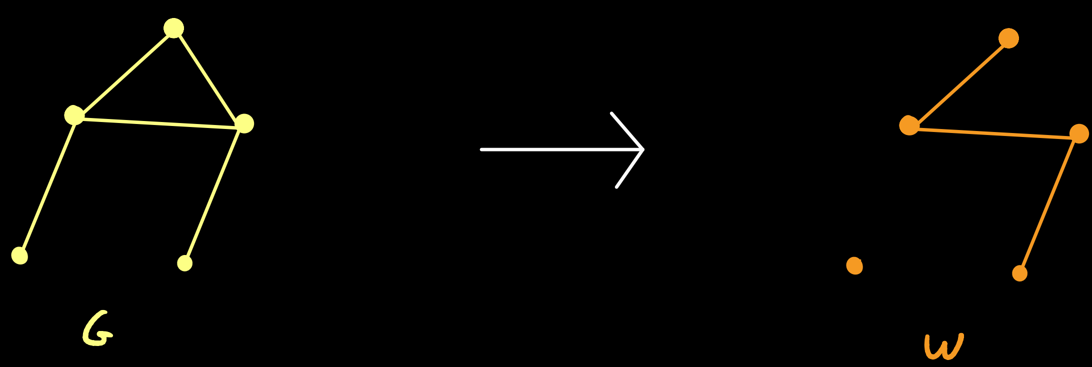
Percolation on finite transitive graphs

Joint with Tom Hutchcroft

Let $G=(V,E)$ be a graph. [countable, locally finite, connected]

Let $p\in[0,1]$.

Bernoulli bond percolation $\mathbb{P}_p^G := \left\{ \begin{array}{c} \text{law of random spanning subgraph } \omega : E\to\{0,1\} \\ (\omega(e))_{e\in E} \sim \text{iid Bernoulli}(p). \end{array} \right\}$



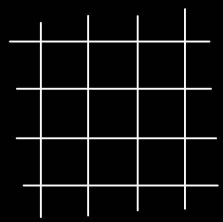
Connected components of ω are called clusters.

G infinite transitive

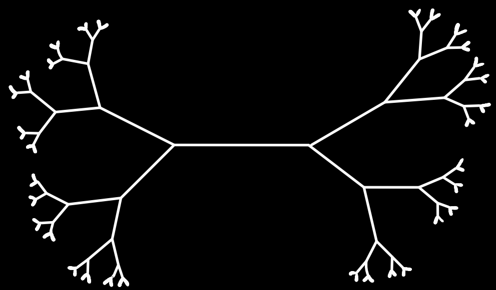
(Benjamini–Schramm '96)

Transitive means $\forall u,v \in V \ \exists \phi \in \mathrm{Aut}\, G : \ \phi(u) = v$.

Eg:



$\mathbb{Z}^d$



Regular tree

[Cayley graph of infinite finitely generated group]

Fact: $\exists p_c \in [0,1]$ such that
$$\mathbb{P}_p^G(\exists \infty \text{ cluster}) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \\ ? & p = p_c \end{cases}$$

Questions: When is $p_c \in (0,1)$?

How many ∞ clusters?

Is $\mathbb{P}_{p_c}^G(\exists \infty \text{ cluster}) = 0$?

Questions: When is $p_c \in (0,1)$?

Thm: (Duminil-Copin, Goswami, Raoufi, Severo, Yadin 2018)

$p_c \in (0,1)$ iff G is not 1-dimensional.

How many ∞ clusters?

Say G is amenable if $\displaystyle\inf_{\substack{W \subseteq V \\ W \text{ finite}}} \frac{|\partial W|}{|W|} = 0.$

Thm: (Aizenman, Kesten, Newman '87) (Burton, Keane '89)

If G amenable then $\forall p$: "$\exists \infty$ cluster $\Rightarrow \exists! \infty$ cluster" $\mathbb{P}_p$-a.s.

Converse remains open.

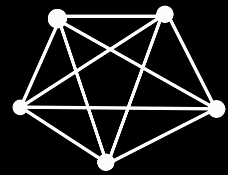
Is $\mathbb{P}_{p_c}^G(\exists \infty \text{ cluster}) = 0$?

