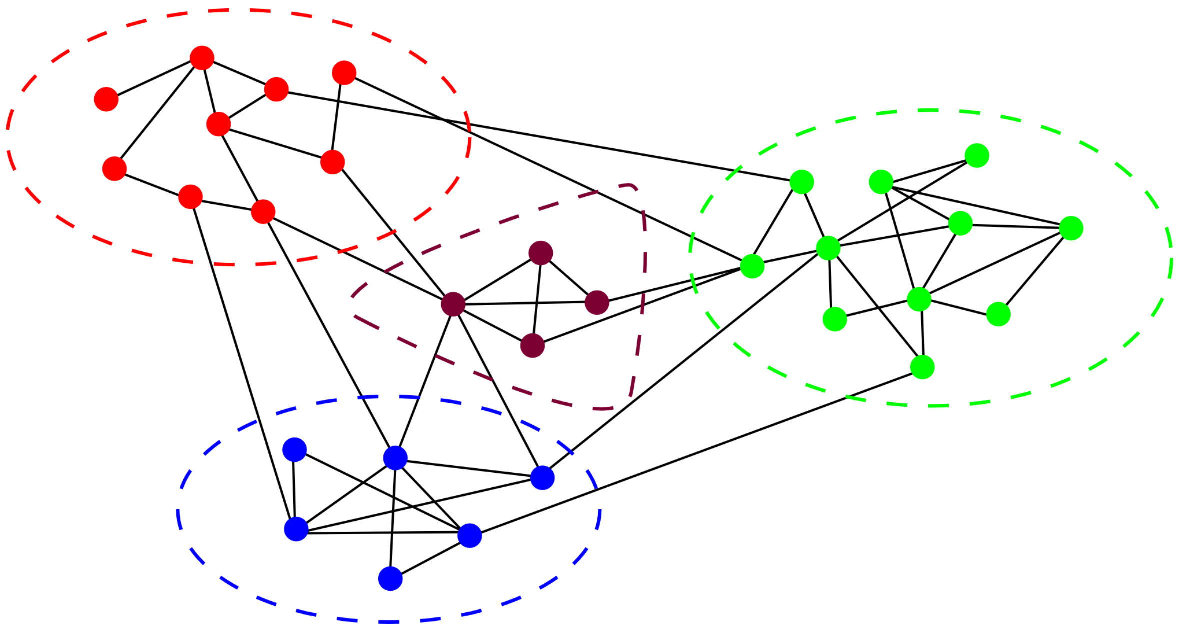
Political blogs in the US, prior to the 2004 elections (Adamic, Glance, 2005)

Facebook: 2.9+ billion monthly active users
2+ billion daily active users



Drosophila protein-protein interaction network (Guruharsha et al., 2011)

Stochastic block model (SBM)



Holland, Laskey, Leinhardt (1983)

Many works in physics, statistics, probability, CS, info theory… including:

- Decelle, Krzakala, Moore, Zdeborová (2011)
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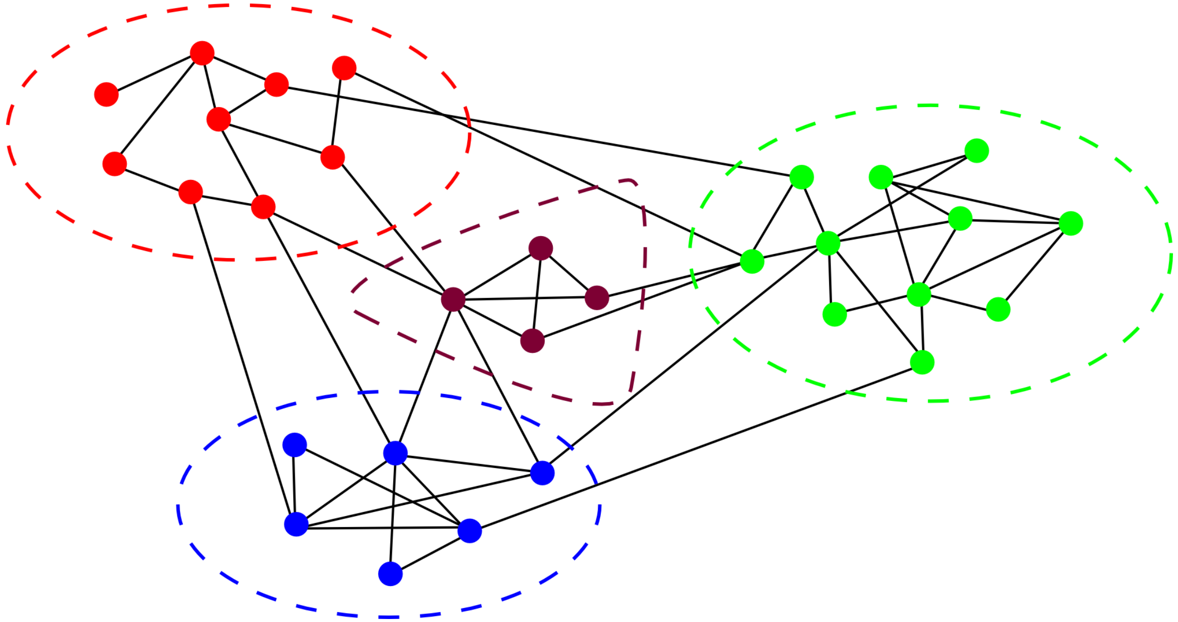

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- Partial recovery?
- Almost exact recovery?
- Exact recovery?

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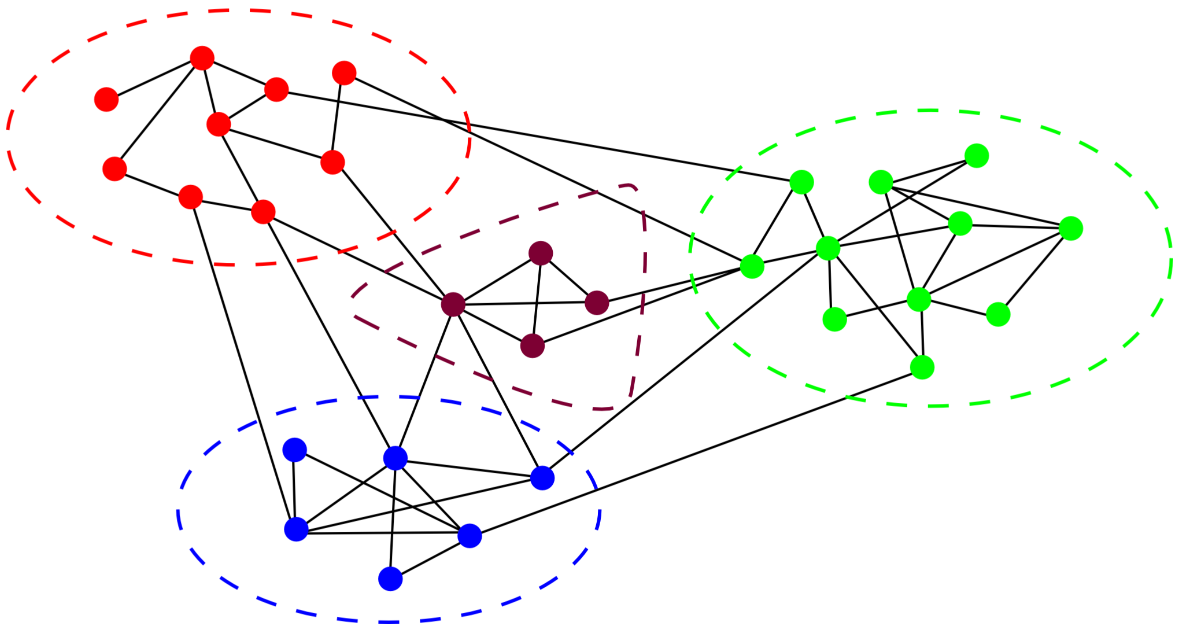

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This talk: two balanced communities

- n nodes
- $\sigma_i \in \{+1, -1\}$ i.i.d. uniform community labels
- Given $\boldsymbol{\sigma} = \{\sigma_i\}$, edges drawn independently:
  - If $\sigma_i = \sigma_j$, then $i \sim j$ with prob. p
  - If $\sigma_i \neq \sigma_j$, then $i \sim j$ with prob. q

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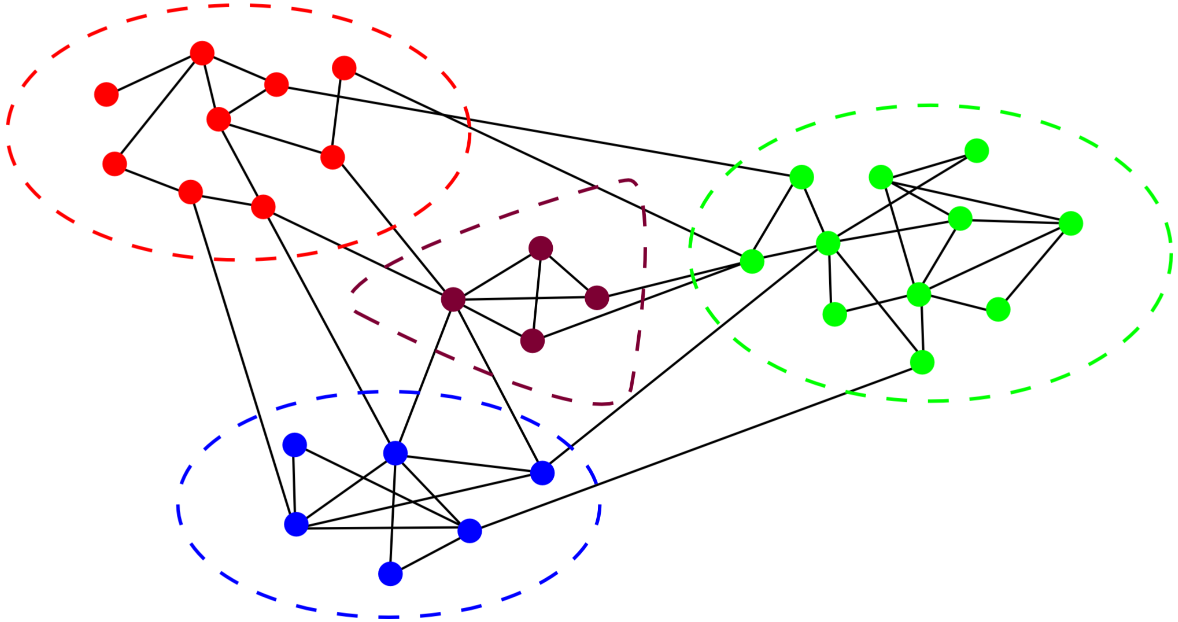

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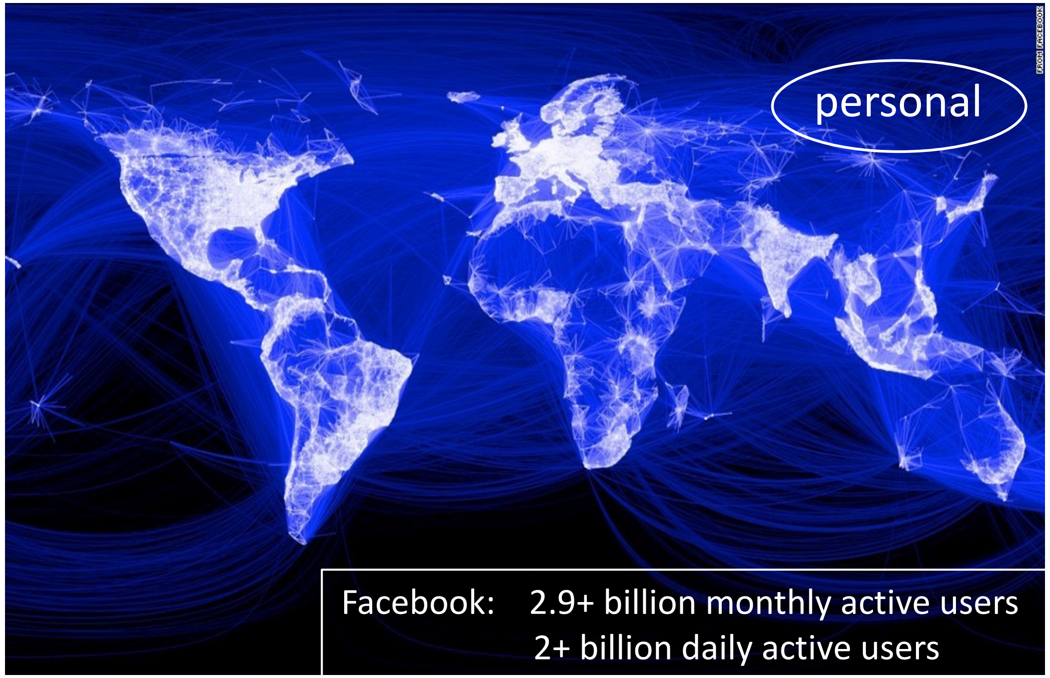
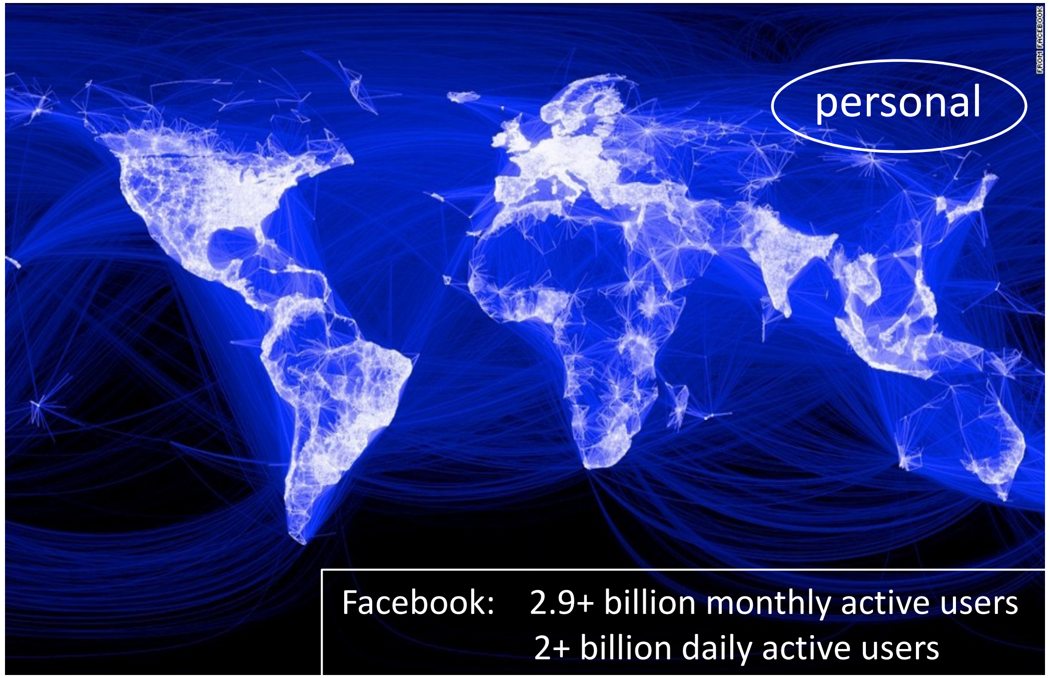
Multiple correlated networks



Q: can we synthesize information from multiple correlated networks to better recover communities?

