# A TALE OF TWO BALLOONS

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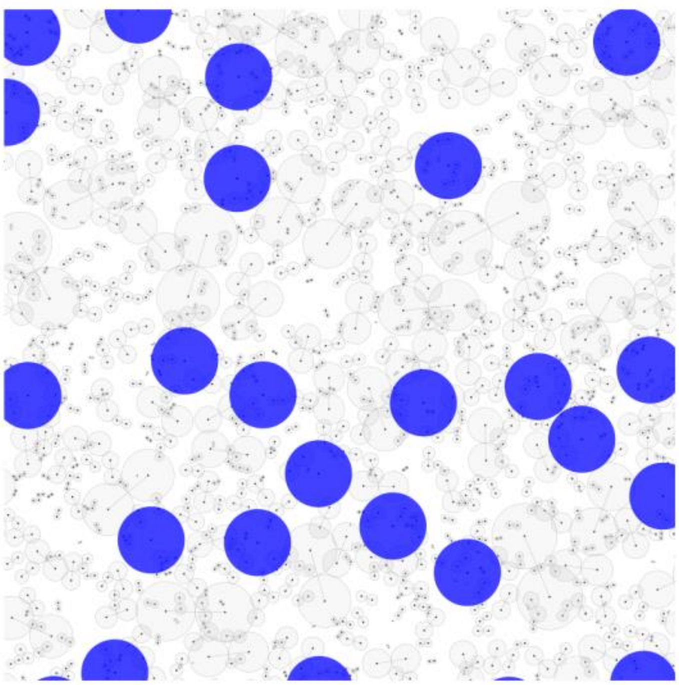
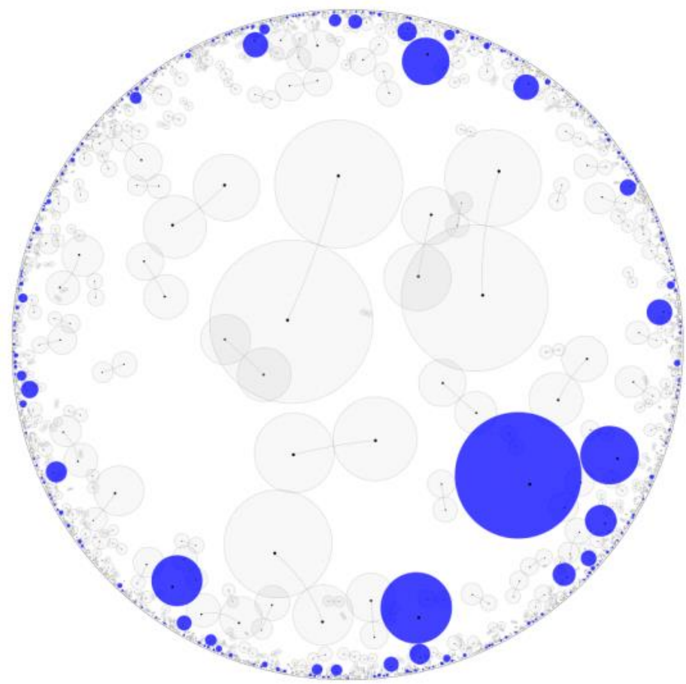
Joint with Omer Angel and Gourab Ray