Conjectured to hold iff G is not 1-dimensional.

Open for $\mathbb{Z}^3$!

G_n = complete graph on n vertices

(Erdős, Rényi '59)



Enumerate clusters $K_1, K_2, \dots$ with $|K_1| \geq |K_2| \geq \dots$ ($|K|$:= # vertices in K)

let $\|K\| := \frac{|K|}{|V(G)|}$.

- (G_n) has a percolation threshold at $(1/n)$:

  (p_n) supercritical [i.e. $\liminf_{n\to\infty} \frac{p_n}{n} > 1$] $\Rightarrow$ $\exists \varepsilon > 0$: $\lim_{n\to\infty} \mathbb{P}_{p_n}^{G_n}(\|K_1\| \geq \varepsilon) = 1$.

  (p_n) subcritical [i.e. $\limsup_{n\to\infty} \frac{p_n}{n} < 1$] $\Rightarrow$ $\forall \varepsilon > 0$: $\lim_{n\to\infty} \mathbb{P}_{p_n}^{G_n}(\|K_1\| \geq \varepsilon) = 0$.

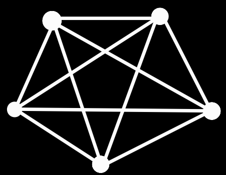
- (G_n) has the supercritical uniqueness property:

  $\forall$ supercritical (p_n): $\|K_2\| \xrightarrow{P} 0$ under $\mathbb{P}_{p_n}^{G_n}$.

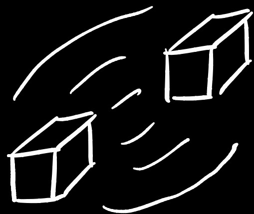
- (G_n) has the supercritical concentration property:

  $\forall$ supercritical (p_n): $\|K_1\| - \mathbb{E}_{p_n}^{G_n}\|K_1\| \xrightarrow{P} 0$ under $\mathbb{P}_{p_n}^{G_n}$.

G_n finite transitive, $|V(G_n)| \to \infty$

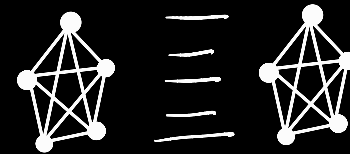


K_n



$\{0,1\}^n$

$(\mathbb{Z}/n\mathbb{Z})^d$



$K_n \square K_2$

[Cayley graph of a finite group]

Questions: When does (G_n) have ...

- percolation threshold?
- supercritical uniqueness property?
- supercritical concentration property?

⚠ Supercritical uniqueness comes first!

Say (p_n) supercritical if $\exists \varepsilon > 0 \ \forall n$: $\mathbb{P}^{G_n}_{(1-\varepsilon)p_n}\left(\|K_1\| \geq \varepsilon\right) \geq \varepsilon$.

Warm-up: (G_n) has bounded vertex degrees

Conjecture: (Benjamini '01)

(G_n) has the supercritical uniqueness property.

(Previously known for tori and (not necessarily transitive) expanders)

Thm: (E., Hutchcroft)

(G_n) has … supercritical uniqueness property ('21)

percolation threshold ('22)

supercritical concentration property ('22+)

Thm: (Hutchcroft, Tointon '21)

If (G_n) has "at least $(1+\varepsilon)$-dimensional growth" then $\exists \varepsilon > 0$ $\forall n$: $\mathbb{P}_{1-\varepsilon}^{G_n}(\|K_1\| \geq \varepsilon) \geq \varepsilon$.

Say $(G_n) \to G$ **locally** if $\forall R \;\exists N \;\forall n \geq N: \; B_R^{G_n} \cong B_R^{G}$.

Thm: (E., Hutchcroft '22+)

If $(p, p, p, \dots)$ is supercritical and $G_n \to G$ locally, then

$$\mathbb{P}_p^{G_n}(0 \in K_1) \longrightarrow \mathbb{P}_p^{G}(0 \in \text{infinite cluster}).$$

Conjecture: (Schramm)

Let (G_n) be a sequence of infinite transitive graphs with $\limsup_{n\to\infty} p_c(G_n) < 1$.

If $G_n \to G$ locally, then $p_c(G_n) \to p_c(G)$.