Multiple correlated networks



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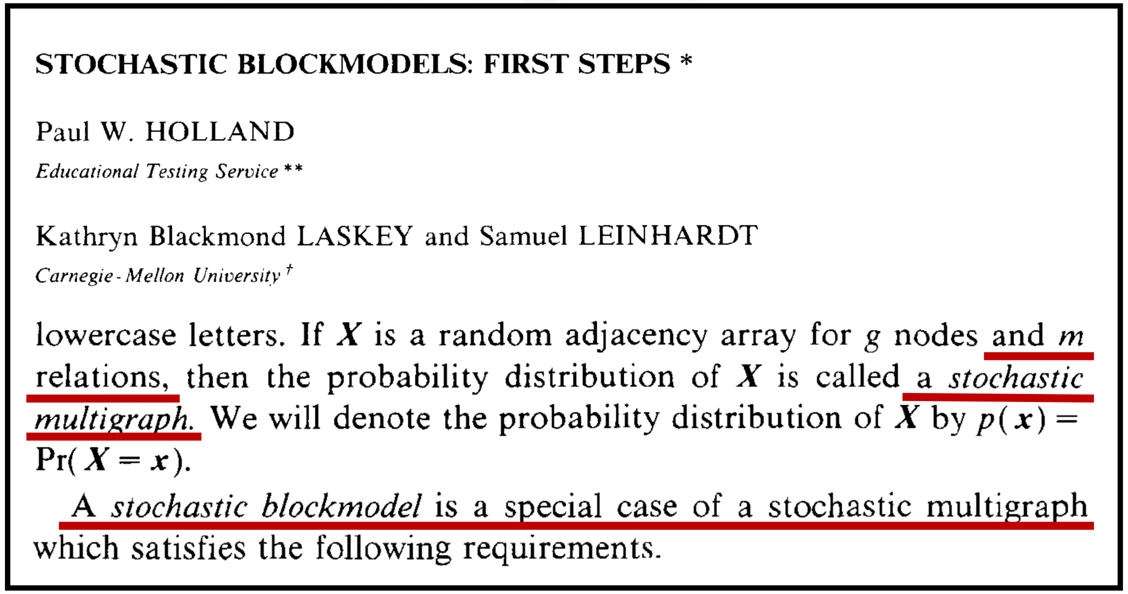


STOCHASTIC BLOCKMODELS: FIRST STEPS *

Paul W. HOLLAND

Educational Testing Service **

Kathryn Blackmond LASKEY and Samuel LEINHARDT

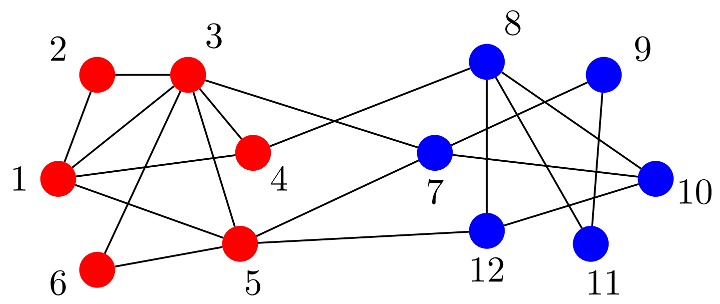
Carnegie-Mellon University †

lowercase letters. If X is a random adjacency array for g nodes and m relations, then the probability distribution of X is called a *stochastic multigraph*. We will denote the probability distribution of X by $p(x) = \Pr(X = x)$.

A *stochastic blockmodel* is a special case of a stochastic multigraph which satisfies the following requirements.

Q: can we synthesize information from multiple correlated networks to better recover communities?

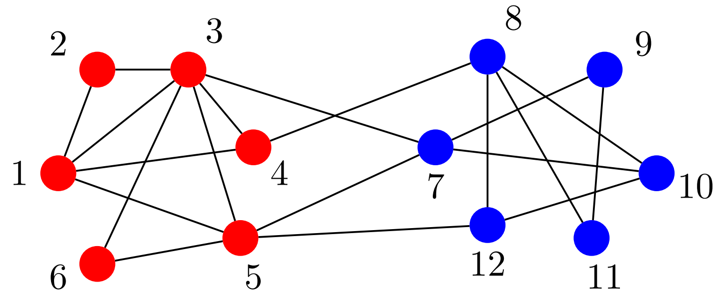
Correlated stochastic block model



$$G \sim \mathrm{SBM}(n, p, q)$$

Correlated stochastic block model

subsample



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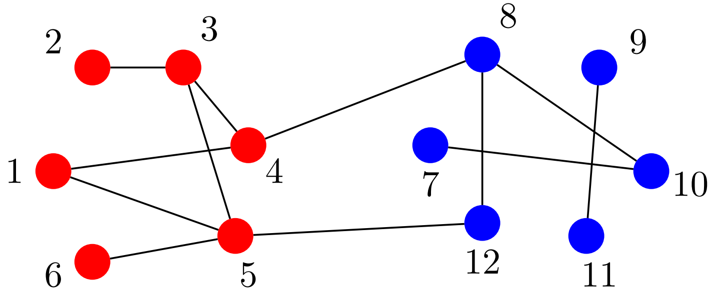
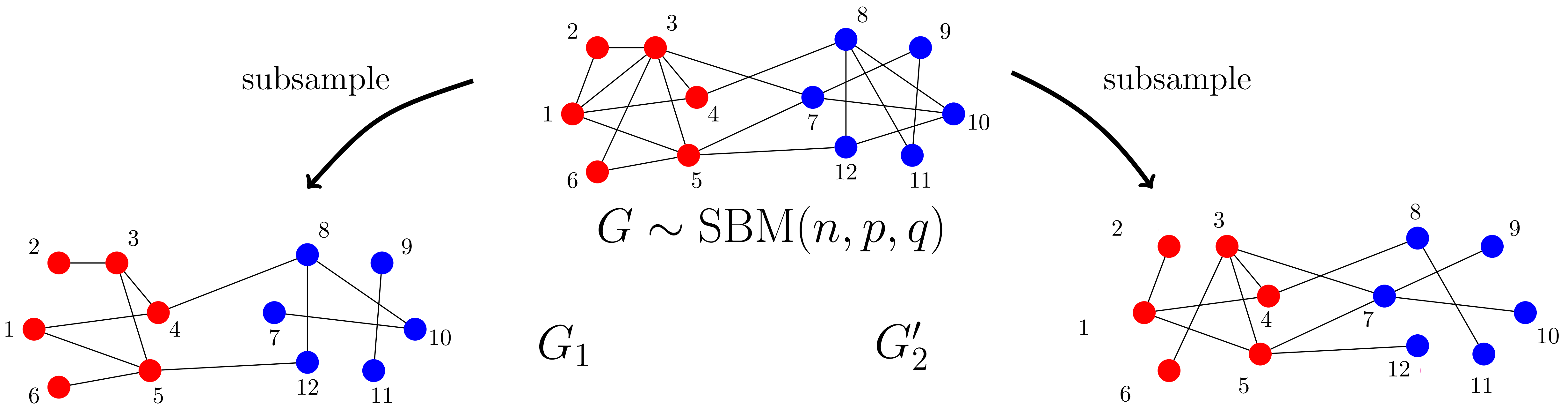


$$G_1$$

- Subsampling probability $s \in [0,1]$

Correlated stochastic block model



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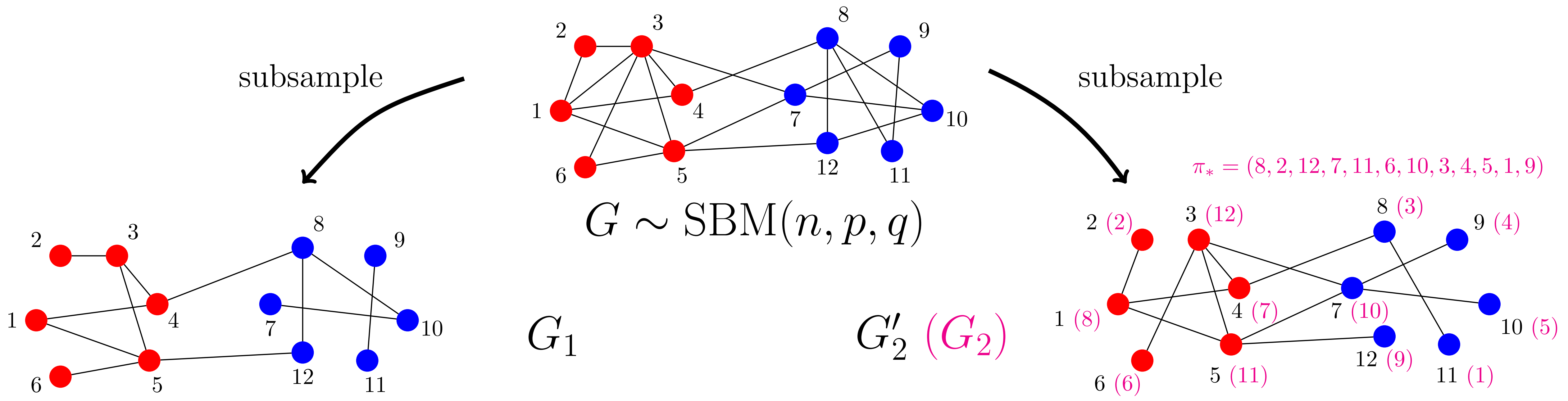

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- π_* uniformly random permutation of $[n]$

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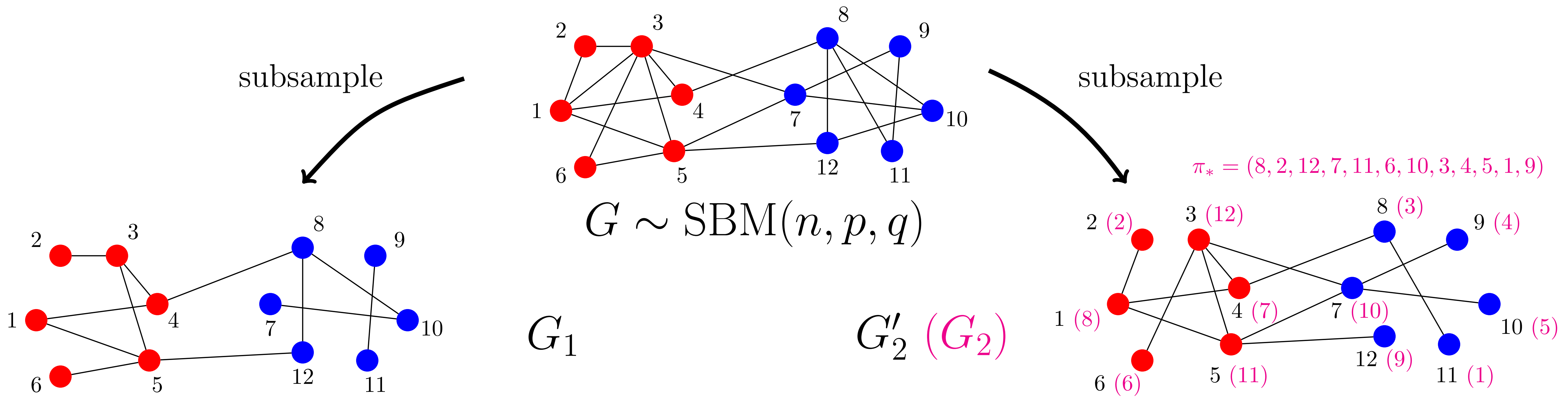

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$$\boxed{(G_1, G_2) \sim \mathrm{CSBM}(n,p,q,s)}$$

- Marginally $G_1, G_2 \sim SBM(n, ps, qs)$
- Corresponding edges are *correlated*

(Onaran, Garg, Erkip, 2016)

HLL83: (G_1, G_2') is a “pair-dependent SBM”

Correlated stochastic block model

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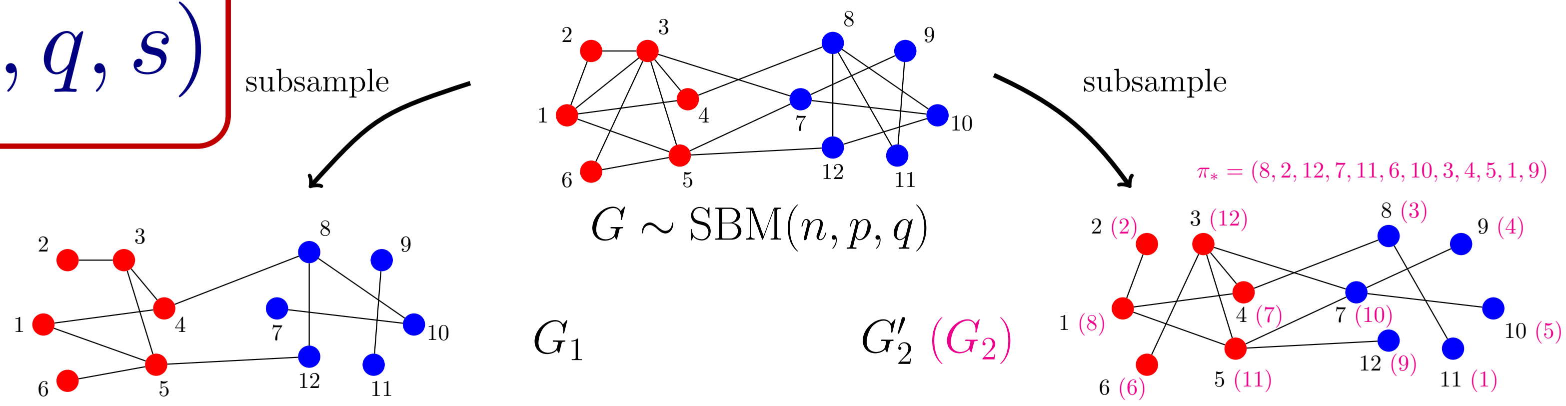

Main Q:

- given (G_1, G_2), when can we (exactly) recover the communities?
- can we do so in regimes where it is impossible to do so using only G_1?

Exact community recovery in the SBM

Need no isolated vertices $\Longrightarrow$ logarithmic degree regime: $p = a\log(n)/n$ and $q = b\log(n)/n$

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Theorem (Abbé, Bandeira, Hall, 2014; Mossel, Neeman, Sly, 2014)

Consider the balanced two-community SBM: $G \sim \mathrm{SBM}\left(n, \frac{a\log n}{n}, \frac{b\log n}{n}\right)$

Exact recovery is **possible** (in polynomial time) if $\boxed{\left|\sqrt{a}-\sqrt{b}\right| > \sqrt{2}}$

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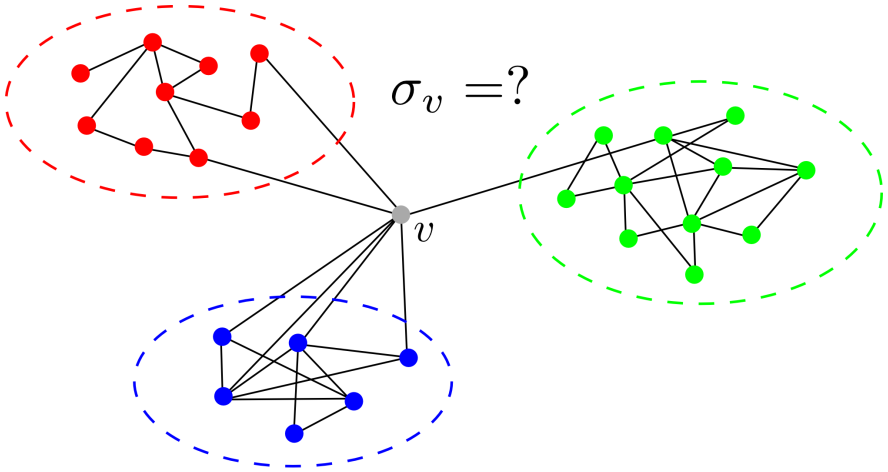
Abbé, Sandon (2015): threshold for general SBMs