**The balloon process:**

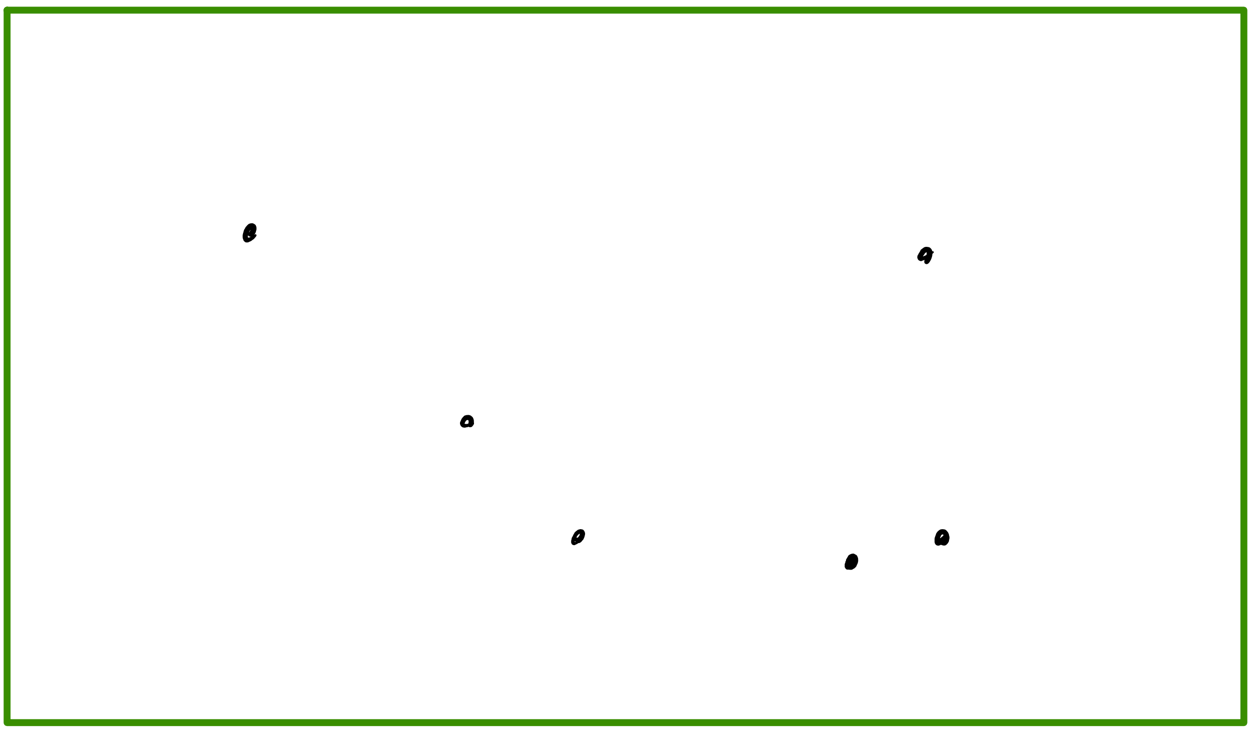
(*) Metric space $(\Omega,\rho)$

(*) Metric space $(\Omega, \rho)$.

(*) Initial set of points $\Pi \subset \Omega$.

(*) Grow balls at rate 1.

(*) When two touch, they pop and disappear.



## Cases of interest:

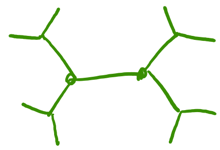
| Metric space $(\Omega, \rho)$ | Initial points $\Pi$ |
|---|---|
| (1) Euclidean space $\mathbb{R}^d$ $d \geq 1$. | Poisson |

(1) Euclidean space $\mathbb{R}^d$, $d \geq 1$.

(2) Hyperbolic plane $\mathbb{H}$.
(constant curvature)

(3) Regular tree $\mathbb{T}_d$, $d \geq 3$.

edges $\cong [0,1]$



Poisson Point Process

**Question: Recurrent or Transient?**

a.s. every point is visited $\infty$-often.
(unbounded times)

a.s. every point is visited finitely often
(bounded times)

**Thm:** (Angel, Ray, S. '21)

The balloon process in $\left\{ \mathbb{R}^d \right.$ is $\left\{ \text{recurrent} \right.$

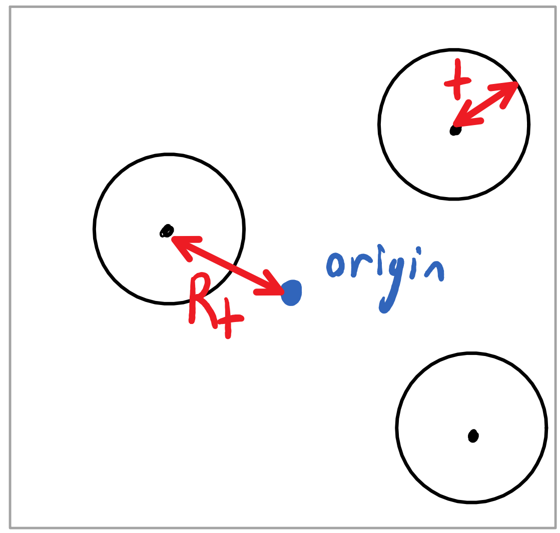
The balloon process in $\left\{\begin{matrix}\mathbb{R}^d\\ \mathbb{H}\\ \mathbb{T}_d\end{matrix}\right.$ is $\left\{\begin{matrix}\text{recurrent}\\ \text{transient}\\ \text{transient}\end{matrix}\right.$ .