The analogue for finite G_n should hold for "nice" sequences.
↳ not 1-dimensional ?

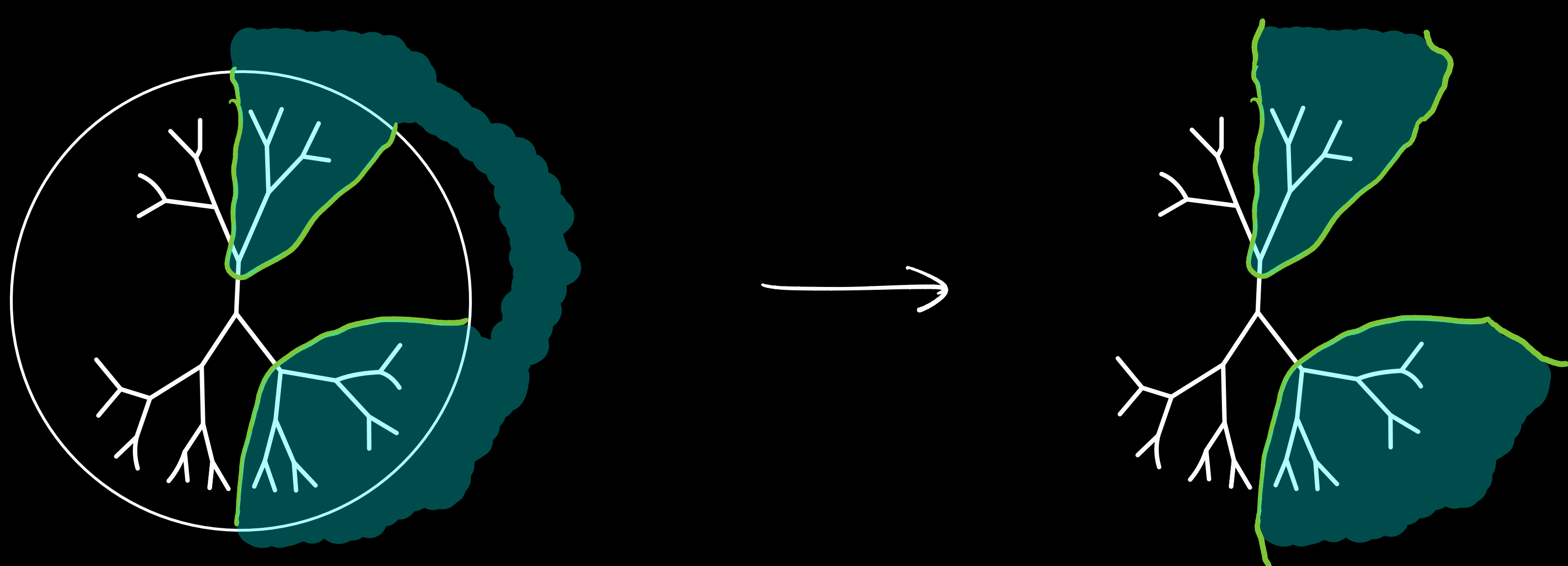
A curious case...

Suppose

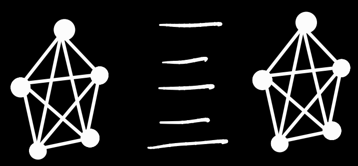
- (G_n) is "nice"
- G is nonamenable

  ↳ $\exists p > p_c(G)$ s.t. $\mathbb{P}_p^G(\exists \text{multiple } \infty\text{-clusters}) = 1$.
- $G_n \to G$ locally.

How are $\left\{ \begin{array}{l} \text{uniqueness of the giant cluster in } \mathbb{P}_p^{G_n} \\ \text{non-uniqueness of the } \infty \text{ cluster in } \mathbb{P}_p^G \end{array} \right\}$ consistent?



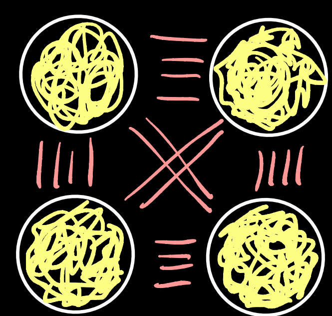
General case: (G_n) may have unbounded vertex degrees



does <u>not</u> have the supercritical uniqueness property.