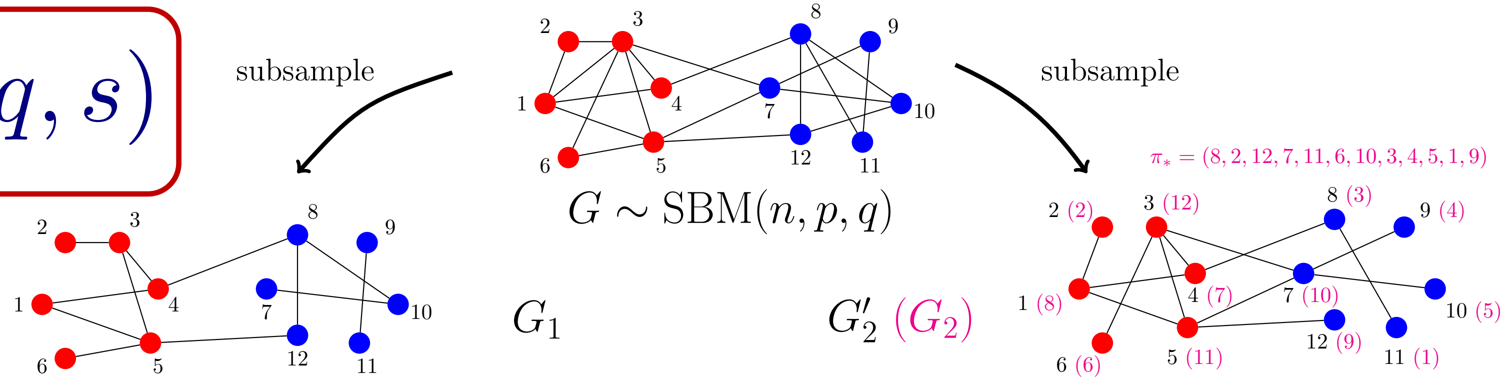


Intuition:

- Testing multivariate Poisson distributions
- Want error probability $n^{-1+o(1)}$
- Error exponent given by Chernoff-Hellinger divergence

Exact community recovery in correlated SBMs

$$\boxed{(G_1, G_2) \sim \mathrm{CSBM}(n, p, q, s)}$$



Since $G_1 \sim SBM(n, ps, qs)$, exact community recovery is possible from G_1 iff

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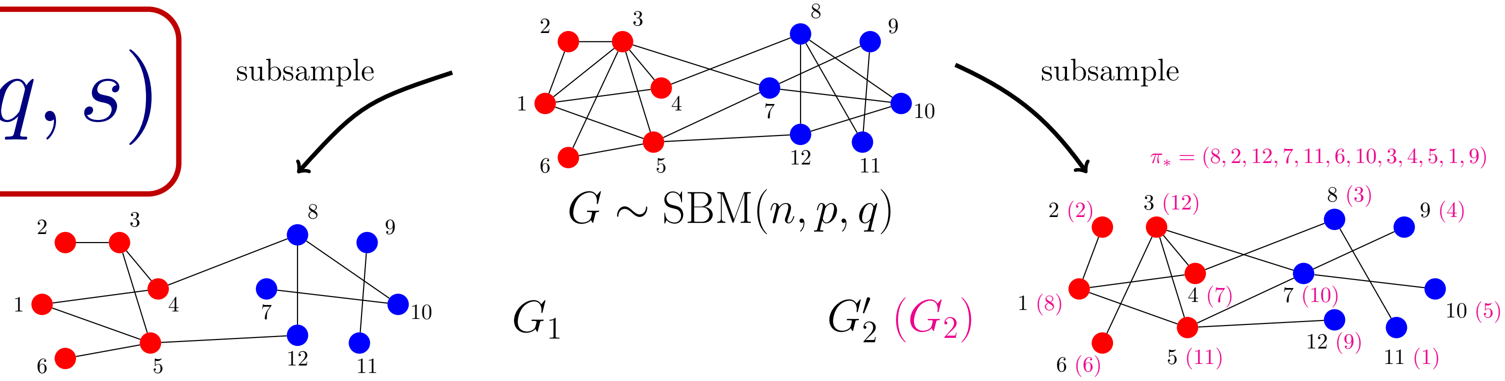

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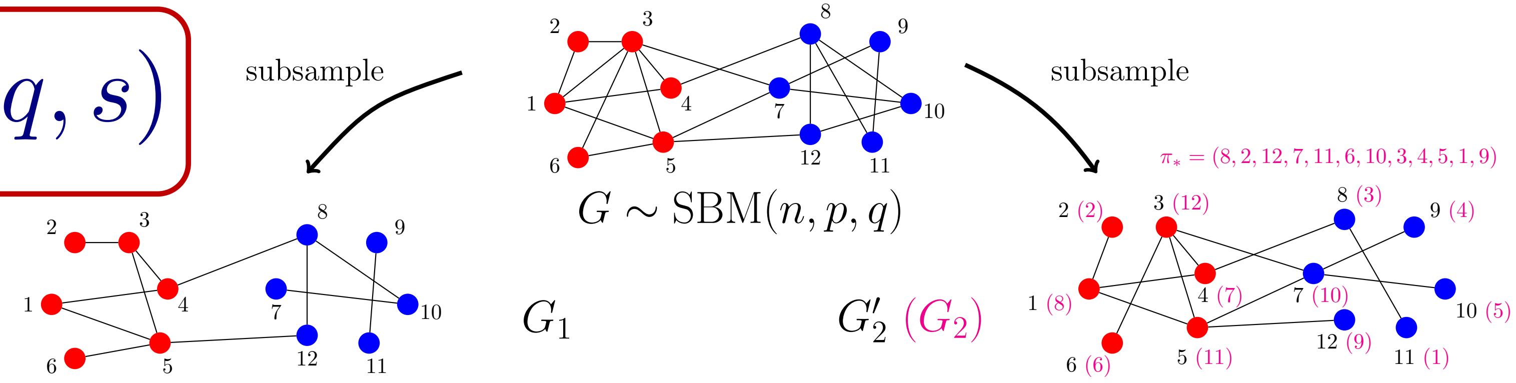

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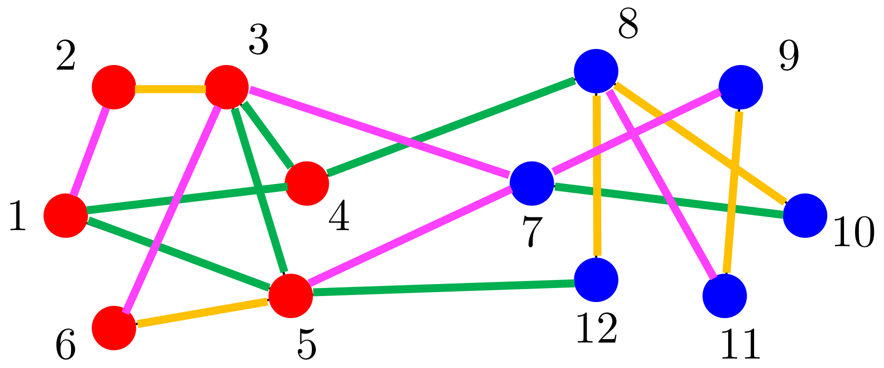
How can we use both G_1 and G_2? **Suppose that π_* is known.** Then:

- in G_1 and G_2
- in G_1, not in G_2
- not in G_1, in G_2



$$G_1 \vee_{\pi_*} G_2 \sim \mathrm{SBM}\left(n, \frac{a(1-(1-s)^2)\log n}{n}, \frac{b(1-(1-s)^2)\log n}{n}\right)$$

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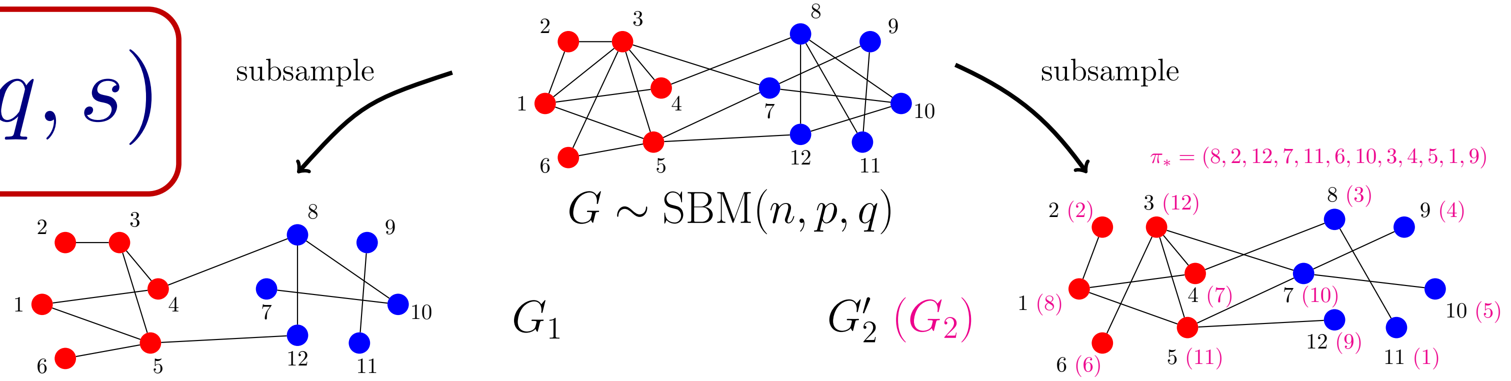

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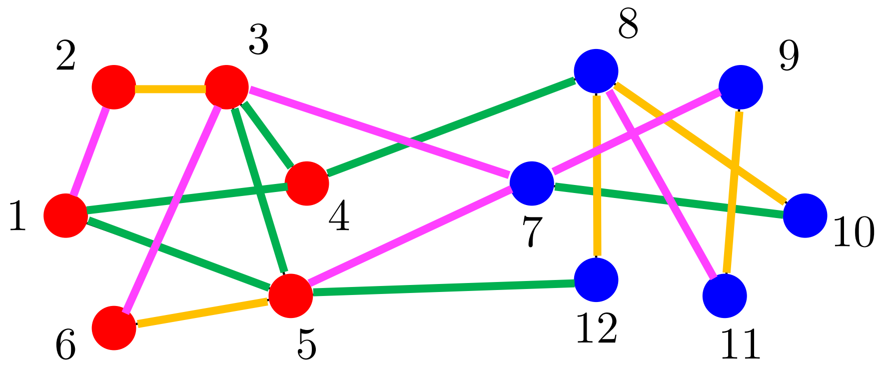

Thus exact community recovery is possible iff

$$\boxed{\left|\sqrt{a} - \sqrt{b}\right| > \sqrt{2/(1-(1-s)^2)}}$$

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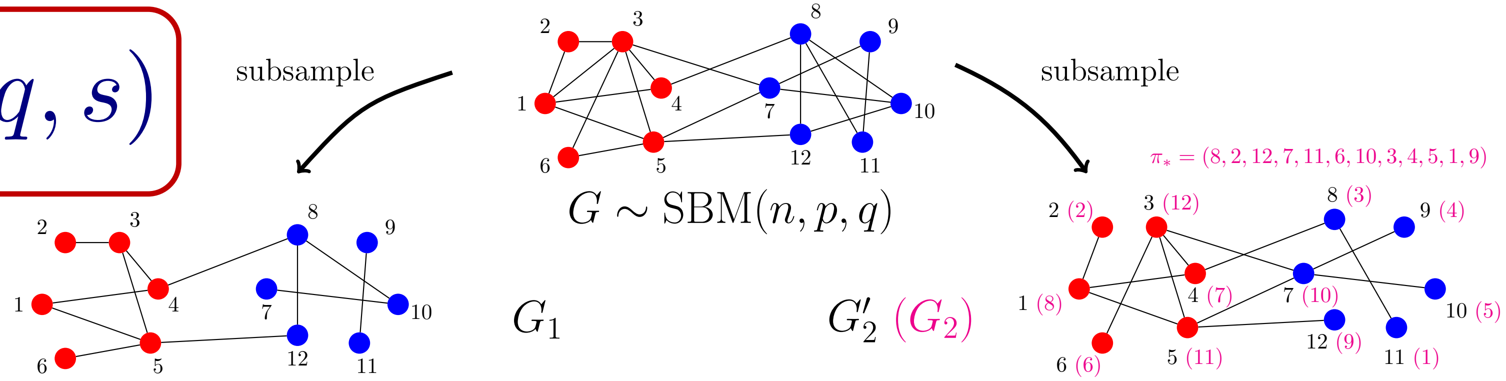


Sin

H

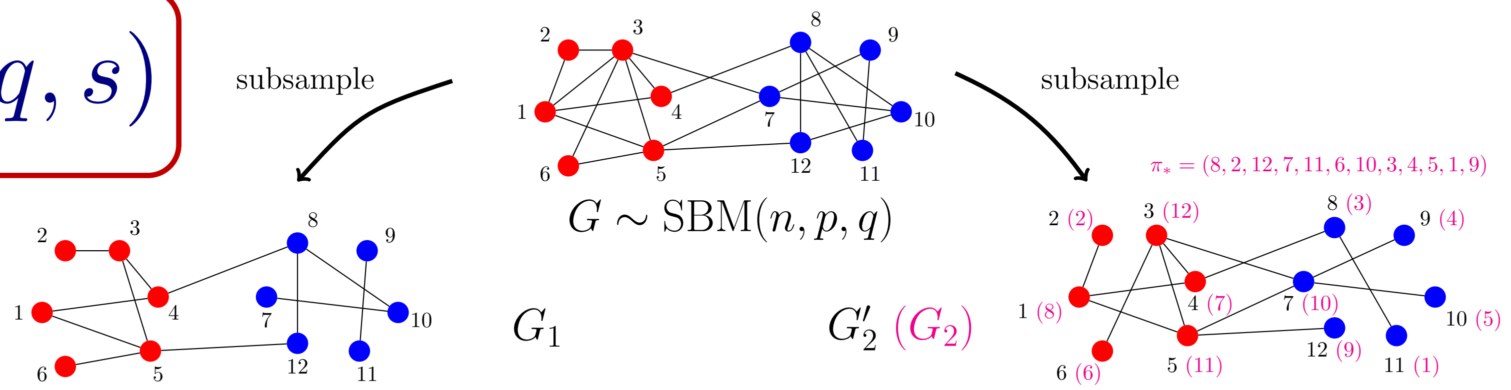
G_1

In particular, **if $\boldsymbol{\pi_*}$ is known** and $\boxed{\sqrt{2/s} > \left|\sqrt{a} - \sqrt{b}\right| > \sqrt{2/(1-(1-s)^2)}}$

then exact community recovery is possible from G_1 and G_2,
even though it is impossible from G_1 alone

Graph matching

$$(G_1, G_2) \sim \mathrm{CSBM}(n, p, q, s)$$



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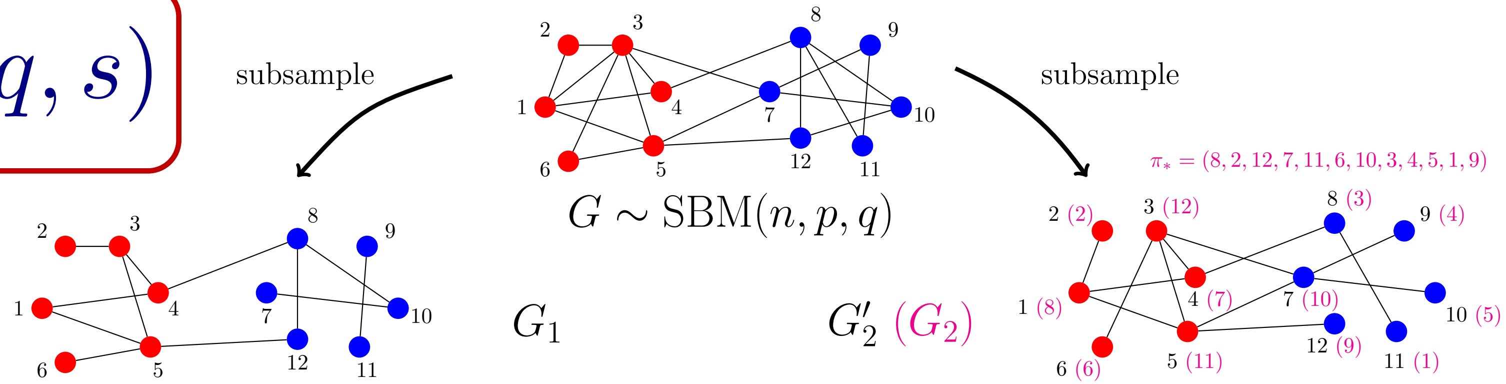