**Def:**

$\Pi_t$ = centers of balloons active at time $t$.

$R_t = \rho(\text{origin}, \Pi_t)$

closest balloon center




**Thm:**

$$\text{a.s.} \quad \liminf_{t\to\infty} \frac{R_t}{t} \begin{cases} = 0 & \text{in } \mathbb{R}^d \\ \geq \dfrac{\log 2}{\log \frac{1+\sqrt{5}}{2}} \approx 1.44 & \text{in } \mathbb{H} \\ = 2 & \text{in } \mathbb{T}_d \end{cases}.$$

**Background:**

## Background:

(*) For balloon process to be well defined, enough that $\Pi$:

(Holroyd, Pemantle, Peres, Schramm '09)

(1) is discrete

(2) has no two pairs of points at equal distance.

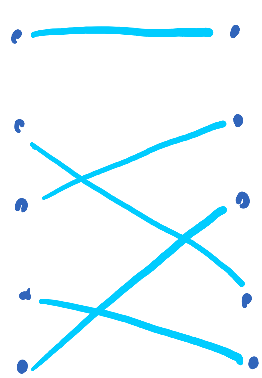
(3) has no infinite descending chain.

(*) PPP satisfies (1)-(3) a.s.

(Häggström, Meester '96 for $\mathbb{R}^d$)

(*) Stable matchings

(I) The marriage problem

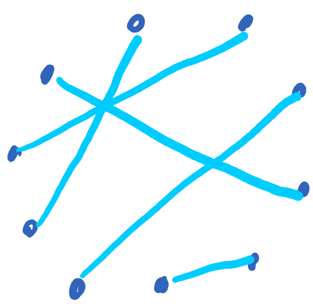


unstable if $\exists$ a man and a woman which are not matched to each other, but each prefer the other over their current partner.


over their current partner.

Gale-Shapley ('62): stable matchings exist.

## (II) The room-mate problem



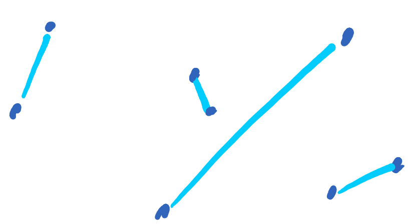
Homogeneous population
(of even size)

Stable matchings might not exist.

## (III) Our metric setting

Homogeneous population $\Pi$

Rank according to distance



HPPS: under assumptions (1)-(3),
$\exists !$ stable matching
and it is generated by greedy algorithm.
(iteratively match mutually closest points)

(*) HPPS study typical matching distance in $\mathbb{R}^d$.
Related to density of $\Pi_t$.

distances in $\Pi_t$ are at least $2t$ $\Rightarrow$ $\mathrm{dens}(\Pi_t) \leq \frac{1}{\mathrm{Vol}(B_t)}$

