Optimizing the Campos-Griffiths-Morris-Sahasrabudhe upper bound on Ramsey numbers

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Joint work with Parth Gupta, Ndiame Ndiaye, and Louis Wei.

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Definition

The Ramsey number $R(k,\ell)$ is the smallest positive integer N such that in any red-blue coloring of the edges of the complete graph on N vertices there exists either a complete subgraph on k vertices with all edges colored red (a red K_k) or a complete subgraph on ℓ vertices with all edges colored blue (a blue K_ℓ).

$$R(k,\ell) = R(\ell,k), \qquad R(1,\ell) = 1$$
$$R(2,\ell) = \ell \text{ for } \ell \geq 2, \qquad R(3,3) = 6.$$

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Theorem (Ramsey, 1929)

$R(k, \ell)$ exists for all positive integers k and ℓ.

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Theorem (Erdős-Szekeres, 1935)

For all integers $k, \ell \geq 2$

$$R(k, \ell) \leq \binom{k+\ell-2}{k-1}.$$

In particular

$$R(k, k) < 4^k$$

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In particular

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Theorem (Erdős, 1947)

For all integers $k \geq 2$

$$R(k, k) \geq \sqrt{2}^k.$$

Upper bounds on the diagonal Ramsey numbers

Upper bounds on $R(k+1, k+1)$.

Erdős-Szekeres	1935	$\binom{2k}{k}$	
Rödl	1980s	$\log^{-c} k \binom{2k}{k}$	for some $c > 0$
Thomason	1988	$k^{-c} \binom{2k}{k}$	
Conlon	2009	$\exp\left(-c\frac{\log^2 k}{\log\log k}\right)\binom{2k}{k}$	
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Theorem (Campos, Griffiths, Morris, and Sahasrabudhe, 2023+)

There exists $\delta > 0$ such that

$$R(k,k) \le (4-\delta)^k.$$

(One can take $\delta = 0.007$ for sufficiently large k.)

Our main result

Theorem (Gupta, Ndiaye, N., Wei 2024+)

For all positive integers $\ell \leq k$

$$R(k,\ell) \leq e^{G(\ell/k)k+o(k)}\binom{k+\ell}{\ell},$$

where

$$G(\lambda) = \left(-0.25\lambda + 0.03\lambda^2 + 0.08\lambda^3\right)e^{-\lambda}.$$

In particular,

$$R(k,k) < (3.8)^k$$

for sufficiently large k.

Erdős-Szekeres proof

Theorem

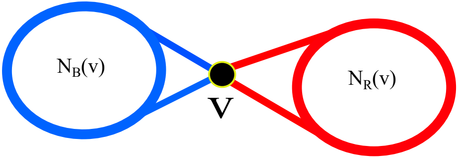
For all $x \in (0,1)$ *and integers* $k, \ell \geq 1$

$$R(k,\ell) \leq x^{-k}(1-x)^{-\ell}.$$

Proof

By induction on $k + \ell$.

Let $N = R(k,\ell) - 1$. Let v be an arbitrary vertex of K_N colored with no red K_k or blue K_ℓ.



Erdős-Szekeres proof

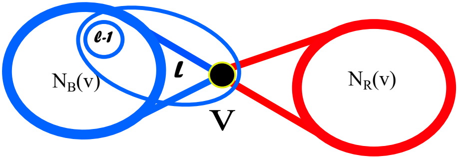
Theorem

For all $x \in (0,1)$ *and integers* $k, \ell \geq 1$

$$R(k,\ell) \leq x^{-k}(1-x)^{-\ell}.$$

Proof

$$|N_B(v)| \leq R(k,\ell-1) - 1 \leq x^{-k}(1-x)^{-\ell+1} - 1.$$



Erdős-Szekeres proof

Theorem

For all $x \in (0,1)$ *and integers* $k, \ell \geq 1$

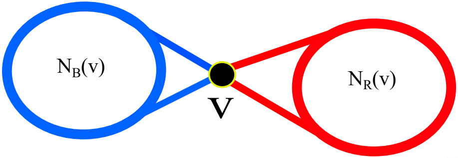
$$R(k,\ell) \leq x^{-k}(1-x)^{-\ell}.$$

Proof

$$|N_B(v)| \leq x^{-k}(1-x)^{-\ell+1} - 1, \qquad |N_R(v)| \leq x^{-k+1}(1-x)^{-\ell} - 1.$$