At $p_n = \frac{2}{n}$:

Same argument works for similar constructions, called molecules.



<u>Thm</u>: (E., Hutchcroft '21, 22, 22+)

These are the only obstacles to

- existence of percolation threshold
- supercritical uniqueness property
- supercritical concentration property

Sketch of proof of Benjamini's conjecture

Fix (G_n) with bounded vertex degrees and $|V(G_n)| \to \infty$.

Cor: (Talagrand '94)

Every increasing and $\mathrm{Aut}\, G_n$-invariant event has threshold width $O\left(\frac{1}{\log |V(G_n)|}\right)$.

Fix (p_n) supercritical.

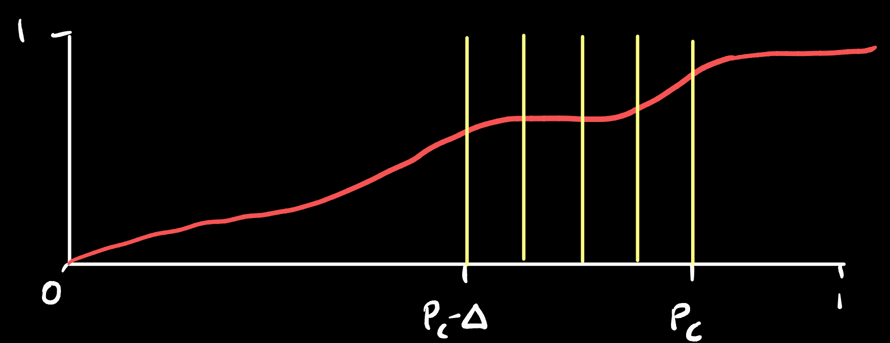
Pick $\Delta > 0$ s.t. $(p_n - \Delta)$ supercritical.

WTS: $\|K_2\| \xrightarrow{P} 0$ under $\mathbb{P}_{p_n}^{G_n}$.

Claim: $\exists (q_n)$ such that
- $\forall n: \quad p_n - \Delta \le q_n \le p_n$
- $\|K_1\| - \mathbb{E}^{G_n}_{q_n} \|K_1\| \xrightarrow{P} 0 \quad$ under $\mathbb{P}^{G_n}_{q_n}$

Proof: Let $n \ge 1$.

Define $M(p) :=$ median of $\|K_1\|$ under $\mathbb{P}^{G_n}_p$.



$$\forall N \quad \exists q \in (p_c - \Delta, p_c) : \quad \left| M\left(q + \tfrac{\Delta}{N}\right) - M\left(q - \tfrac{\Delta}{N}\right) \right| \le \frac{2}{N}.$$

Note that:
- $\mathbb{P}^{G_n}_{q - \frac{\Delta}{N}} \left(\|K_1\| \ge M(q) - \frac{2}{N} \right) \ge \frac{1}{2}$
- $\mathbb{P}^{G_n}_{q + \frac{\Delta}{N}} \left(\|K_1\| \le M(q) + \frac{2}{N} \right) \ge \frac{1}{2}$

- $\mathbb{P}^{G_n}_{q-\frac{\Delta}{N}}\left(\|K_1\| \geq M(q) - \frac{2}{N}\right) \geq \frac{1}{2}$
- $\mathbb{P}^{G_n}_{q+\frac{\Delta}{N}}\left(\|K_1\| \leq M(q) + \frac{2}{N}\right) \geq \frac{1}{2}$

Set $N = \sqrt{\log |V(G_n)|}$ so that

$$[\text{threshold width}] \underset{\text{Talagrand}}{\lesssim} \frac{1}{\log |V(G_n)|} \ll \frac{\Delta}{N}.$$

Therefore:

- $\mathbb{P}^{G_n}_{q}\left(\|K_1\| \geq M(q) - \frac{1}{N}\right) = 1 - o(1)$
- $\mathbb{P}^{G_n}_{q}\left(\|K_1\| \leq M(q) + \frac{1}{N}\right) = 1 - o(1).$

$\square$

Claim: $\|K_2\| \xrightarrow{P} 0$ under $\mathbb{P}_{q_n}^{G_n}$.