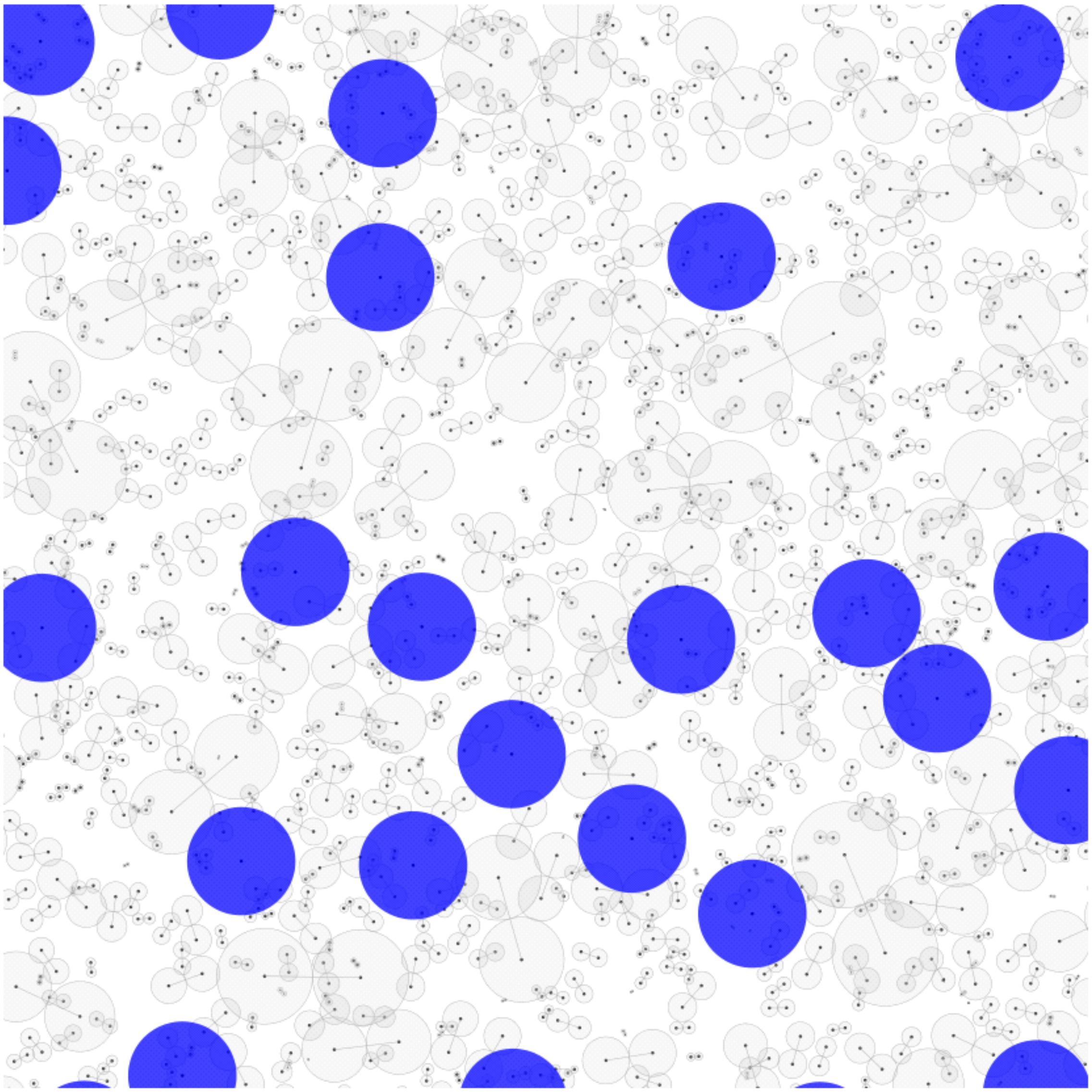
Prop: a.s. $\liminf_{t\to\infty} (R_t - 2t) \leq 0$.

$\left[ \Rightarrow \liminf \frac{R_t}{t} \leq 2. \right]$

balloons reach half-way to origin, $\infty$-often.

[Idea: insertion/deletion tolerance + ergodicity]

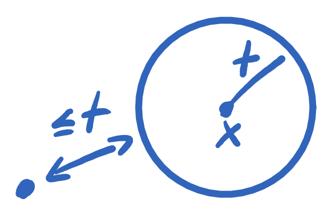
# Recurrence in $\mathbb{R}^d$:



$$T_x = \sup\{t : x \in \Pi_t\} = \text{time when balloon at } x \text{ popped}$$

Prop $\Longrightarrow$

$$\limsup_{\|x\| \to \infty} \frac{T_x}{\|x\|} \geq \frac{1}{2}.$$



$\|x\|\to\infty \qquad \|x\|$


$T_x \geq t$

$\|x\| \leq 2t$

**Thm:** For any stationary process $(X_n)_{n\in\mathbb{Z}^d}$,

$$\limsup_{\|n\|\to\infty} \frac{X_n}{\|n\|} \in \{0,\infty\} \text{ a.s.}$$

$$\Rightarrow \quad \limsup_{\|x\|\to\infty} \frac{T_x}{\|x\|} = \infty .$$

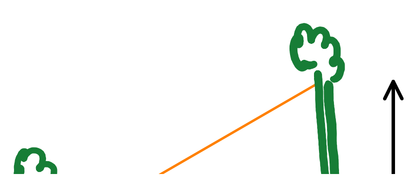
$$\Leftrightarrow \quad \liminf_{t\to\infty} \frac{R_t}{t} = 0 .$$