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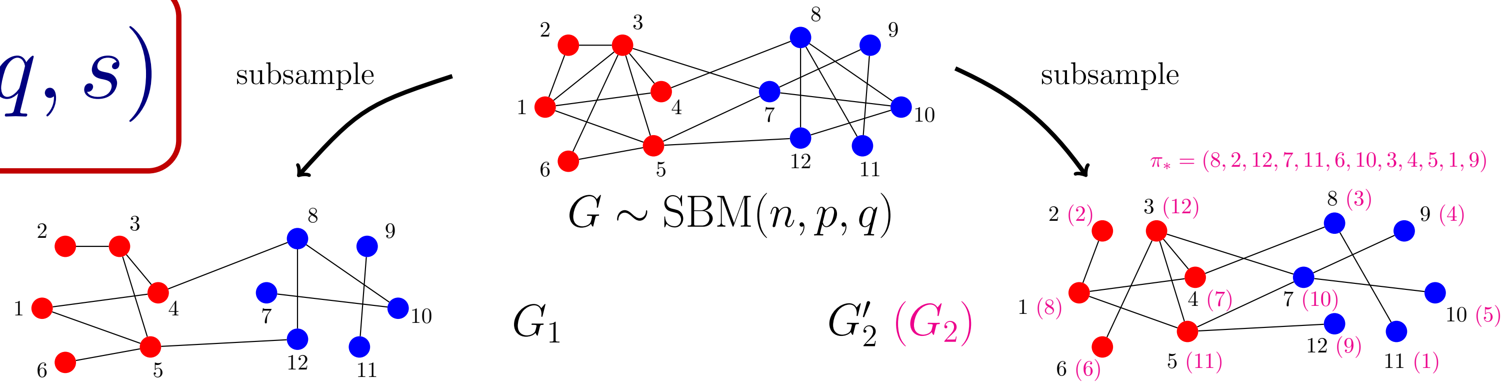

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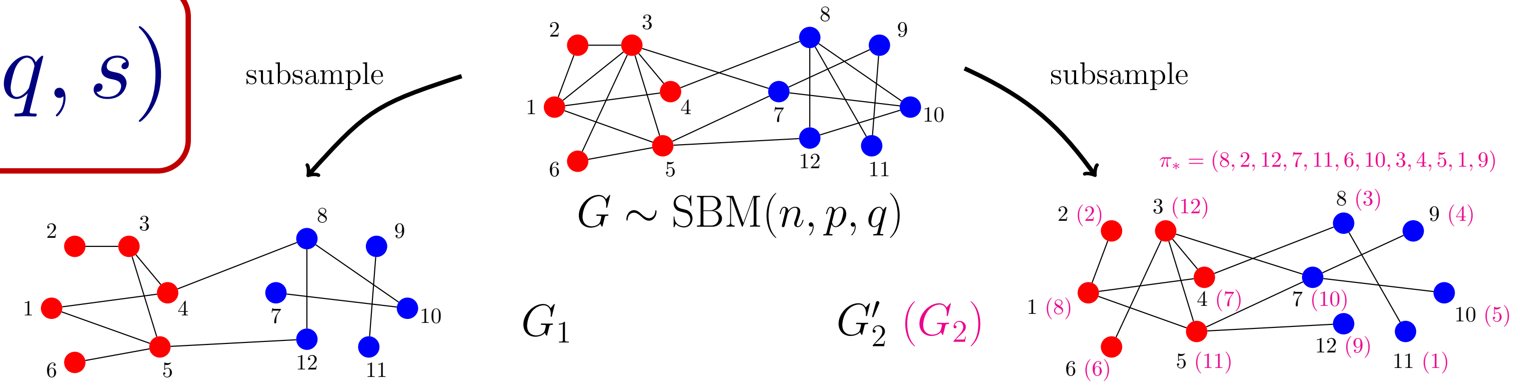

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- Many works in statistics/probability/CS/info theory… including:
  - Cullina, Kiyavash (2016, 2017)
  - Barak, Chou, Lei, Schramm, Sheng (2019)
  - Ding, Ma, Wu, Xu (2018)
  - Mossel, Xu (2019)
  - Fan, Mao, Wu, Xu (2019a,b)
  - Ganassali, Massoulié (2020)
  - Wu, Xu, Yu (2020, 2021)
  - Cullina, Kiyavash, Mittal, Poor (2020)
  - Mao, Rudelson, Tikhomirov (2021a,b)
  - Ganassali, Lelarge, Massoulié (2021)
  - Mao, Wu, Xu, Yu (2021,2022)
  - Ding, Du (2022a,b)

Correlated SBMs: graph matching and community recovery

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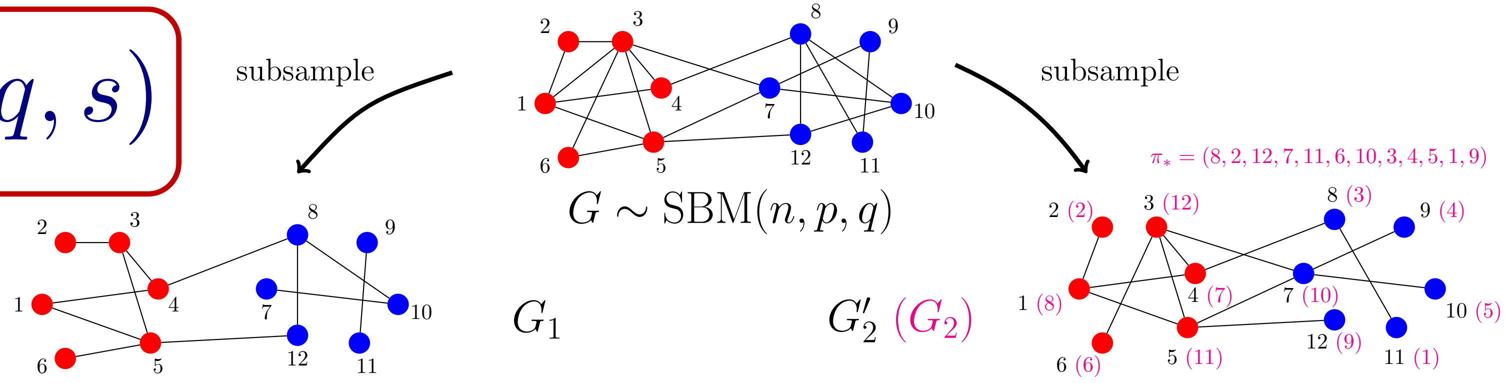

Main Q1 (community recovery):

- given (G_1, G_2), when can we (exactly) recover the communities?
- can we do so in regimes where it is impossible to do so using only G_1?

Main Q2 (graph matching):

- given (G_1, G_2), when can we (exactly) recover the latent permutation π_*?

Related work

Multi-layer networks/SBMs

- Holland, Laskey, Leinhardt (1983)
- Han, Xu, Airoldi (2015)
- Paul, Chen (2016, 2020a,b, 2021)
- Ali et al. (2019)
- Lei, Chen, Lynch (2019)
- Arroyo et al. (2020)
- Bhattacharyya, Chatterjee (2020)
- Chen, Liu, Ma (2020)
- …

Contextual block models

- Kanade, Mossel, Schramm (2016)
- Mossel, Xu (2016)
- Zhang, Levina, Zhu (2016)
- Binkiewicz, Vogelstein, Rohe (2017)
- Deshpande, Sen, Montanari, Mossel (2018)
- Abbé, Fan, Wang (2020)
- Lu, Sen (2020)
- …

- Mayya, Reeves (2019)
- Ma, Nandy (2021)

Correlated SBMs: graph matching and community recovery

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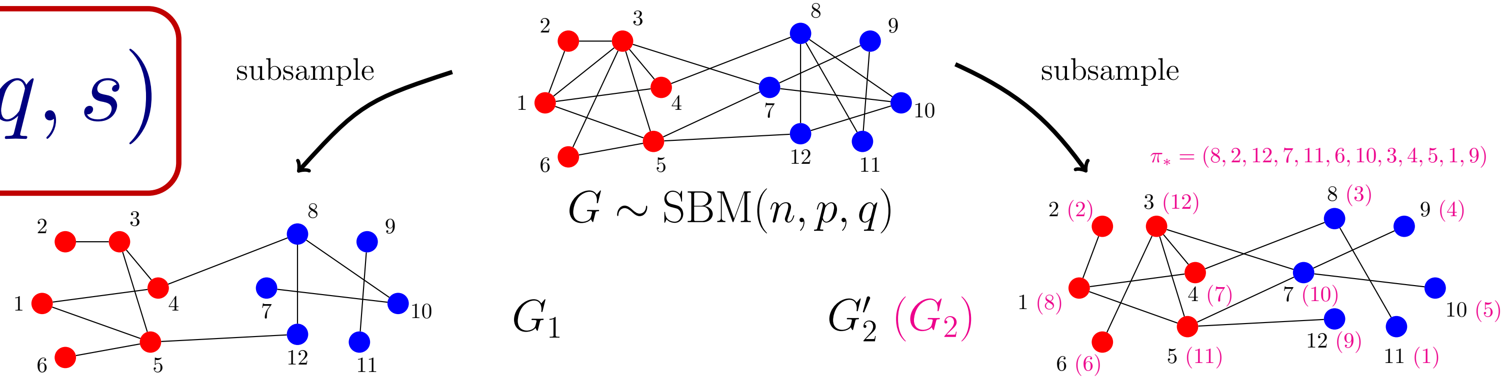

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Main Q2 (graph matching):

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Results

Exact graph matching

Theorem (R., Sridhar, 2021)

Let $\hat{\pi}(G_1, G_2)$ be a vertex mapping that maximizes the number of agreeing edges between G_1 and G_2.

$$\widehat{\pi}(G_1, G_2) \in \arg\max_{\pi \in \mathcal{S}_n} \sum_{(i,j) \in \mathcal{E}} A_{i,j} B_{\pi(i),\pi(j)}$$

If $s^2 \left(\frac{a+b}{2} \right) > 1$ then $\lim_{n \to \infty} \mathbb{P}\left(\widehat{\pi}(G_1, G_2) = \pi_* \right) = 1$

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- Cullina, Kiyavash (2016, 2017): exact graph matching for the correlated Erdős-Rényi model; see also Wu, Xu, Yu (2021)

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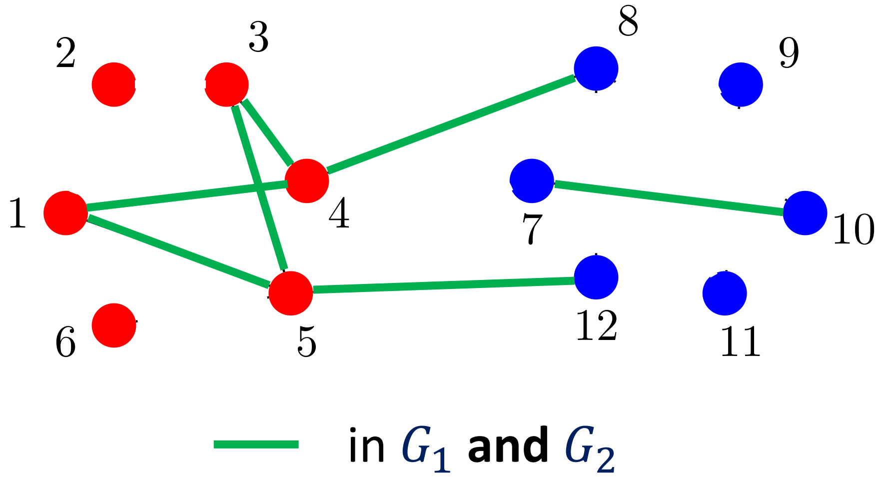

Exact graph matching

Theorem (R., Sridhar, 2021)

Let $\hat{\pi}(G_1, G_2)$ be a vertex mapping that maximizes the number of agreeing edges between G_1 and G_2.

$$\widehat{\pi}(G_1, G_2) \in \arg\max_{\pi \in \mathcal{S}_n} \sum_{(i,j) \in \mathcal{E}} A_{i,j} B_{\pi(i),\pi(j)}$$

If $\quad s^2 \left(\frac{a+b}{2} \right) > 1 \quad$ then $\quad \lim_{n \to \infty} \mathbb{P}\left(\widehat{\pi}(G_1, G_2) = \pi_* \right) = 1$

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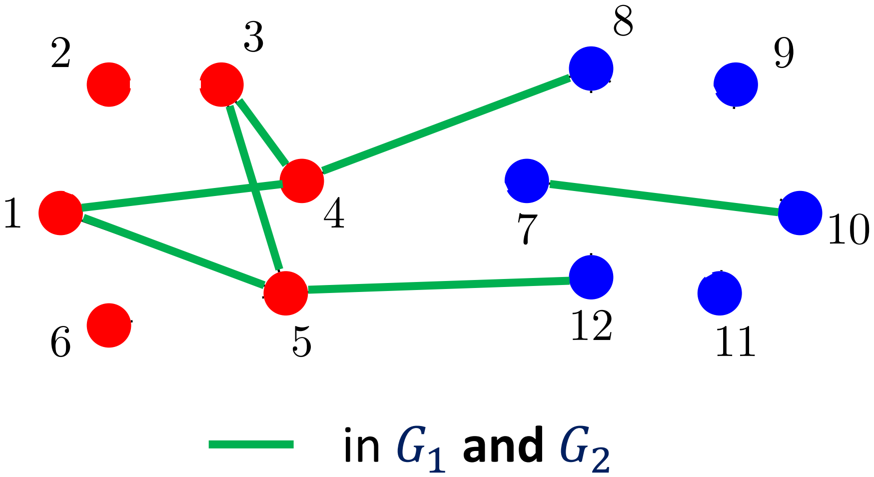
- Onaran, Garg, Erkip (2016): same conclusion under stronger parameter assumptions and assuming all community labels are known



Exact graph matching – converse

Theorem (Cullina, Singhal, Kiyavash, Mittal, 2016)

If $s^2\left(\frac{a+b}{2}\right) < 1$ then $\lim_{n\to\infty} \mathbb{P}\left(\widetilde{\pi}(G_1, G_2) = \pi_*\right) = 0$ for every estimator $\tilde{\pi}$

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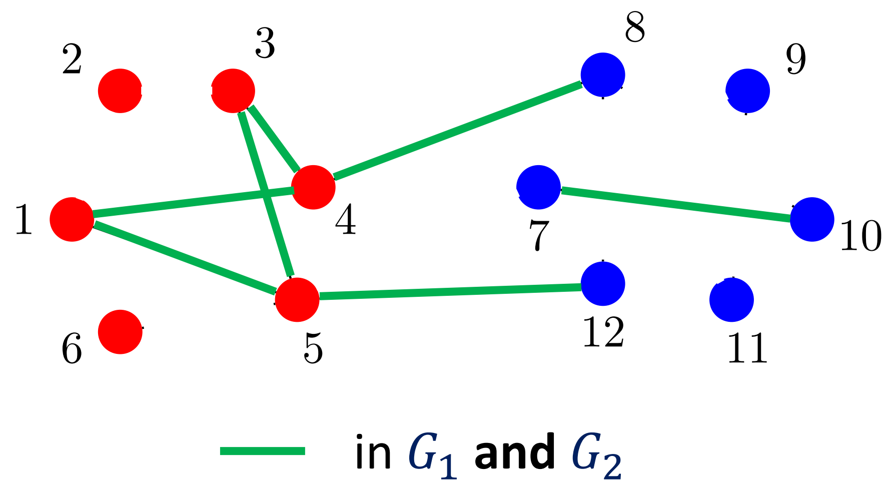