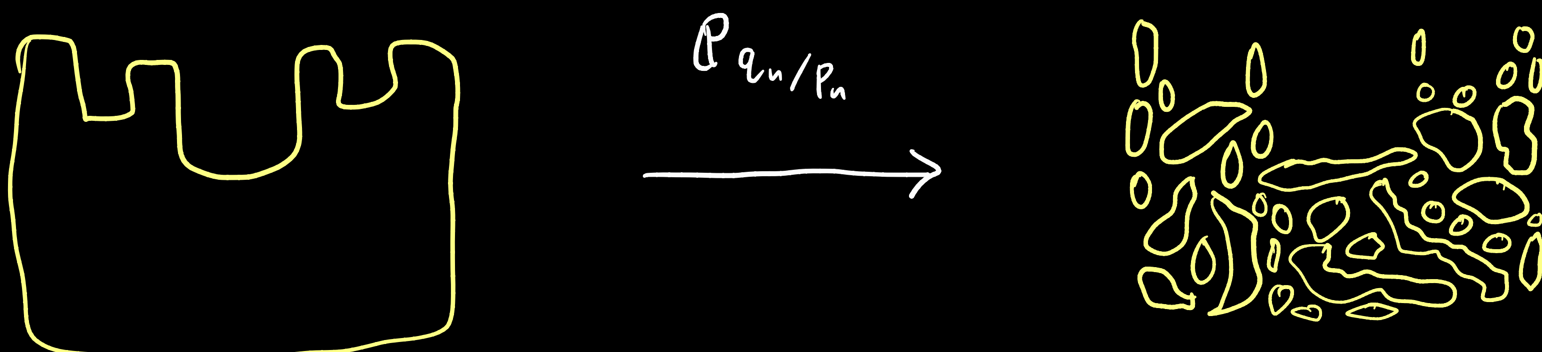
Proof: "concentration $\Rightarrow$ uniqueness"

Assume for contradiction:

- $\exists \varepsilon > 0 \ \forall n: \quad \mathbb{P}_{p_n}^{G_n}(\|K_2\| \geq \varepsilon) \geq \varepsilon$
- $\|K_1\| \xrightarrow{P} \alpha \in (0,1)$ under $\mathbb{P}_{q_n}^{G_n}$.

Def: A subgraph $S \subseteq G$ is called a sandcastle if $\|S\| \geq \varepsilon$ and