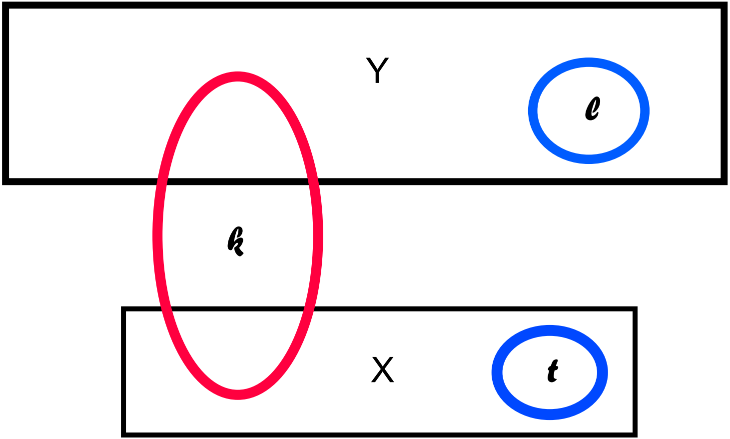
$$R(k,\ell) - 1 \leq (x^{-k}(1-x)^{-\ell+1} - 1) + (x^{-k+1}(1-x)^{-\ell} - 1) + 1$$

$$= (x + (1-x))x^{-k}(1-x)^{-\ell} - 1 = x^{-k}(1-x)^{-\ell} - 1$$



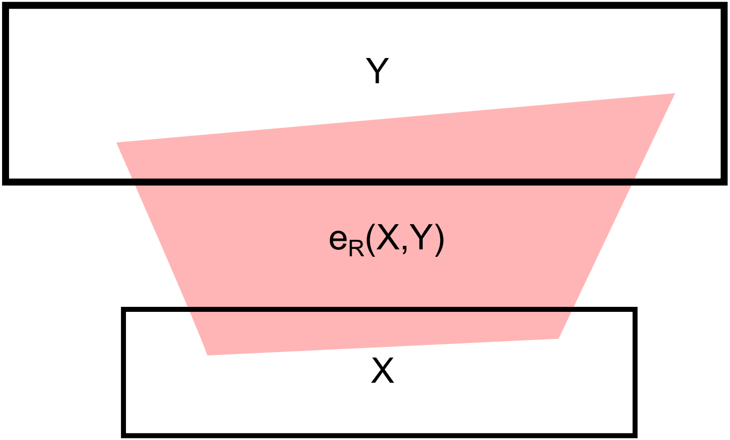
The protagonist

A pair of disjoint sets of vertices (X, Y) is a candidate. A candidate (X, Y) is (k, ℓ, t)-good if $X \cup Y$ contains a red K_k or X contains a blue K_t or Y contains a blue K_ℓ.



The protagonist

A pair of disjoint sets of vertices (X, Y) is a candidate.
$e_R(X, Y)$ is the number of red edges between X and Y.
Let $f_p(X, Y) = e_R(X, Y) - p|X||Y|$ be the excess amount of red edges between X and Y when compared to density p.

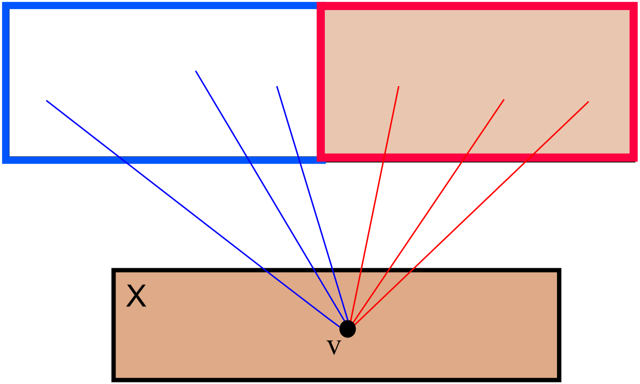


Convexity

Let $f_p(X, Y) = e_R(X, Y) - p|X||Y|$.