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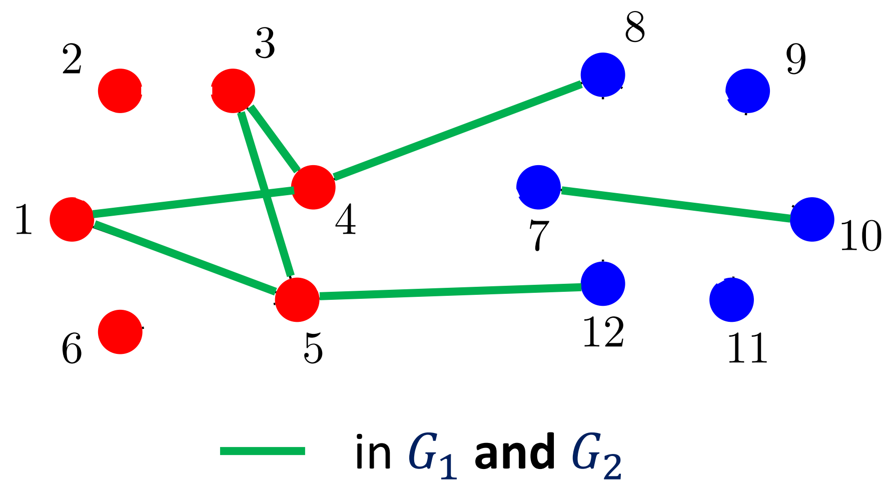

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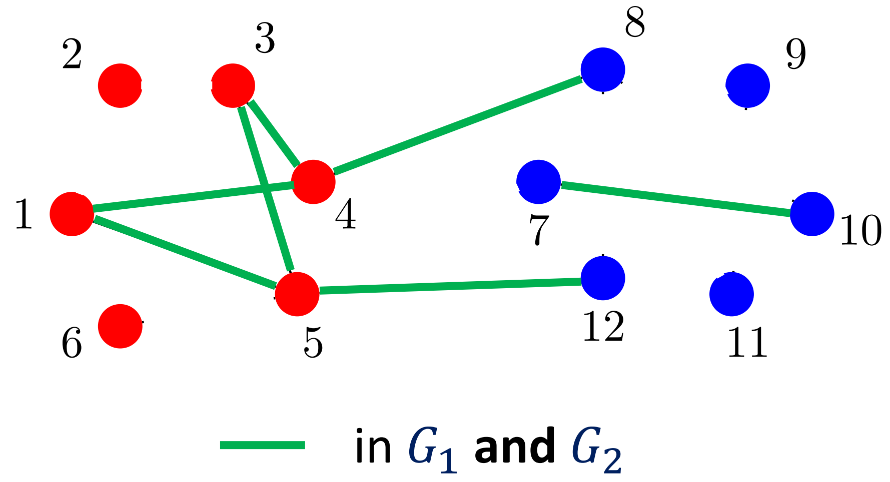

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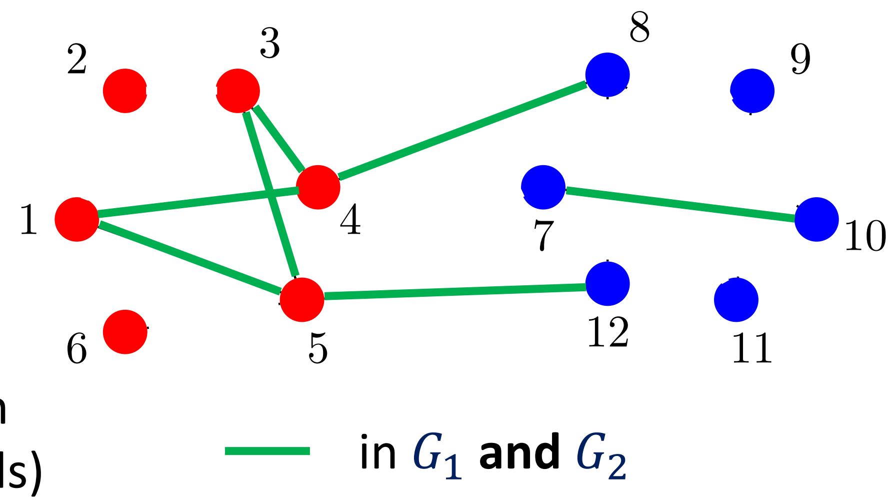
- Such vertices are hard to match due to the lack of shared information (even for optimal estimators that have access to the community labels)



Exact community recovery

Theorem (R., Sridhar, 2021)

Exact community recovery is **possible**

If $s^2\left(\frac{a+b}{2}\right) > 1$ and $|\sqrt{a} - \sqrt{b}| > \sqrt{2/(1-(1-s)^2)}$

then there is an estimator $\widehat{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\sigma}}(G_1, G_2)$ such that $\boxed{\lim_{n\to\infty} \mathbb{P}(\mathrm{ov}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}) = 1) = 1}$

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Proof: can recover π_* whp; then run a community recovery algorithm on the union of the matched graphs.

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Proof: even if π_* is known, it is impossible to exactly recover the communities from $G_1 \vee_{\pi_*} G_2$

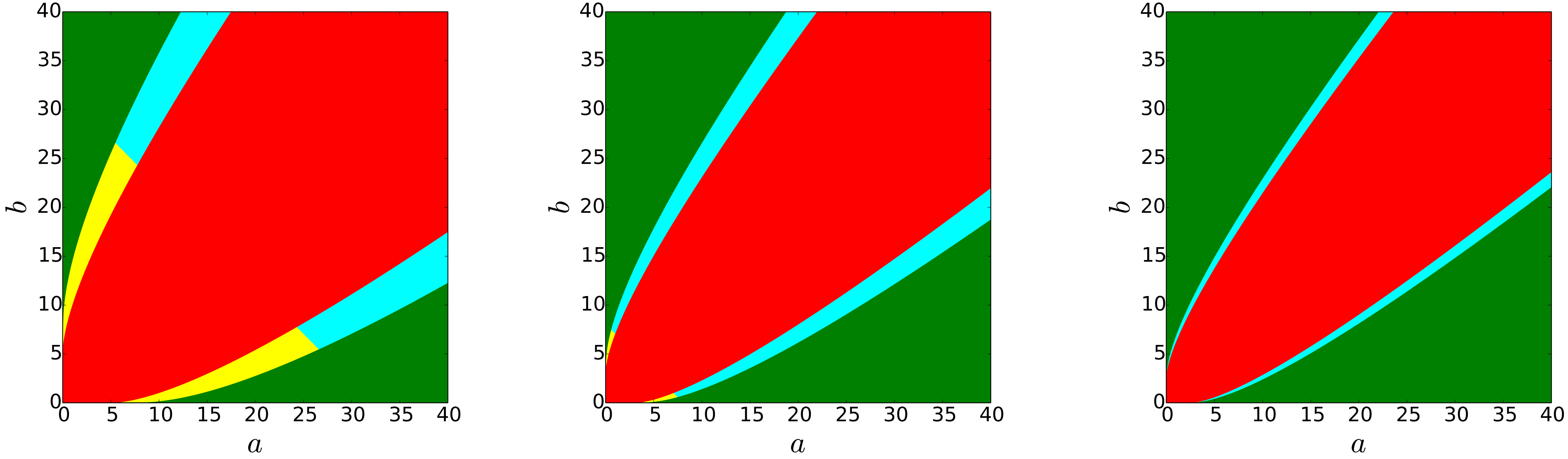
Phase diagrams

Exact community recovery possible from G_1

Exact community recovery impossible from (G_1, G_2)

Exact community recovery impossible from G_1, possible from (G_1, G_2)

Exact community recovery impossible from G_1, exact recovery of π_* impossible



$s = 0.25$ $s = 0.5$ $s = 0.75$

Phase diagrams

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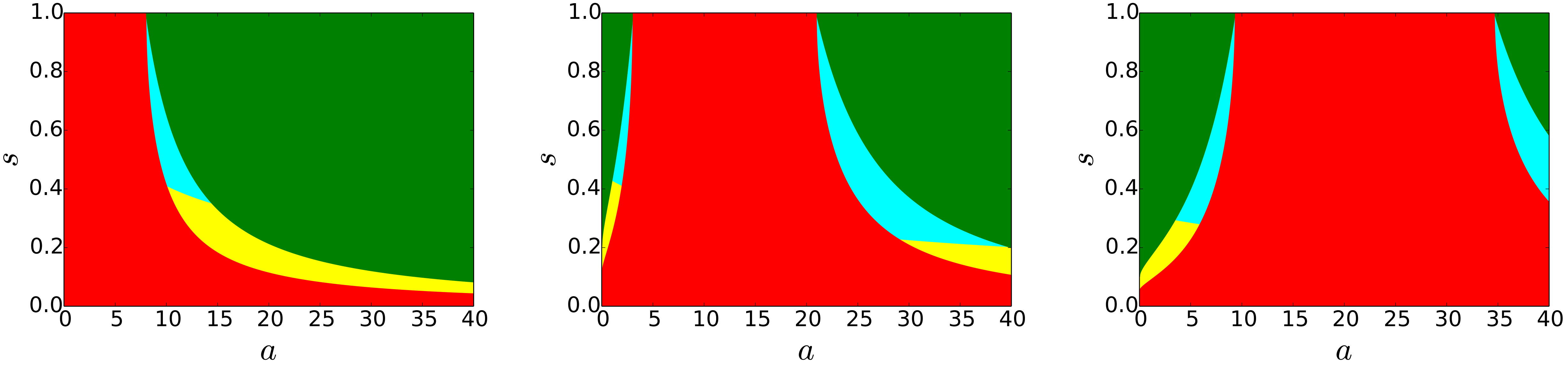

$b = 2$ $\qquad$ $b = 10$ $\qquad$ $b = 20$

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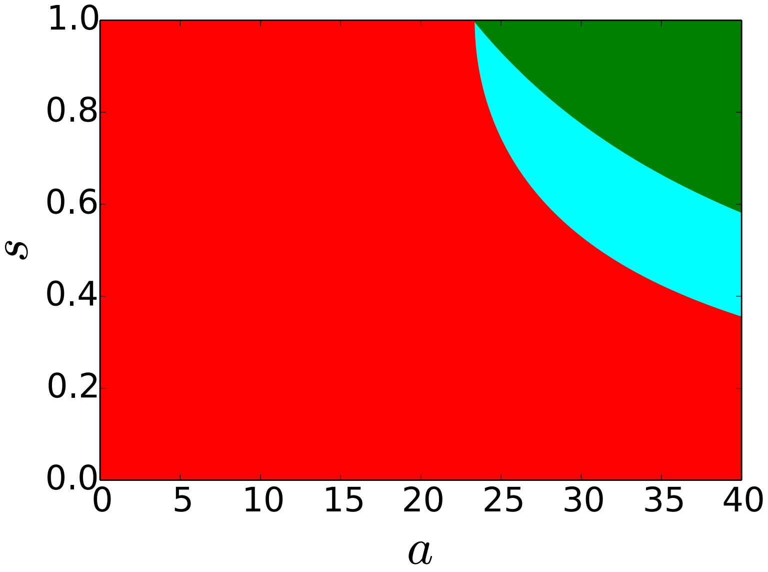
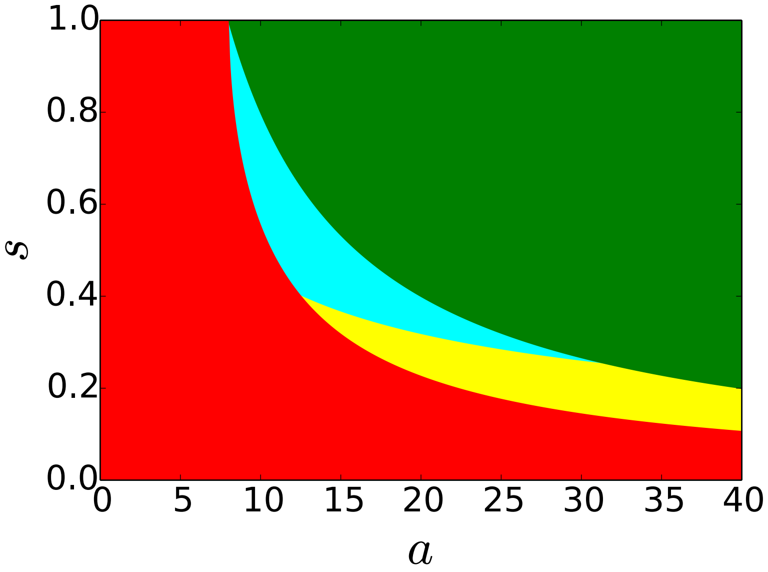
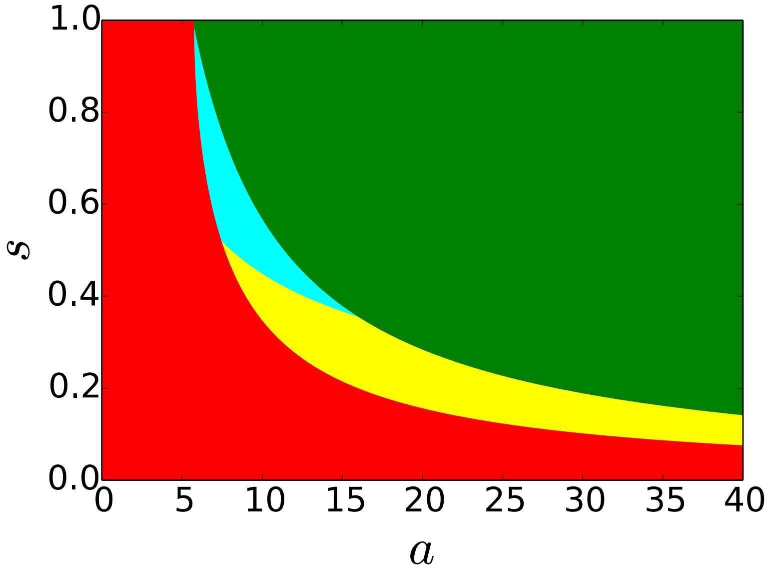


$a/b = 2$



$a/b = 4$



$a/b = 6$

Proof (graph matching)

Estimator

A, B: adjacency matrices of G_1, G_2

$$\widehat{\pi}(G_1, G_2) \in \arg\max_{\pi \in \mathcal{S}_n} \sum_{(i,j) \in \mathcal{E}} A_{i,j} B_{\pi(i), \pi(j)}$$

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Permutation $\pi \in \mathcal{S}_n$ on vertices

$\Updownarrow$ $\tau = \ell(\pi)$

Lifted permutation $\tau: \mathcal{E} \to \mathcal{E}$ on vertex pairs

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$$X(\tau) := \sum_{e \in \mathcal{E}} A_e B_{\tau_*(e)} - \sum_{e \in \mathcal{E}} A_e B_{\tau(e)} = \sum_{e \in \mathcal{E} : \tau(e) \neq \tau_*(e)} \left(A_e B_{\tau_*(e)} - A_e B_{\tau(e)} \right)$$

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If $X(\tau) > 0$ for every $\tau \neq \tau_*$, then $\hat{\pi} = \pi_*$

Permutations

Let S_{k_1,k_2} denote the set of lifted permutations such that

- k_1 vertices are mismatched in V_+
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(relative to π_*)

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Assume that the communities are approximately balanced (this happens whp).

$$\mathcal{F}_\epsilon := \left\{\left(1 - \frac{\epsilon}{2}\right)\frac{n}{2} \leq |V_+|, |V_-| \leq \left(1 + \frac{\epsilon}{2}\right)\frac{n}{2}\right\}$$

Lemma

When $k_1 \leq \frac{\epsilon}{2}|V_+|$ and $k_2 \leq \frac{\epsilon}{2}|V_-|$:

$$M^+(\tau) \geq (1-\epsilon)\frac{n}{2}(k_1 + k_2),$$
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In general:

$$M^+(\tau) \geq (1-\epsilon)\frac{n}{4}(k_1 + k_2),$$
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Analysis

Claim

If $s^2\left(\frac{a+b}{2}\right) > 1$ then there exists $\delta > 0$ such that

$$\boxed{\mathbb{P}\left(\widehat{\tau} \in S_{k_1,k_2} \,|\, \boldsymbol{\sigma}, \tau_*\right) \mathbf{1}(\mathcal{F}_\epsilon) \leq n^{-\delta(k_1+k_2)}.}$$