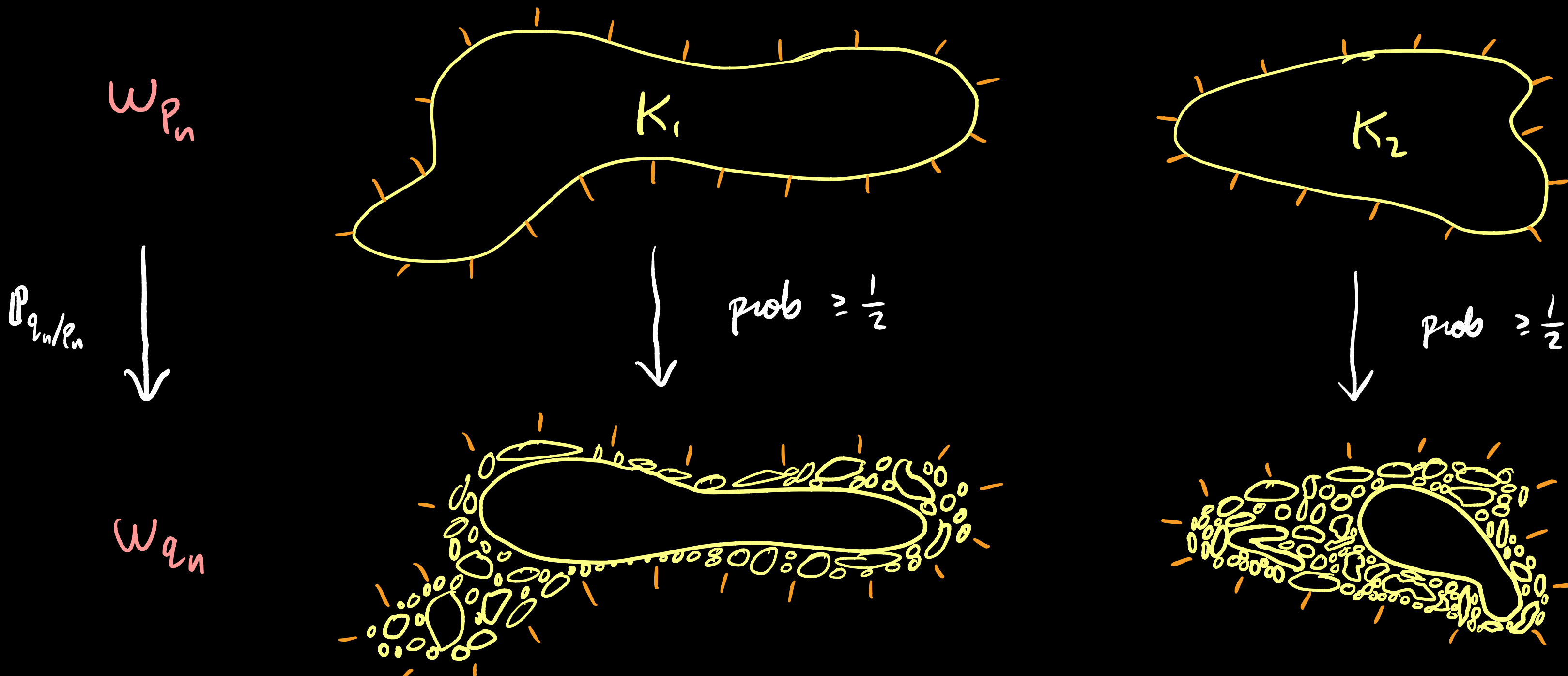
$$\mathbb{P}_{q_n/p_n}^{S}\left(|K_1| \leq \frac{\alpha}{2} \cdot |V(G_n)|\right) \geq \frac{1}{2}$$



Claim: $\exists N \ \forall n \geq N$: $\mathbb{P}_{p_n}^{G_n}(K_1 \text{ or } K_2 \text{ is a sandcastle}) \geq \frac{\varepsilon}{2}$.

Proof: Suppose false at n.

Then $\mathbb{P}_{p_n}^{G_n}(\|K_1\|, \|K_2\| \geq \varepsilon \text{ but } K_1, K_2 \text{ not sandcastles}) \geq \frac{\varepsilon}{2}$.