Lemma

$$\frac{1}{|X|} \sum_{v \in X} f_p(X, N_R(v) \cap Y) \geq p \cdot f_p(X, Y).$$



Analogue of Erdős-Szekeres induction

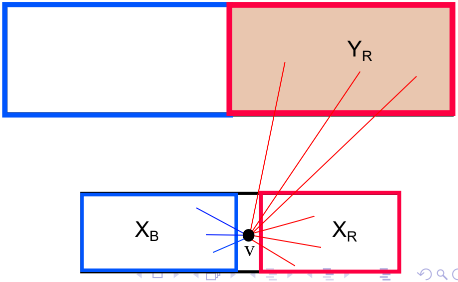
Lemma

Let $0 < x < p < 1$, let k, ℓ and t be positive integers and let (X, Y) be a candidate such that

$$f_p(X, Y) \geq (k+t)x^{-k}(1-x)^{-\ell}(p-x)^{-t}$$

then (X, Y) is (k, ℓ, t)-good.

Let $v \in X$ be such that
$f_p(X, N_R(v) \cap Y) \geq p \cdot f_p(X, Y)$.

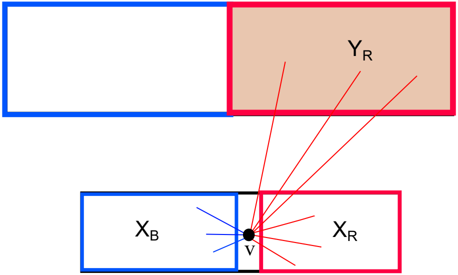


Analogue of Erdős-Szekeres induction

Lemma

If $f_p(X,Y) \geq (k+t)x^{-k}(1-x)^{-\ell}(p-x)^{-t}$ then (X,Y) is (k,ℓ,t)-good.

We may assume (X_R, Y_R) is not $(k-1,\ell,t)$-good. Then $f_p(X_R,Y_R) < (k+t-1)x^{-k+1}(1-x)^{-\ell}(p-x)^{-t}$ and so $f_p(X_R,Y_R) < \frac{k+t-1}{k+t} x f_p(X,Y)$.

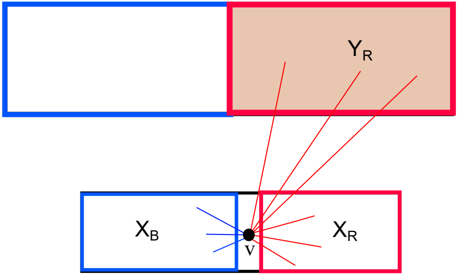


Analogue of Erdős-Szekeres induction

Lemma

If $f_p(X,Y) \geq (k+t)x^{-k}(1-x)^{-\ell}(p-x)^{-t}$ then (X,Y) is (k,ℓ,t)-good.

Similarly (X_B, Y_R) is not $(k,\ell,t-1)$-good and and so $f_p(X_B,Y_R) < \frac{k+t-1}{k+t}(p-x)f_p(X,Y)$.



Analogue of Erdős-Szekeres induction

Lemma

If $f_p(X,Y) \geq (k+t)x^{-k}(1-x)^{-\ell}(p-x)^{-t}$ then (X,Y) is (k,ℓ,t)-good.

$$\begin{aligned}
f_p(X_R, Y_R) &< \frac{k+t-1}{k+t} x\, f_p(X,Y), \\
f_p(X_B, Y_R) &< \frac{k+t-1}{k+t} (p-x) f_p(X,Y), \\
p f_p(X,Y) \leq f_p(X, Y_R) &= f_p(X_R, Y_R) + f_p(X_B, Y_R) + f_p(\{v\}, Y_R) \\
&< \frac{k+t-1}{k+t} p f_p(X,Y) + |Y|,
\end{aligned}$$

So $|Y| \geq p x^{-k}(1-x)^{-\ell}(p-x)^{-t} \geq x^{-k}(1-x)^{-\ell} \geq R(k,\ell)$.