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- Union bound gives factor of $|S_{k_1,k_2}| \leq n^{k_1+k_2}$

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Proof sketch:

- Union bound gives factor of $\left|S_{k_1,k_2}\right| \leq n^{k_1+k_2}$
- Individual bound boils down to bounds on the probability-generating function:

$$\mathbb{P}\left(\widehat{\tau} = \tau \,|\, \boldsymbol{\sigma}, \tau_*\right) \leq \mathbb{P}\left(X(\tau) \leq 0 \,|\, \boldsymbol{\sigma}, \tau_*\right) = \mathbb{P}\left(n^{-X(\tau)/2} \geq 1 \,\middle|\, \boldsymbol{\sigma}, \tau_*\right)$$
$$\leq \mathbb{E}\left[\left(1/\sqrt{n}\right)^{X(\tau)} \,\middle|\, \boldsymbol{\sigma}, \tau_*\right]$$

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&\le \mathbb{E}\left[\left(1/\sqrt{n}\right)^{X(\tau)} \,\middle|\, \boldsymbol{\sigma}, \tau_*\right] \\
&\le \exp\left(-(1-\epsilon)s^2\left(aM^+(\tau) + bM^-(\tau)\right)\frac{\log n}{n}\right)
\end{aligned}$$

Generating function

$$M^+(\tau) := \left|\left\{e \in \mathcal{E}^+(\boldsymbol{\sigma}) : \tau(e) \neq \tau_*(e)\right\}\right|,$$
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Joint generating function

$$\boxed{\Phi^\tau(\theta, \omega, \zeta) := \mathbb{E}\left[\theta^{X(\tau)} \omega^{Y^+(\tau)} \zeta^{Y^-(\tau)} \,\middle|\, \boldsymbol{\sigma}, \tau_*\right]}$$

The PGF of only $X(\tau)$ only works when $s^2(a+b)/2 > 2$

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Lemma

For any $\varepsilon \in (0,1)$ and $1 \leq \omega, \zeta \leq 3$, and for all n large enough:

$$\Phi^\tau\left(1/\sqrt{n}, \omega, \zeta\right) \leq \exp\left(-(1-\epsilon)s^2\left(\alpha M^+(\tau) + \beta M^-(\tau)\right)\frac{\log n}{n}\right)$$

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Analysis:

- Decompose according to cycles of $\tau_*^{-1} \circ \tau$; independence across cycles
- For correlated Erdős-Rényi: explicit formulas
- For correlated SBM: recursive bounds

Lemma

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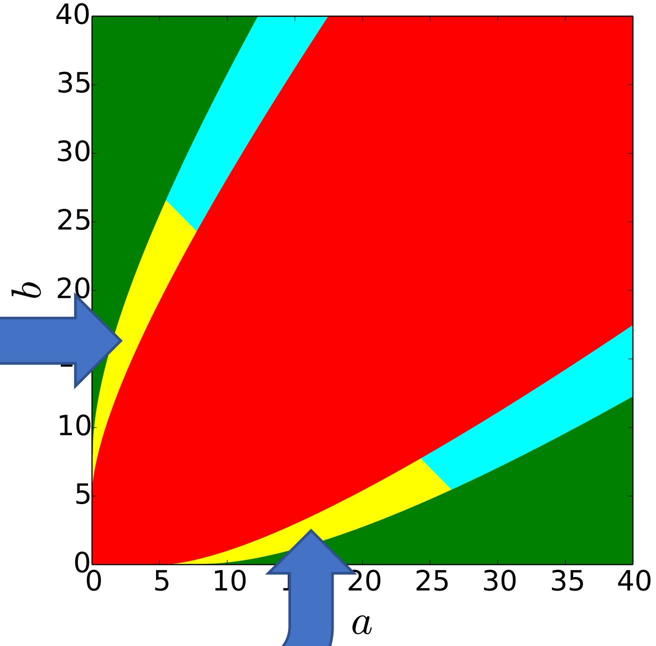
$$\Phi^\tau\left(1/\sqrt{n}, \omega, \zeta\right) \leq \exp\left(-(1-\epsilon)s^2\left(\alpha M^+(\tau) + \beta M^-(\tau)\right)\frac{\log n}{n}\right)$$

The interplay between community recovery and graph matching

Closing the gap for exact community recovery



$s = 0.25$

What happens in this region?

- Exact community recovery possible from G_1
- Exact community recovery impossible from (G_1, G_2)
- Exact community recovery impossible from G_1, possible from (G_1, G_2)
- Exact community recovery impossible from G_1, exact recovery of π_* impossible

- Exact community recovery is impossible from G_1
- Exact graph matching is impossible
- **Q:** is exact community recovery from (G_1, G_2) possible?

Interplay btw community recovery and graph matching

Theorem (Gaudio, R., Sridhar, 2022)

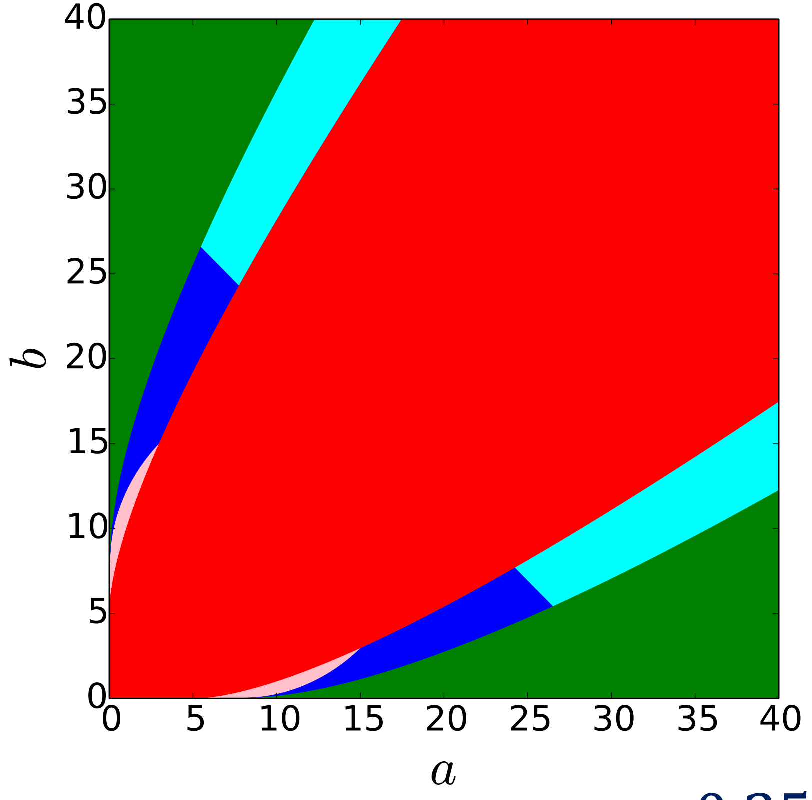
In the regime where $\left|\sqrt{a}-\sqrt{b}\right| > \sqrt{2/(1-(1-s)^2)}$,

the threshold for exact community recovery is given by:

$$\boxed{s^2\left(\frac{a+b}{2}\right) + s(1-s)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2 = 1}$$

graph matching

community recovery



$s = 0.25$

Phase diagrams

Exact graph matching:

possible

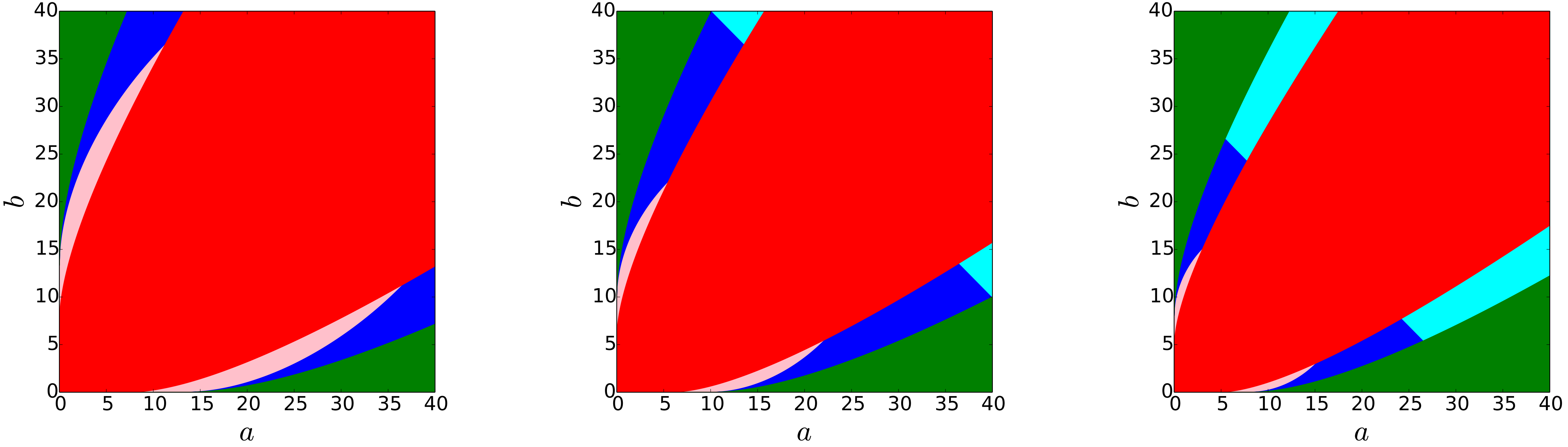
impossible

Exact community recovery possible from G_1

Exact community recovery impossible from G_1, possible from (G_1, G_2)

Exact community recovery impossible from (G_1, G_2)

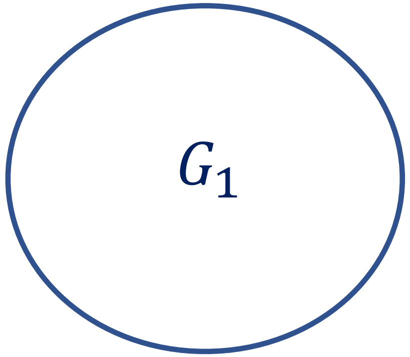
(though possible if π_* were known)



$s = 0.15$ $s = 0.20$ $s = 0.25$

Algorithm

1. Almost exact labeling of G_1
   [Mossel, Neeman, Sly, 2014]



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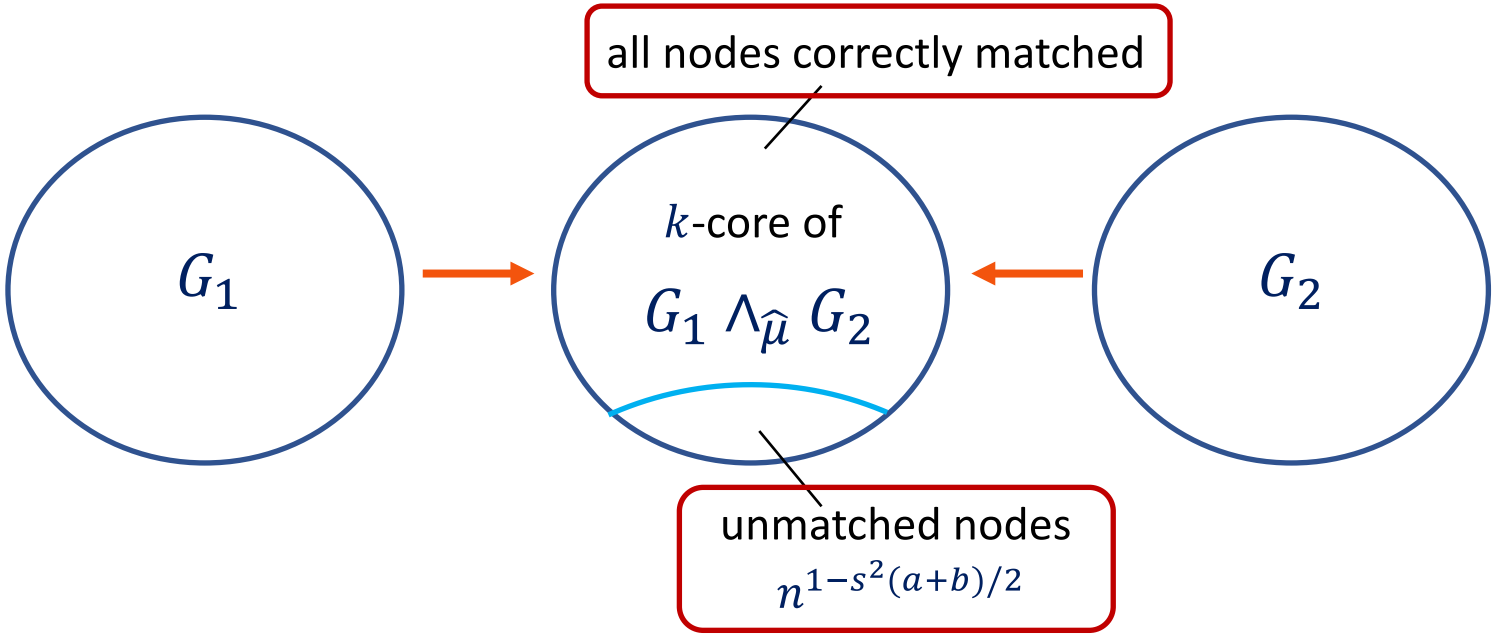

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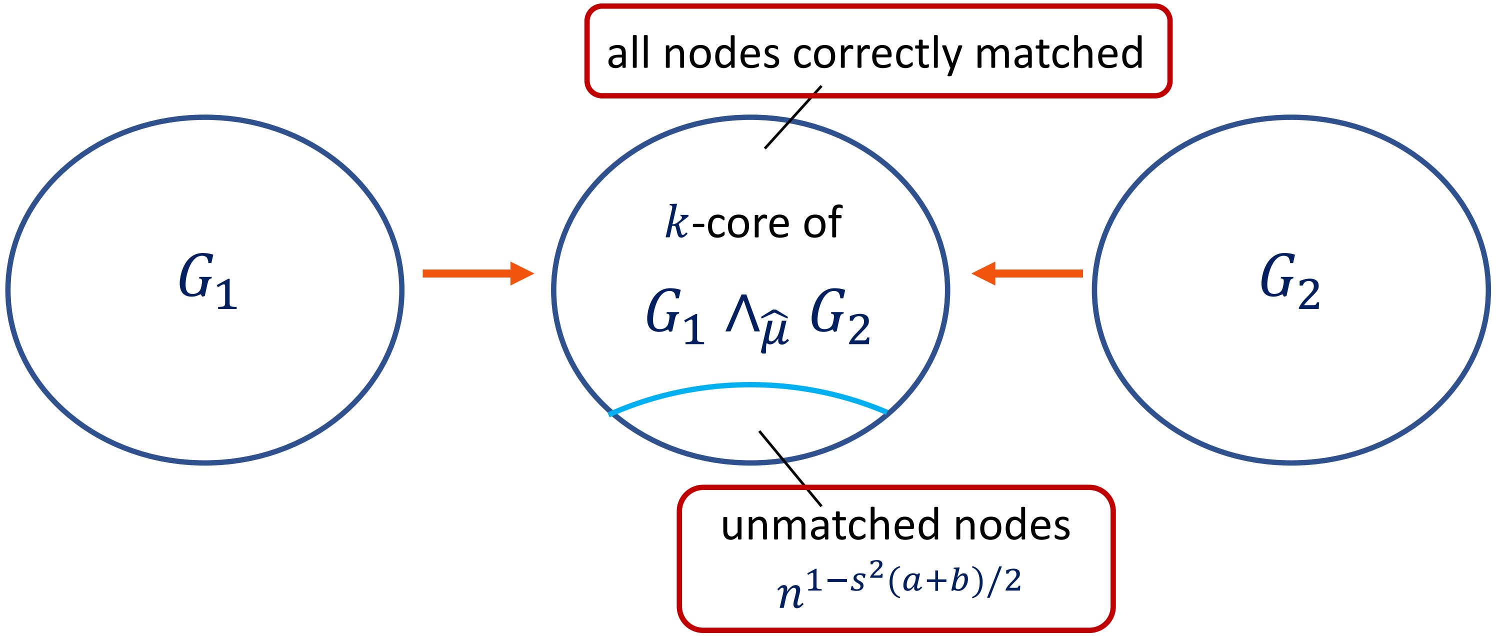