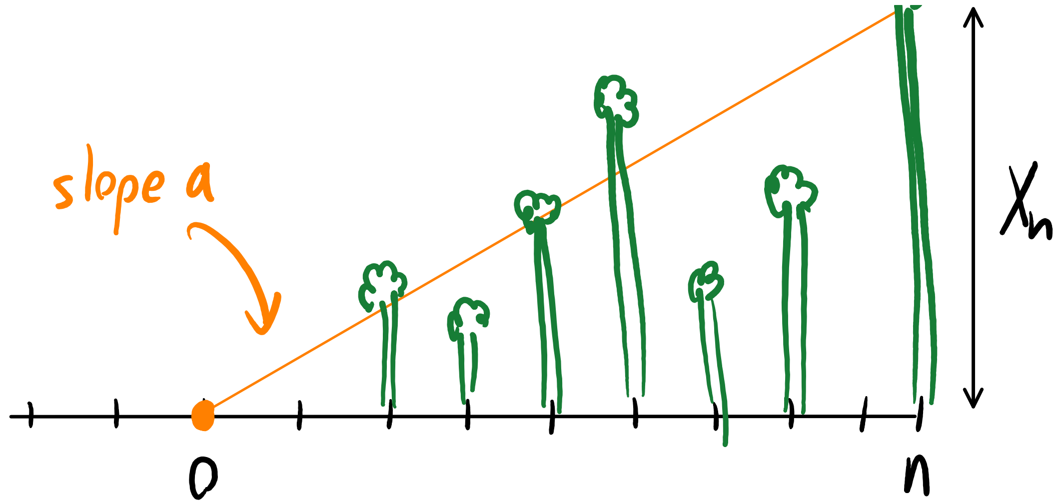
$$\left[\text{Apply Thm to } X_n = \max_{\|x\|=n} T_x\right]$$

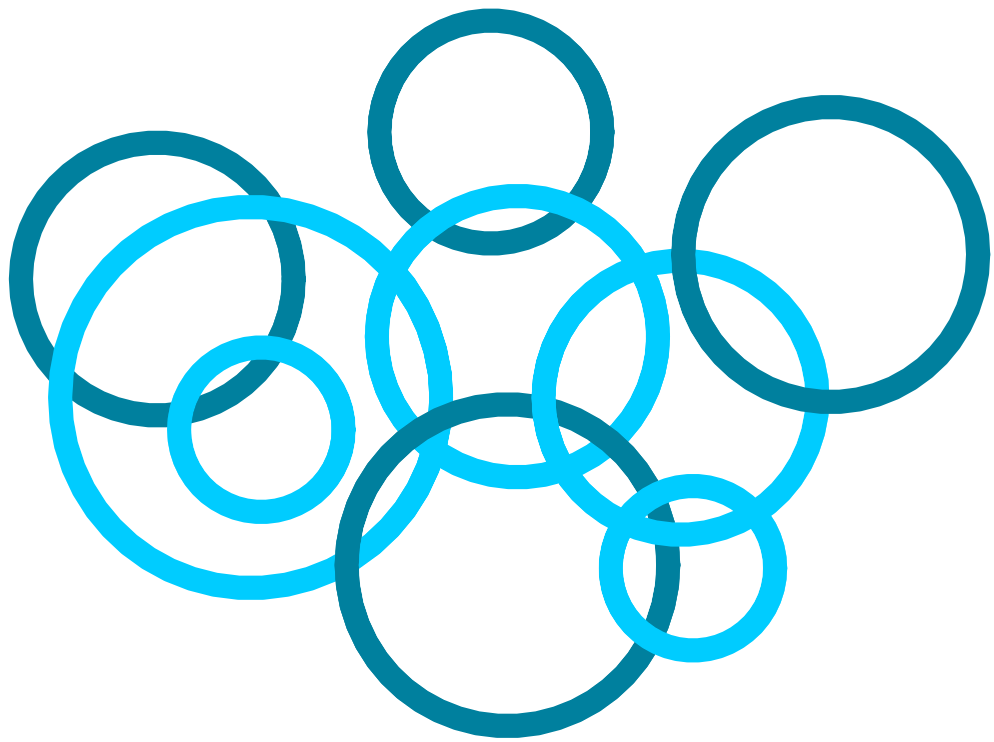
**Proof idea:**

(*) Can assume ergodicity

(*) Suppose $\limsup \frac{X_n}{\|n\|} > a$ .