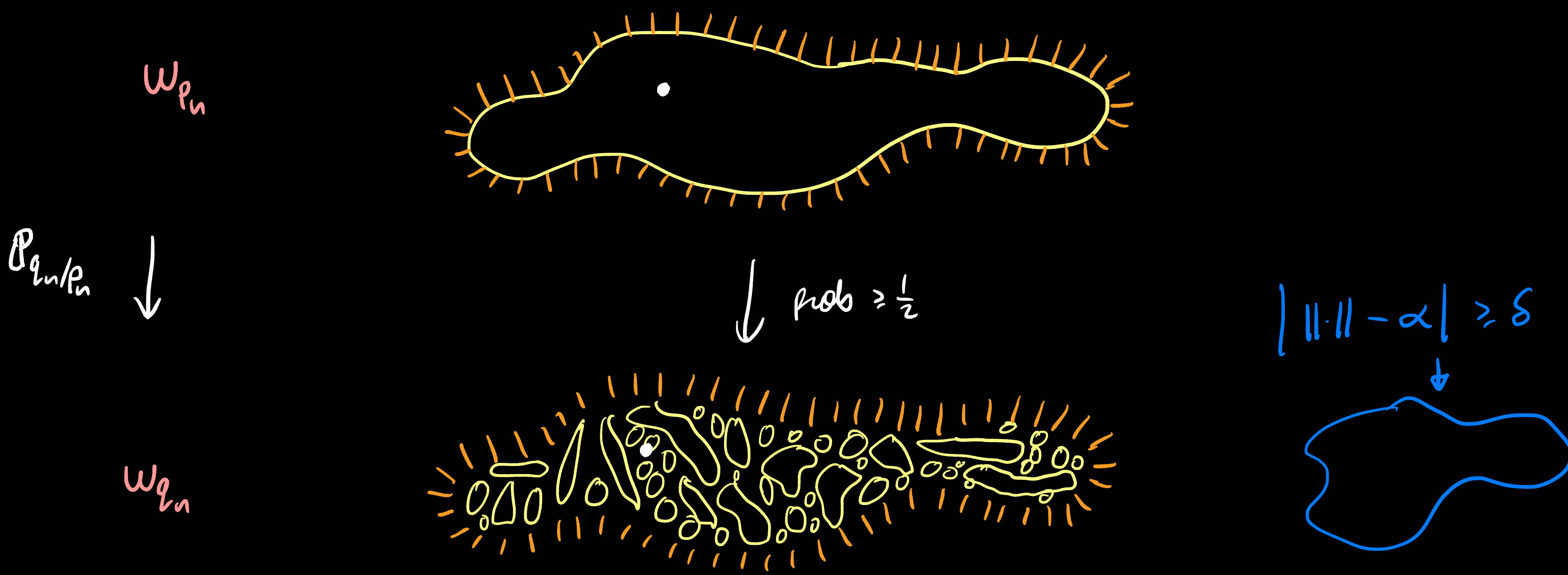


So $\mathbb{P}_{q_n}^{G_n}(\|K_2\| \geq \frac{\alpha}{2}) \geq \frac{\varepsilon}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$. ※ Contradiction for n large.

Claim: $\exists (s_n)$ such that

- $\forall n:\quad s_n \subseteq G_n$ and $\|s_n\| \geq \varepsilon$
- $\|K_1(\omega \setminus \bar{s}_n)\| \xrightarrow{P} \alpha$ under $\mathbb{P}_{q_n}^{G_n}$.

Proof: Otherwise $\exists\, \delta > 0$ s.t. with good probability:



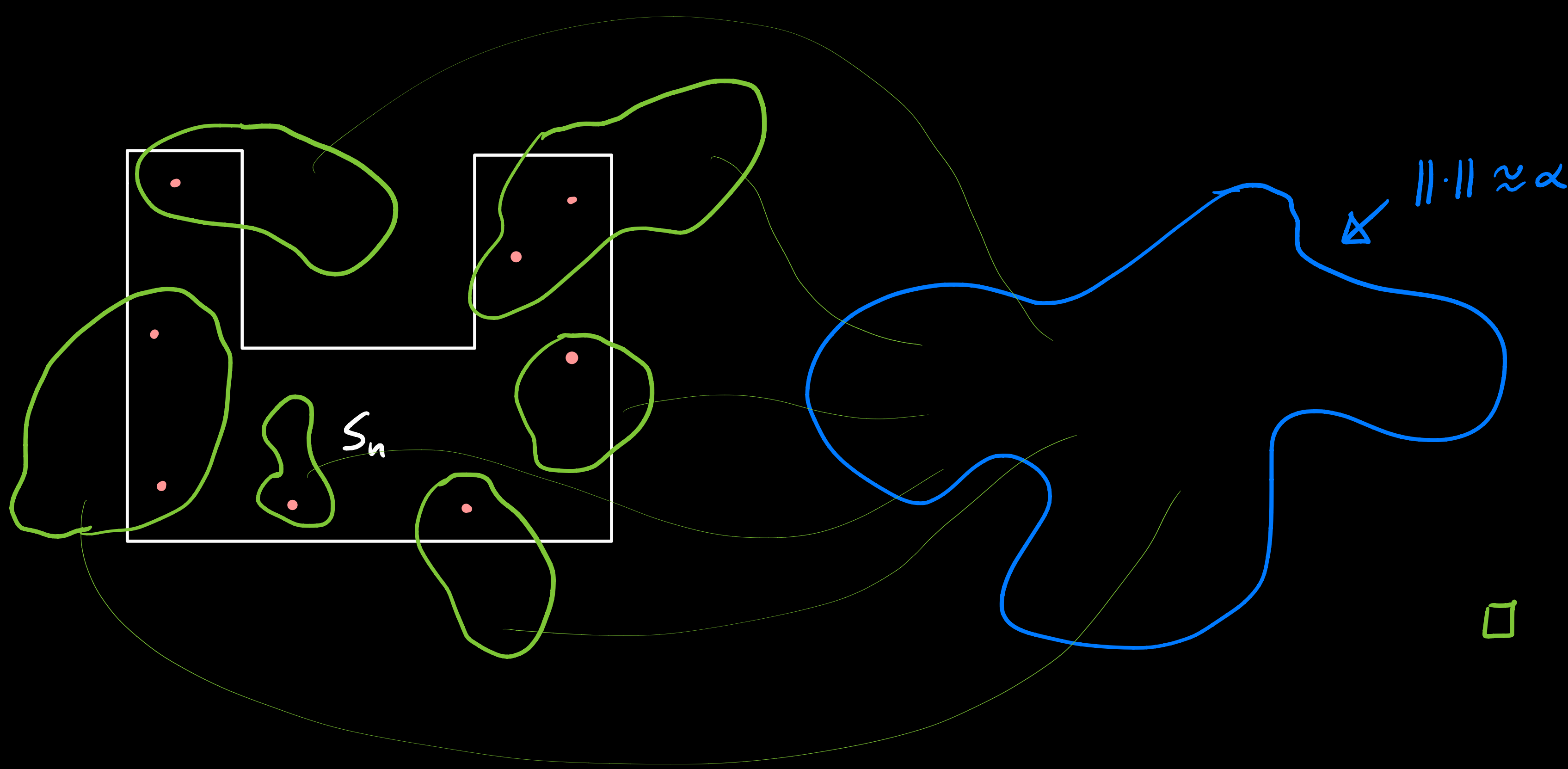
So $\big|\, \|K_1(\omega_q)\| - \alpha \,\big| \geq \delta$.