Remarks on the k-core estimator:

- Works well for correlated inhomogeneous random graphs [R., Sridhar, 2023]
- Closely related to densest subgraph estimator [Ding, Du, 2022a,b]

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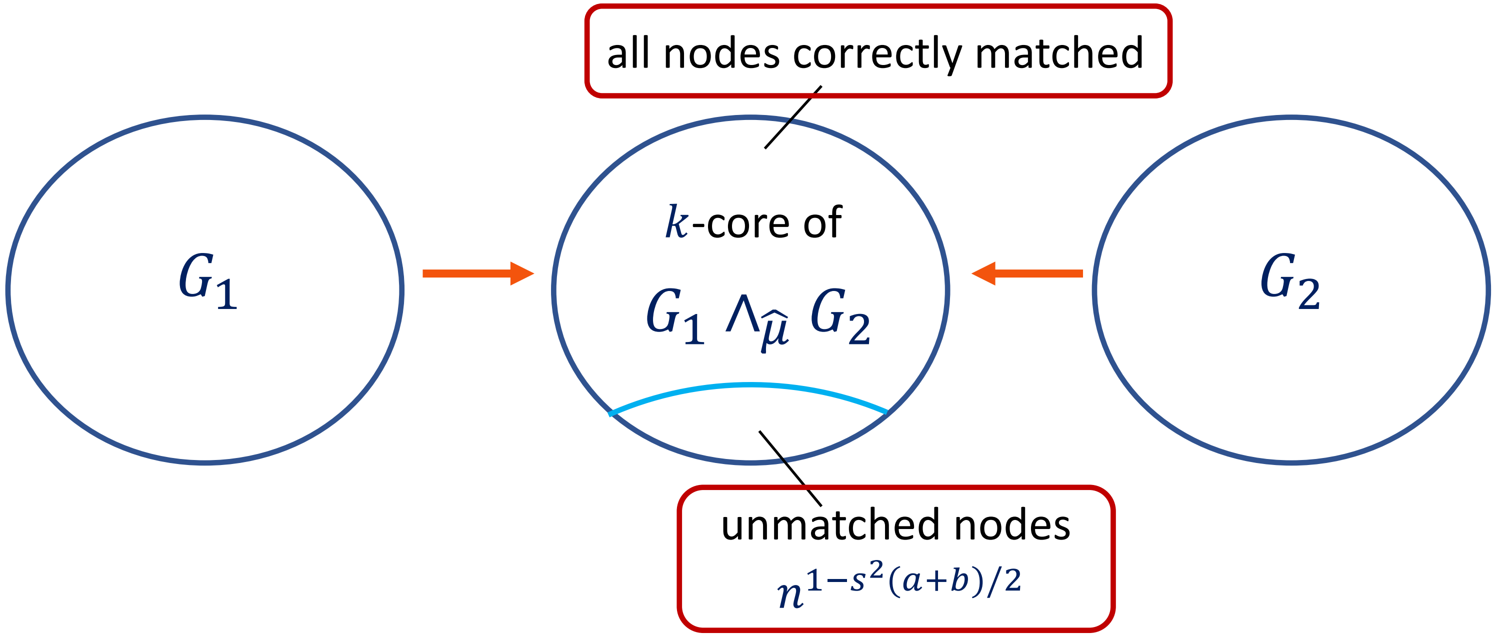

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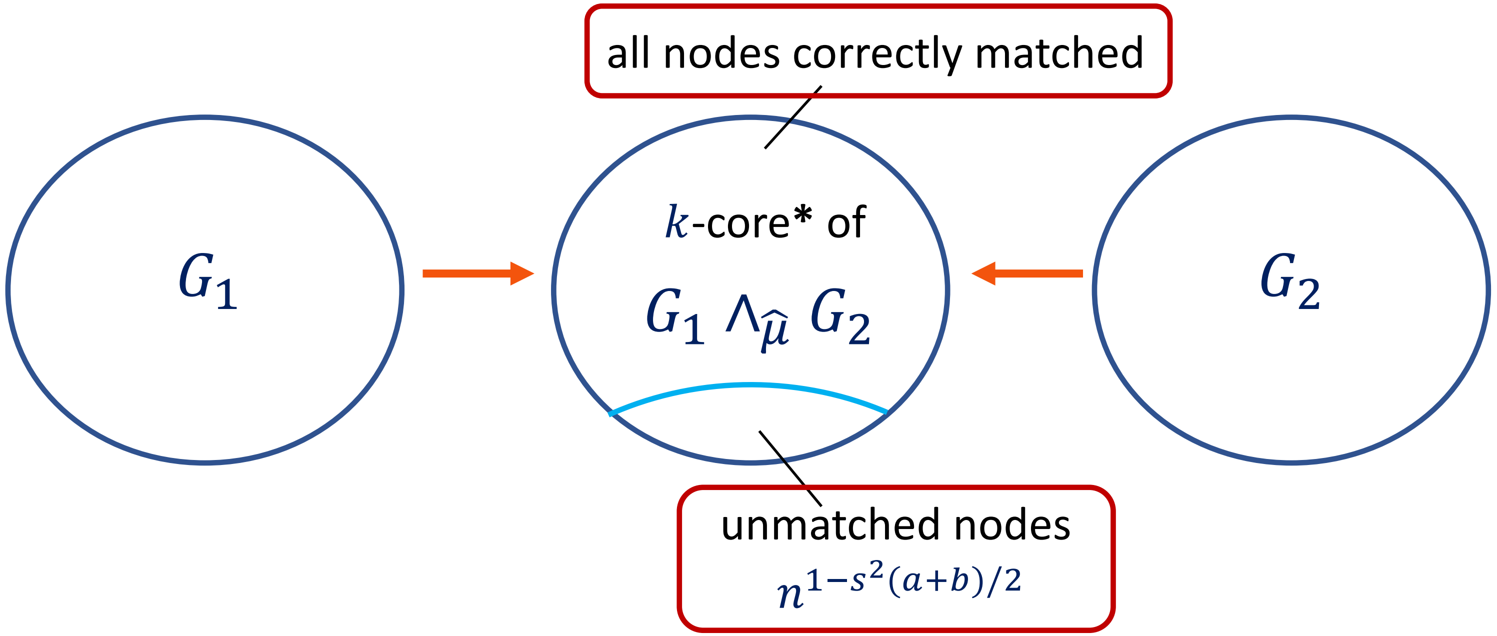

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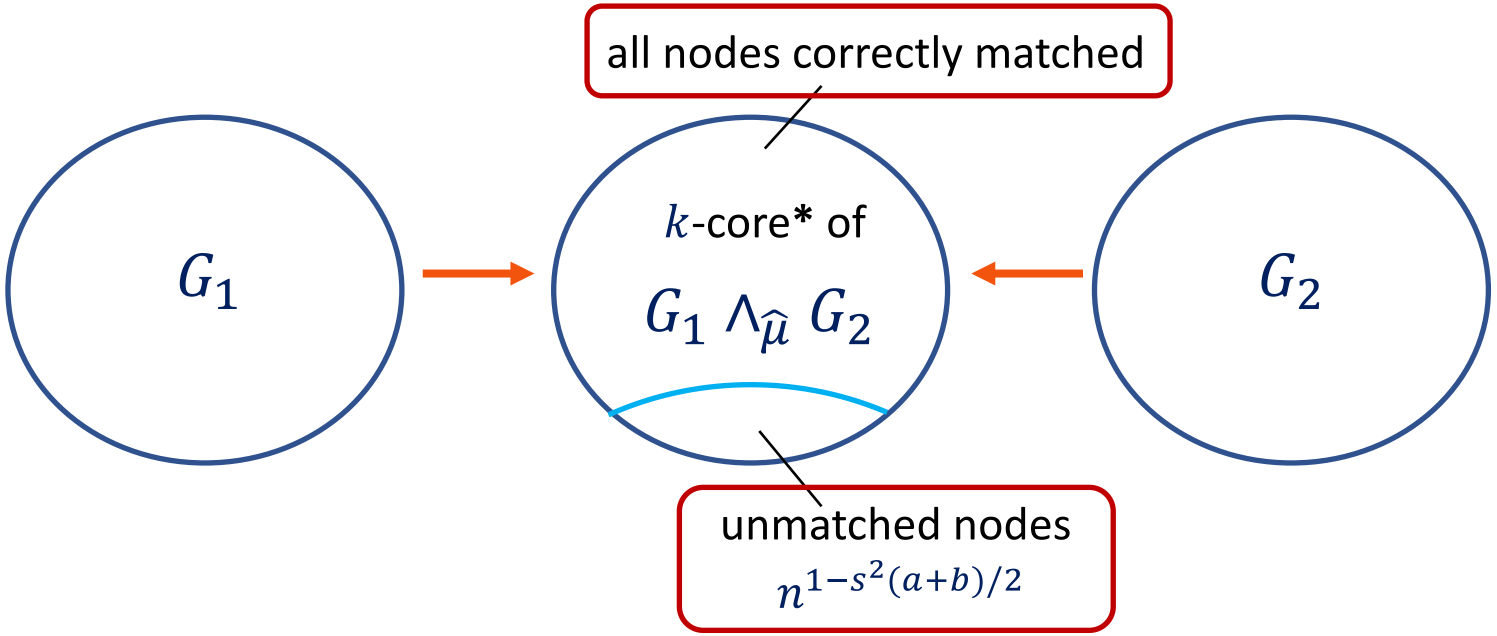

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- Consider $G_1 \vee_{\hat{\mu}} G_2$
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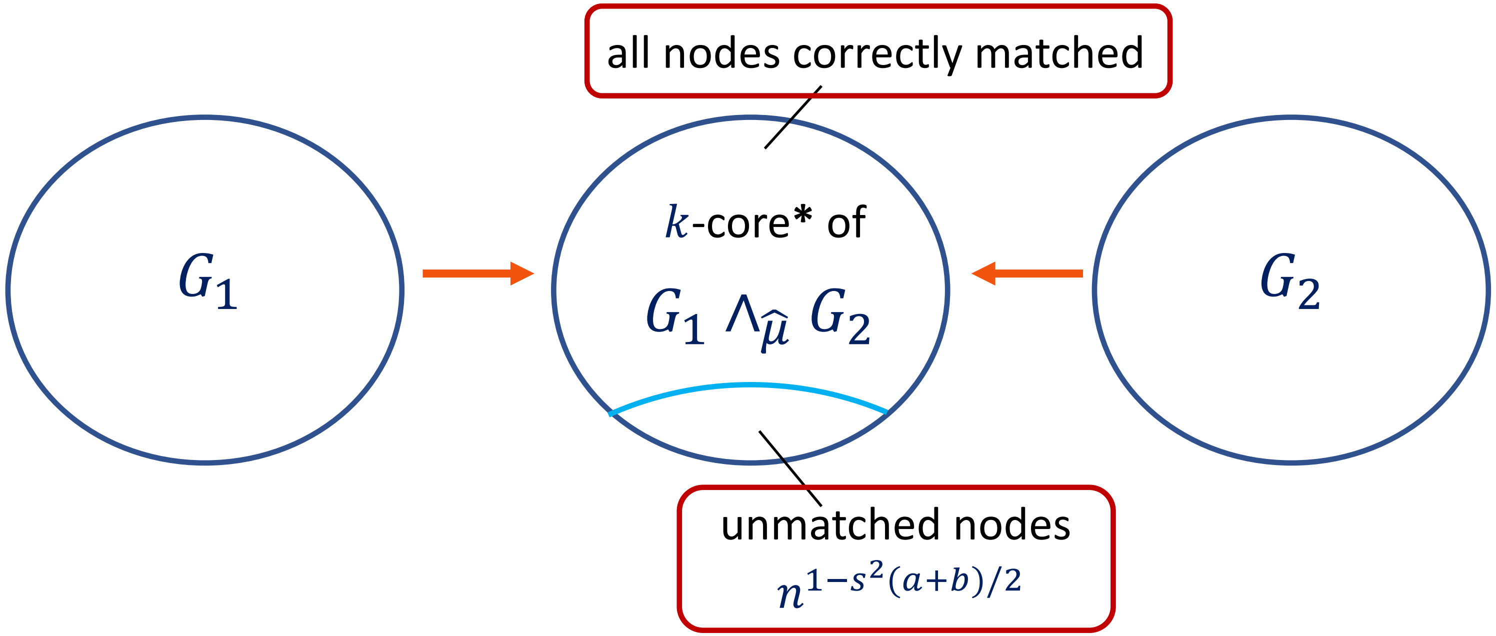

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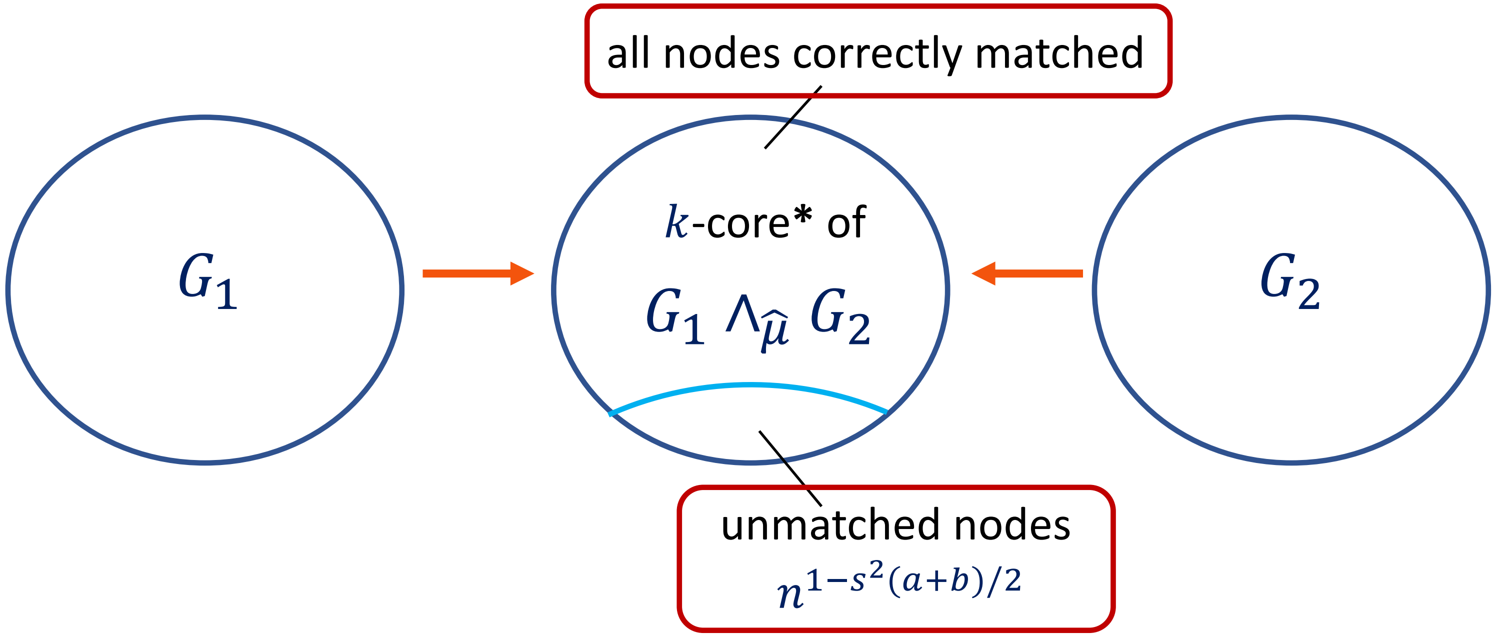