(*) Vitali Covering Lemma:

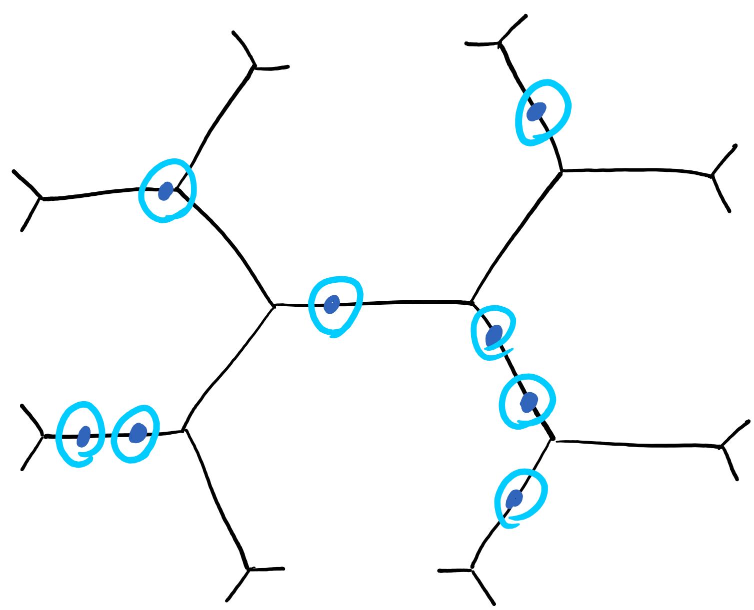
$\exists$ disjoint balls whose 5 x blow-up covers everything

$\Rightarrow$ their 2 × blow-down
covers positive proportion

$$\Rightarrow \quad \limsup \frac{X_n}{\|n\|} > 2a.$$

## Transience in $\mathbb{T}_d$:

d-regular tree
$\downarrow$
graph
OR
continuous space



**Def:** $\tilde{\pi}_t$ = projection of $\pi_t$ to vertices

$\Rightarrow \quad \tilde{\Pi}_t$ is (1) $(2t-1)$-separated

(2) factor of IID

Question: What is the max. density of such a process $X$ ?

Trivial bound: $\approx \dfrac{1}{\mathrm{Vol}(B_t)} \approx \dfrac{1}{(d-1)^t}$.

Thm: $$\mathrm{dens}(X) \le \frac{Ct}{(d-1)^{2t}}.$$

(*) Previous results for independent sets:

Bollobás (81), McKay (87)

Rahman, Virag ('17)

(*) Enough that the factor is

(*) Enough that the factor is $\Gamma$-equivariant for some transitive $\Gamma < \mathrm{Aut}(T_d)$.

Proof idea:

(1) Approximate factor by block factor
(finite-range map)

(2) Random $d$-regular graph $\xrightarrow{\text{locally}}$ $\mathbb{T}_d$.
[Colored config. model $\Rightarrow$ colored $\mathbb{T}_d$]

(3) Apply block factor to finite graph.

(4) Bound size of largest $2t$-separated set in the finite random regular graph.