↯ Contradiction for large n

Claim: $\exists N \quad \forall n \geq N: \quad \mathbb{P}_{q_n}^{G_n}\left(\|K_1\| \geq \alpha + \frac{\alpha\varepsilon}{3} \right) \geq \frac{\alpha}{4}.$

Proof: By transitivity,

$$\mathbb{P}_{q_n}^{G_n}\left(\left| \{ v \in S_n : \|K_v\| \geq \tfrac{\alpha}{2} \} \right| \geq \frac{\alpha}{3} |S_n| \right) \geq \frac{\alpha}{3}.$$



$\square$

This claim contradicts concentration of $\|K_1\|$ under $\mathbb{P}_{q_n}^{G_n}$!

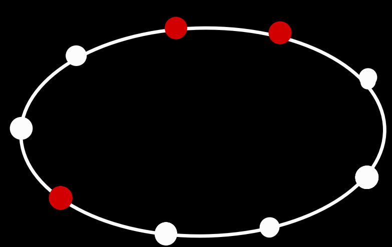
An open problem

Let (G_n) be sequence of finite transitive graphs with $|V(G_n)| \to \infty$.

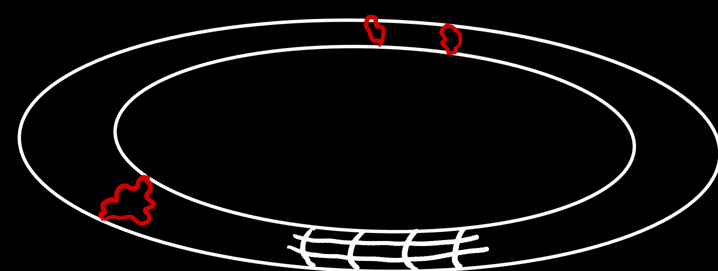
Say (G_n) has the general uniqueness property if

$$\forall (p_n): \quad \|K_2\| \xrightarrow{P} 0 \text{ under } \mathbb{P}_{p_n}^{G_n}.$$

Fails for approximately 1-dimensional examples.



$\mathbb{Z}/n\mathbb{Z}$



$(\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/2^n\mathbb{Z})$

Conjecture: (Alon, Benjamini, Stacey '04)

If $\operatorname{diam} G_n = o\left(\frac{|V(G_n)|}{\log |V(G_n)|}\right)$, then (G_n) has the general uniqueness property.