The off-diagonal result

Lemma

If $f_p(X, Y) \geq (k+t)x^{-k}((1-x)(p-x))^{-\ell}$ *then* (X, Y) *is* (k, ℓ, ℓ)*-good.*

The off-diagonal result

Lemma

If $f_p(X,Y) \geq (k+t)x^{-k}((1-x)(p-x))^{-\ell}$ then (X,Y) is (k,ℓ,ℓ)-good.

Theorem

For all $\frac{\sqrt{5}-1}{\sqrt{5}+1} < p < 1$ and all positive integers k and ℓ

$$R(k,\ell) \leq 4(k+\ell)\left(\frac{1+\sqrt{5}}{2}p + \frac{1-\sqrt{5}}{2}\right)^{-k/2}(1-p)^{-\ell}.$$

By induction on ℓ. Let $(1-x)(p-x) = (1-p)^2$, i.e. $x = \frac{1+\sqrt{5}}{2}p + \frac{1-\sqrt{5}}{2}$. If $|N_B(v)| \gtrsim (1-p)n$ for some v we apply the induction hypothesis to $N_B(v)$, otherwise randomly dividing the vertices we obtain a candidate which is (k,ℓ,ℓ)-good by the lemma.

The off-diagonal result

Theorem

For all $\frac{\sqrt{5}-1}{\sqrt{5}+1} < p < 1$ *and all positive integers* k *and* ℓ

$$R(k,\ell) \le 4(k+\ell)\left(\frac{1+\sqrt{5}}{2}p + \frac{1-\sqrt{5}}{2}\right)^{-k/2}(1-p)^{-\ell}.$$

Substituting $p = \frac{(\sqrt{5}+1)k+(2\sqrt{5}-2)\ell}{\left(\sqrt{5}+1\right)(k+2\ell)}$ we get

Theorem

For all positive integers $k \ge \ell$

$$R(k,\ell) \le 4(k+\ell)\left(\frac{\left(\sqrt{5}+1\right)(k+2\ell)}{4\ell}\right)^{\ell}\left(\frac{k+2\ell}{k}\right)^{k/2}$$

Comparison

Let

$$ES(k,\ell) = \binom{k+\ell-2}{k-1} = e^{O(\log k)} \left(\frac{k+\ell}{k}\right)^k \left(\frac{k+\ell}{\ell}\right)^\ell,$$

denote the Erdős-Szekeres upper bound on the Ramsey numbers. Our result implies

$$\frac{R(k,\ell)}{ES(k,\ell)} \leq e^{O(\log k)} \left(\frac{(\sqrt{5}+1)(k+2\ell)}{4(k+\ell)}\right)^\ell \left(\frac{(k+2\ell)k}{(k+\ell)^2}\right)^{k/2}$$

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This yields an exponential improvement of the Erdős-Szekeres bound whenever $\ell < 0.6989k$.
For $\ell = o(k)$ the improvement is of the order

$$e^{O(\log k)} \left(\frac{\sqrt{5}+1}{4}\right)^\ell < e^{-0.21\ell + O(\log k)}.$$

Improvements

Let $\mathcal{R}$ be the closure of the set of all pairs $(x, y) \in (0, 1)^2$ such that

$$R(k, \ell) \leq x^{-k} y^{-\ell} \qquad \text{for all } k, \ell \in \mathbb{N} \text{ such that } k + \ell \geq N(x, y).$$

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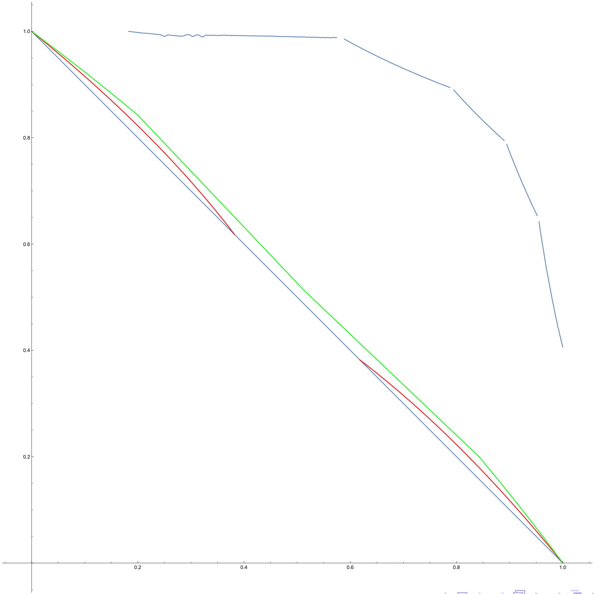
We have $(x, 1-x) \in \mathcal{R}$ by the Erdős-Szekeres argument and we proved