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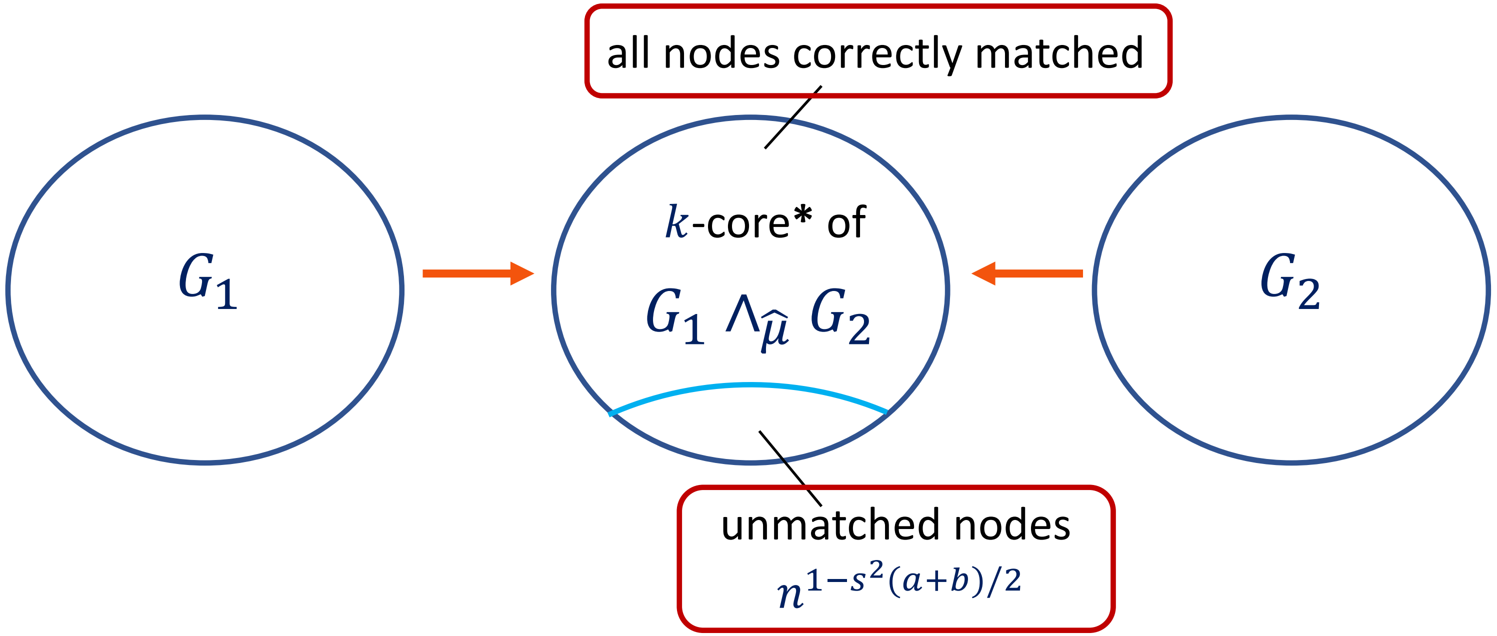

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$$s^2\left(\frac{a+b}{2}\right) + s(1-s)\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2 = 1$$

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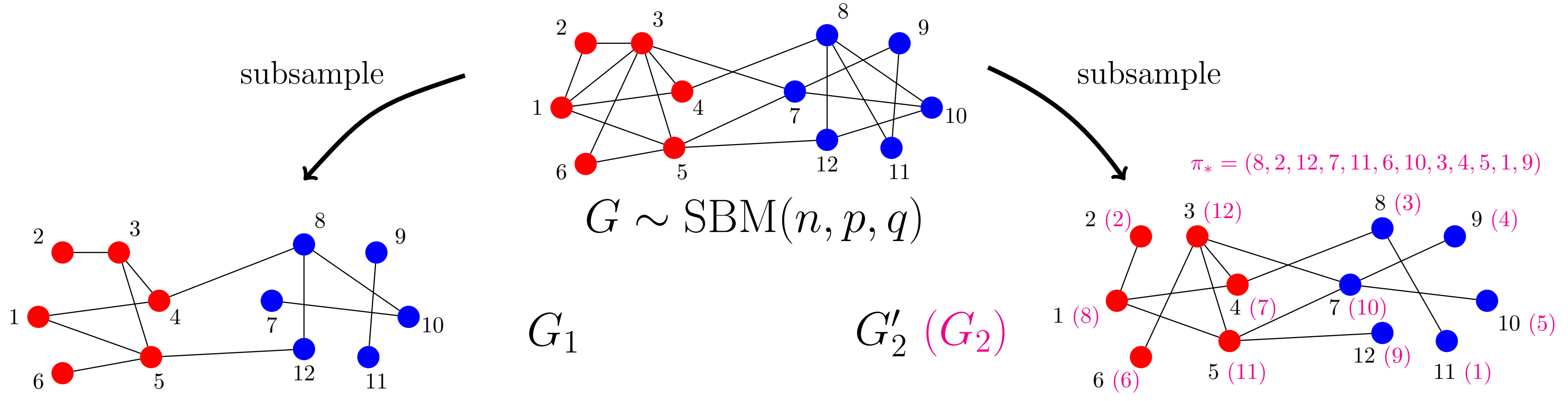
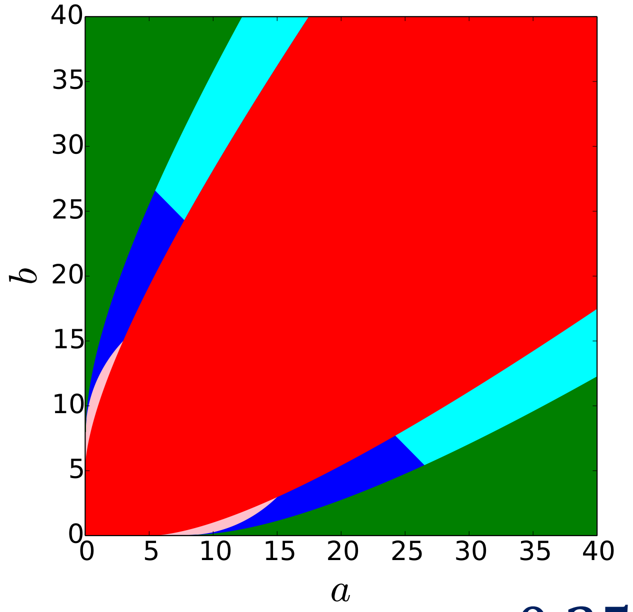
Impossibility argument sketch

- S_*: singletons in the intersection graph $G_1 \wedge_{\pi_*} G_2$
- Key: $|S_*| \asymp n^{1-s^2(a+b)/2}$
- MAP estimator fails even if given:
  - All community labels in G_2
  - S_*
  - π_* on $[n] \setminus S_*$
- Proof uses careful second moment analysis

Open problems / future directions

Efficient algorithms





$s = 0.25$

- Current algorithms for (exact) graph matching are not efficient
- Do there exist efficient algorithms for graph matching?

Exciting and promising recent developments for efficient graph matching for correlated Erdős—Rényi random graphs:

- Mao, Rudelson, Tikhomirov (2021)
- Mao, Wu, Xu, Yu (2022)

Beyond exact community recovery

- Almost exact recovery? → Improved error rate?
- Partial recovery? → Improved fraction recovered?
- Community detection? → Lower threshold?

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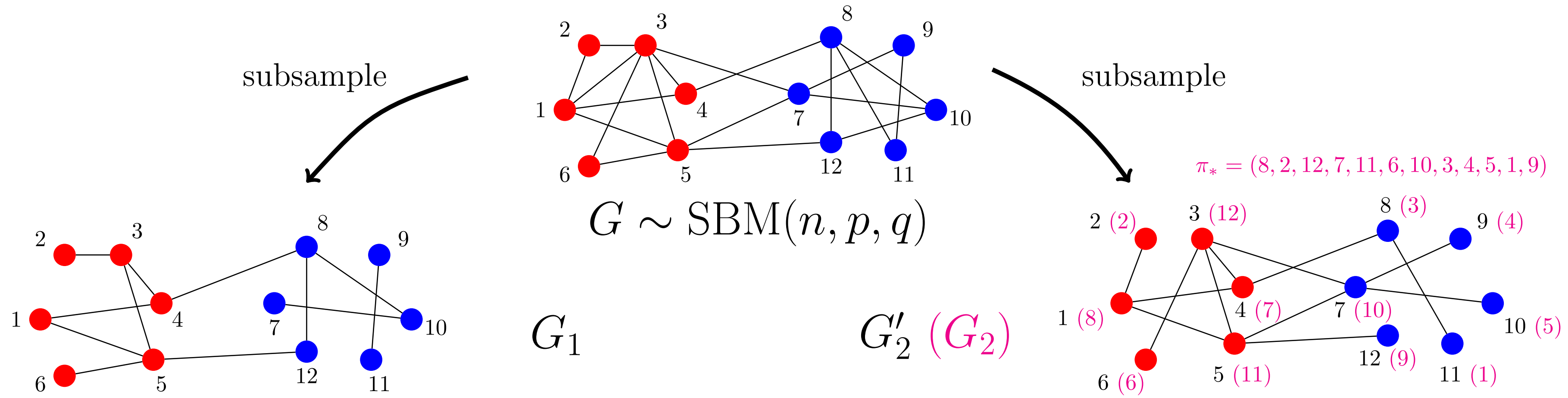
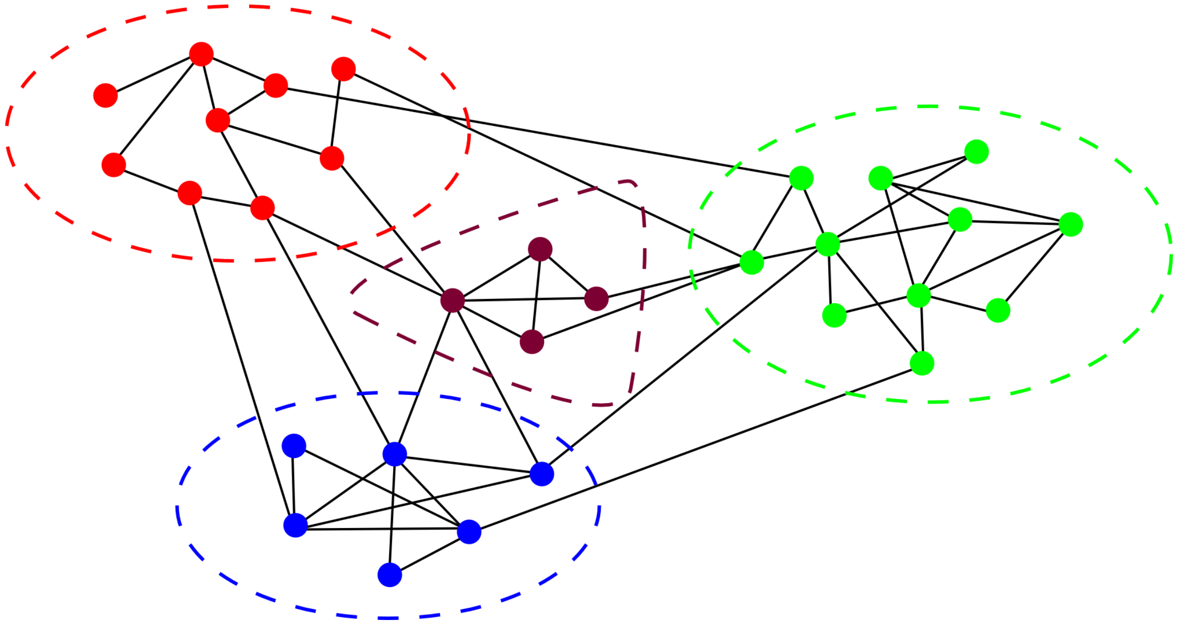
Challenge:
interplay between community recovery and graph matching

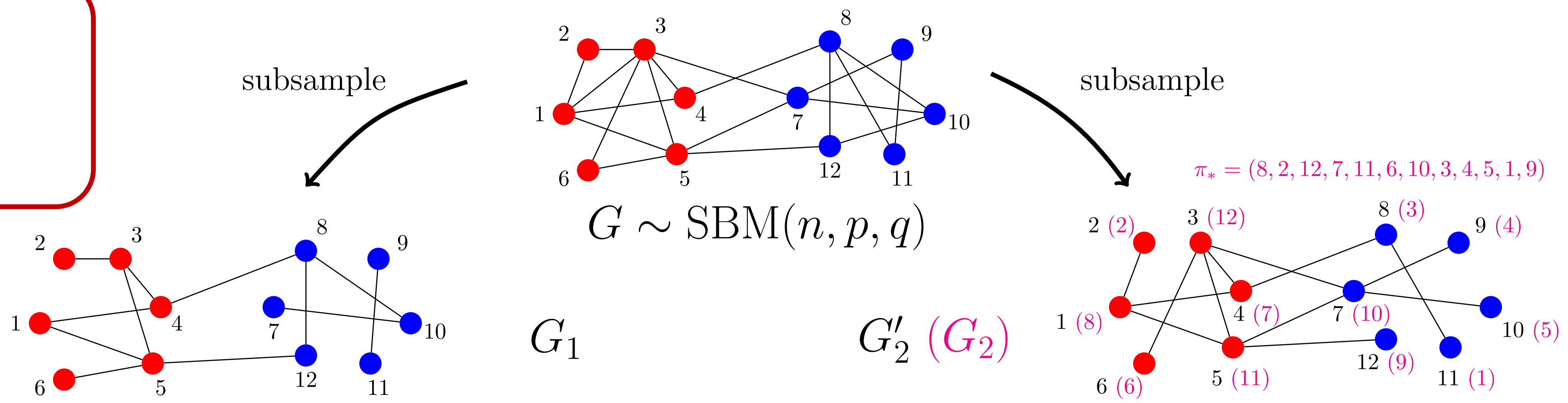
General correlated SBMs





k communities, general parameters:
- graph matching?
- (exact) community recovery?

Summary



- **Correrelated SBMs:** determined the fundamental limits of

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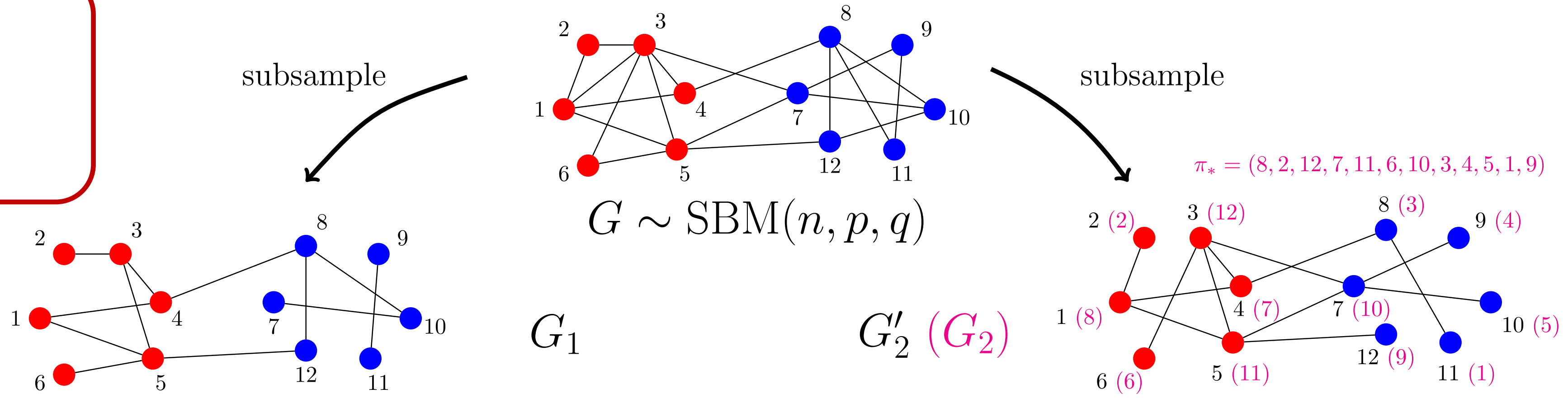
Summary



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Thank you!