Back to balloons:

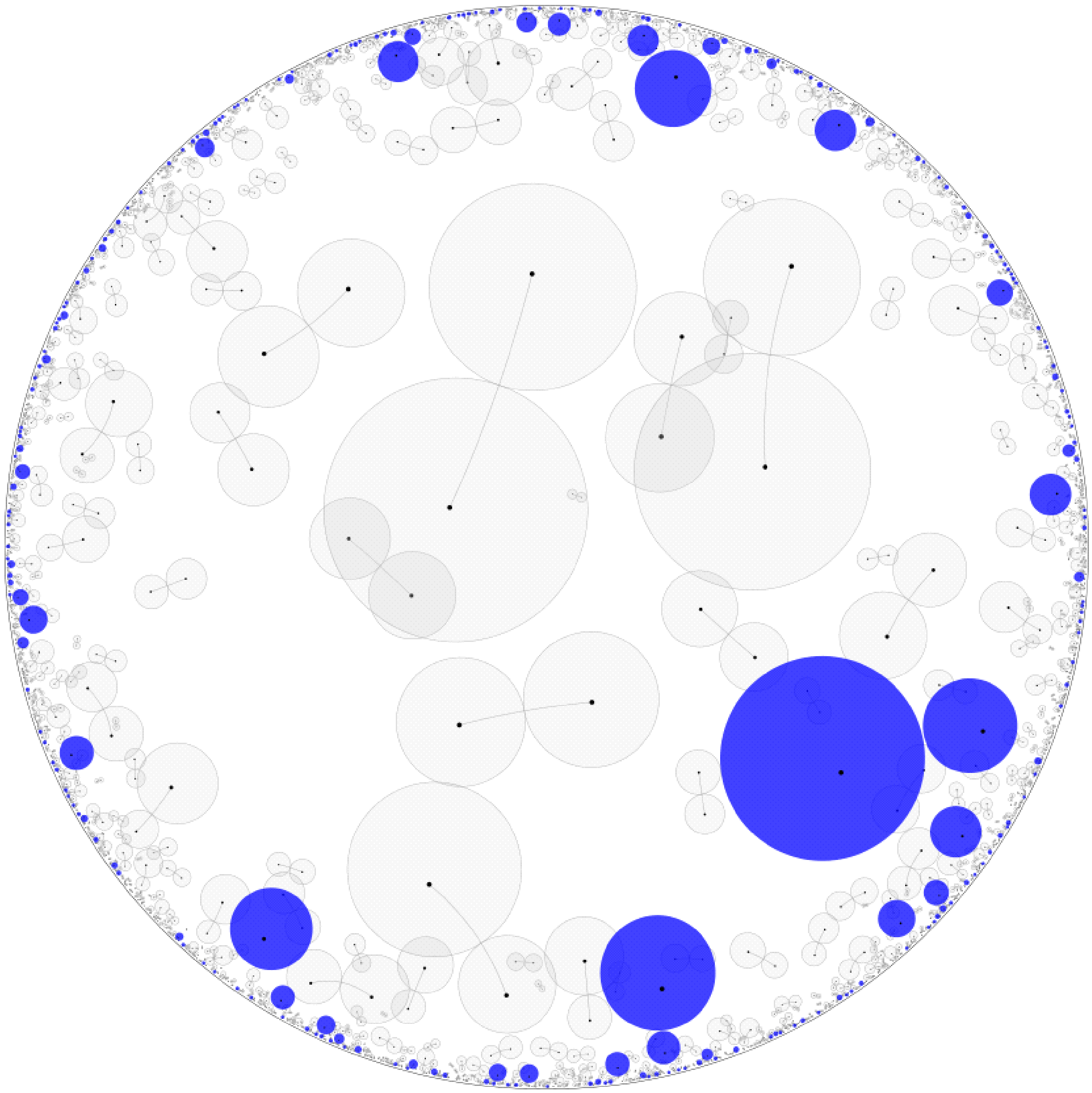
$$\overset{\text{Thm}}{\Longrightarrow} \quad \mathrm{dens}(\widetilde{\Pi}_t) \le \frac{Ct}{(d-1)^{2t}}.$$

$$\overset{\text{union bound}}{\Longrightarrow} \quad \mathbb{P}\left( R_t \le 2t - \Omega(\log t)\right) \le \frac{1}{t^2}.$$

Borel-Cantelli + monotonicity of $R_t$ $\Longrightarrow$ $$R_t \ge 2t - \Omega(\log t) \text{ a.s.}$$