$$\left(\left(\frac{1+\sqrt{5}}{2}p + \frac{1-\sqrt{5}}{2}\right)^{1/2}, 1-p\right) \in \mathcal{R} \qquad \text{for} \qquad \frac{\sqrt{5}-1}{\sqrt{5}+1} < p < 1.$$

The region $\mathcal{R}$



Improvements

Let $\mathcal{R}_*$ be the interior of $\mathcal{R}$.

Lemma

If $f_p(X, Y) \geq (k+t)x^{-k}(1-x)^{-\ell}(p-x)^{-t}$ then (X, Y) is (k, ℓ, t)-good.

$$\Downarrow$$

Lemma

If $(x, y) \in \mathcal{R}_*$ and $f_p(X, Y) \geq x^{-k}y^{-\ell}(p-x)^{-t}$ then (X, Y) is (k, ℓ, t)-good for sufficiently large k, ℓ and t.

Final version of the inductive lemma

Theorem

For all $0 < \mu, x, y, p < 1$ *such that* $x < p^{\frac{1}{1-\mu}}(1-\mu)$ *and* $(x,y) \in \mathcal{R}_*$ *there exists* L_0 *such that for all positive integers* k, ℓ *with* $\ell \geq L_0$ *the following holds. Every red-blue coloring of edges the complete graph on* $N \geq x^{-k/2}(\mu y)^{-\ell/2}$ *with the density of red edges at least* p *contains a red* K_k *or a blue* K_ℓ*.*

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The proof:

- instead of $e_R(X,Y) - p|X||Y|$ we lower bound "higher moments" of density
$$(e_R(X,Y) - p|X||Y|)^r |X|^{1-r}|Y|^{1-r}$$
in the regime $r \to \infty$,
- we use further combinatorial ideas from the Campos-Griffiths-Morris-Sahasrabudhe proof, which, in particular, requires us to occasionally apply the induction hypothesis to a common blue neighborhood of a large set of vertices.

Descending to a candidate

Theorem

For all $0 < \mu, x, y, p < 1$ *such that* $x < p^{\frac{1}{1-\mu}}(1-\mu)$ *and* $(x, y) \in \mathcal{R}_*$ *there exists* L_0 *such that for all positive integers* k, ℓ *with* $\ell \geq L_0$ *the following holds. Every red-blue coloring of edges the complete graph on* $N \geq x^{-k/2}(\mu y)^{-\ell/2}$ *with the density of red edges at least* p *contains a red* K_k *or a blue* K_ℓ*.*

Theorem

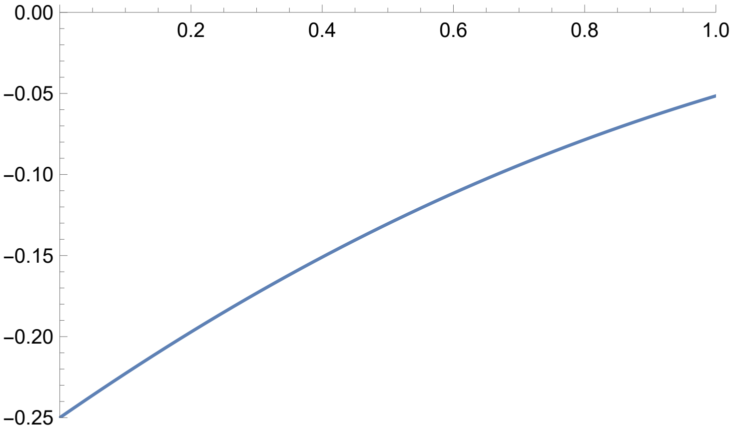
Let $F : (0,1] \to \mathbb{R}_+$ *be smooth and let* $M, X, Y : (0,1] \to (0,1)$ *be such that* $F'(\lambda) < 0, X(\lambda) = (1 - e^{-F'(\lambda)})^{\frac{1}{1-M(\lambda)}}(1 - M(\lambda)), (X(\lambda), Y(\lambda)) \in \mathcal{R}$*, and*

$$F(\lambda) > -\frac{1}{2}(\log X(\lambda) + \lambda \log M(\lambda) + \lambda \log Y(\lambda))$$

for all $0 < \lambda \leq 1$*. Then* $R(k, \ell) \leq e^{F(\ell/k)k + o(k)}$ *for all* $k \geq \ell$*.*

Plot of the bound

Plot of $H(\lambda) = G(\lambda)/\lambda$. Our improvement for $R(k, \ell)$ over the classical bound is of the order $\exp(H(\ell/k)\ell)$.



Multicolor Ramsey numbers

The multicolor Ramsey number $R(k_1, \ldots, k_c)$ is the minimum integer N such that every coloring of edges of the complete graph on N vertices in c colors contains a complete subgraph on k_i vertices with all edges colored with color i.

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The multicolor Ramsey number $R(k_1, \ldots, k_c)$ is the minimum integer N such that every coloring of edges of the complete graph on N vertices in c colors contains a complete subgraph on k_i vertices with all edges colored with color i.

Theorem

For all $x_1 \ldots, x_c \in (0,1)$ such that $x_1 + \ldots + x_c = 1$ and integers $k_1, \ldots, k_c \geq 1$ we have

$$R(k_1, \ldots, k_c) \leq x_1^{-k_1} x_2^{-k_2} \ldots x_c^{-k_c}.$$

The upper bound is minimized for $x_i = \frac{k_i}{k_1 + \ldots + k_c}$.
In particular, it gives

$$R_c(k) := R(\underbrace{k, \ldots, k}_{c \text{ times}}) \leq c^{ck}.$$

Multicolor Ramsey numbers

Theorem (Gupta, Ndiaye, N., Wei 2024+)

For all integers $k, \ell_1, \ldots, \ell_c \geq 1$ with $\ell = \ell_1 + \ldots + \ell_c$ we have

$$R(k, \ell_1 \ldots, \ell_c) \leq 2(k+\ell)\left(\frac{k+2\ell}{k}\right)^{k/2}\left(\frac{\left(\sqrt{5}+1\right)(k+2\ell)}{4\ell}\right)^{\ell} \cdot \prod_{i=1}^{c}\left(\frac{\ell_i}{\ell}\right)^{-\ell_i}.$$

Improves on the classical bound for $\ell \leq 0.69k$.

Multicolor Ramsey numbers

Theorem (Ballister, Bollobás, Campos, Hurley, Griffiths, Morris, and Sahasrabudhe 2024+)

For each $c \geq 2$ *there exists* $\delta > 0$ *such that*

$$R_c(k) \leq e^{-\delta k} c^{ck}.$$

Conclusion

$$\sqrt{2}^k \le R(k,k) \le (3.8)^{k-o(k)}$$

Conclusion

$$(1 - o(1))\frac{\sqrt{2}k}{e}\sqrt{2}^k \leq R(k,k) \leq (3.8)^{k-o(k)}$$

Lower bound due to Spencer, 1975.

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$$(1 - o(1))\frac{\sqrt{2}k}{e}\sqrt{2}^k \leq R(k,k) \leq (3.8)^{k-o(k)}$$

Lower bound due to Spencer, 1975.

Problem (Conlon, Fox, Sudakov)

Can the lower bound be improved by a constant factor?

$$A^{ck} \leq R_c(k) \leq e^{-\delta_c k} c^{ck}$$

for an absolute constant A.

Multicolor Ramsey numbers

$$A^{ck} \leq R_c(k) \leq e^{-\delta_c k} c^{ck}$$

for an absolute constant A.
The value of A has been recently (2021) improved by Conlon and Ferber, and subsequently by Wigderson and by Sawin.

Thank you!