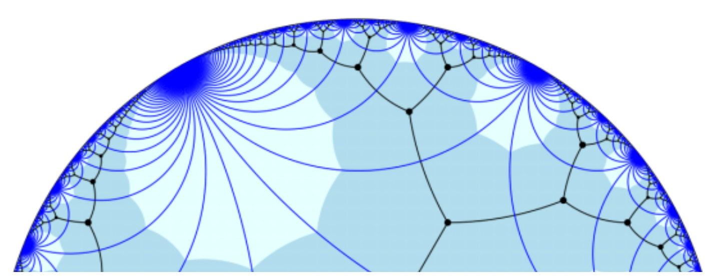
$$\Longrightarrow \quad \liminf_{t\to\infty} \frac{R_t}{t} \ge 2 \text{ a.s.}$$

## Transience in $\mathbb{H}$:

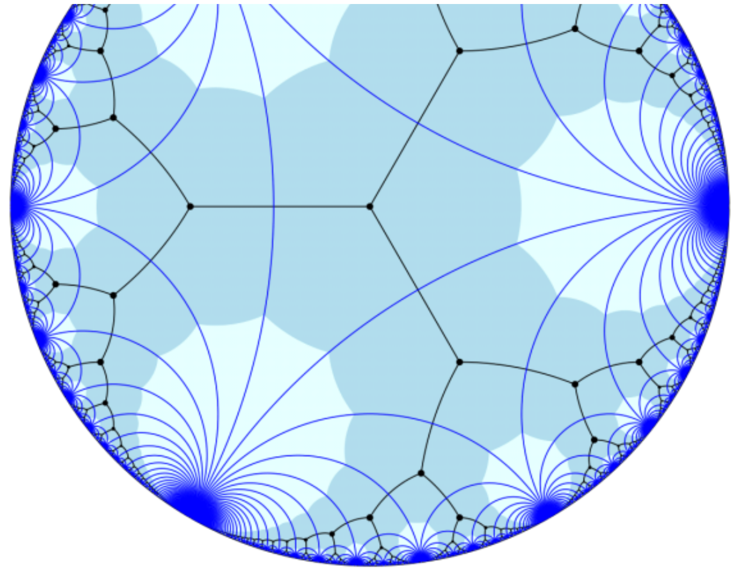


Goal: Bound dens$(\Pi_+)$.

(*) Approximate by 3-regular



by 3-regular tree.



Problem: projection of $\Pi_t$ to tree has unbounded distortion.

(*) Remove caps near cusps.

$$\Longrightarrow \quad \boxed{d_{\mathbb{H}}(x,y) \le a \cdot d_{T_3}(\pi(x),\pi(y)) + c}$$

(*) Now projection of $\Pi_t$ to tree is

(1) $\frac{2(t-c)}{a}$-separated.

(2) factor of IID

[$\Gamma$-equivariant for some trans. $\Gamma < \mathrm{Aut}(T_3)$]

$$\overset{\text{Thm}}{\Longrightarrow} \quad \mathrm{dens}(\Pi_t) \le Ct \cdot 2^{-\frac{2t}{a}}.$$

Markov ineq. + Borel-Cantelli $\Longrightarrow$

$$\boxed{\liminf_{t \to \infty} \frac{R_t}{t} \geq \frac{2 \log 2}{a}.}$$

(*) Can take $a = d_{\mathbb{H}}(u,v)$ for $u, v \in \mathbb{T}_3$ adjacent.

$$\Longrightarrow \quad a = \log 3 \quad \Longrightarrow \quad \frac{2 \log 2}{\log 3} \approx 1.26.$$

(*) Can take $a = \frac{1}{2} d_{\mathbb{H}}(u,v)$ for $u, v \in \mathbb{T}_3$ distance 2.

$$\Longrightarrow \quad a = 2 \log \frac{1+\sqrt{5}}{2} \quad \Longrightarrow \quad \frac{\log 2}{\log \frac{1+\sqrt{5}}{2}} \approx 1.44.$$

## Open problems:

(1) In $\mathbb{R}^d$,

(*) $\mathrm{dens}(\Pi_t) \sim c/t^d$.

(*) Stationary distrib. for $\left(\frac{1}{t}\Pi_t\right)$.

(2) General initial points $\Pi$.

e.g. perturbation of $\mathbb{Z}^d$ in $\mathbb{R}^d$.

(3) Random growth rates.

e.g. 0 or 1.

(4) In $\mathbb{H}$, is $\liminf R_t/t = 2$ ?

(5) Other spaces.

e.g. hyperbolic space $\mathbb{H}^d$.