Minimum Weight Disk Triangulations & Fillings

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+ Zur Luria

Simplicial complexes:

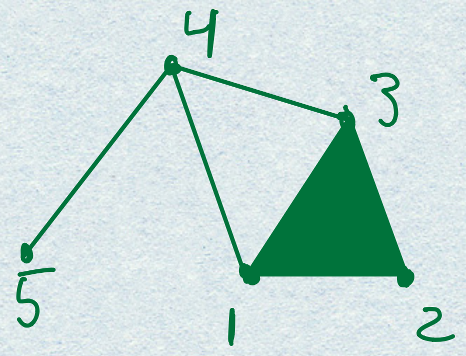
Abstract:

Downward - Closed $Y \subseteq 2^V$

$$\left\{ \begin{array}{c} \emptyset, \\ 1, 2, 3, 4, 5, \\ 12, 13, 23, 14, 34, 45, \\ 123 \end{array} \right\}$$

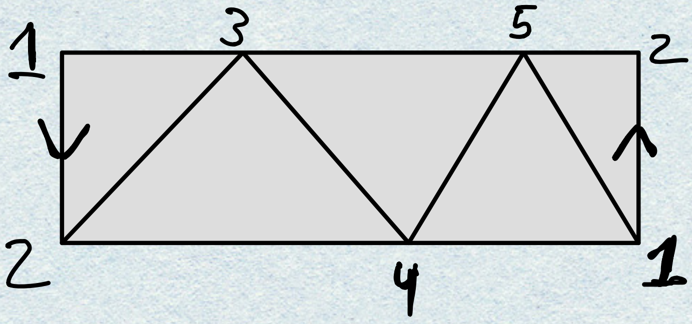
Geometric Realization:

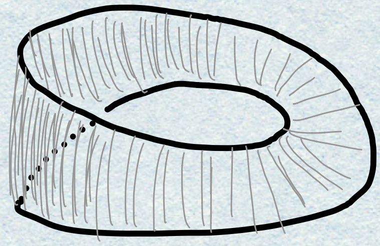
Gluing of simplices



Example:

$\{\ldots, 123, 234, 345, 145, 125\}$





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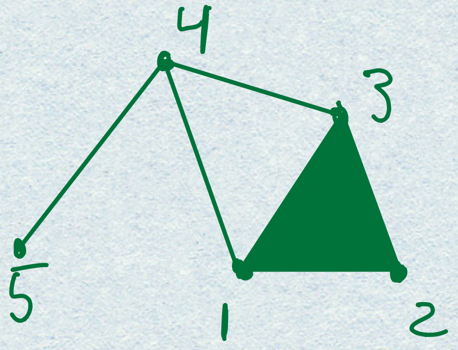
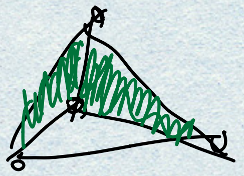


Random Simplicial 2-Complexes

Generalizing $G(n,p)$

$Y_2(n,p)$ [Linial-Meshulam]

n=4



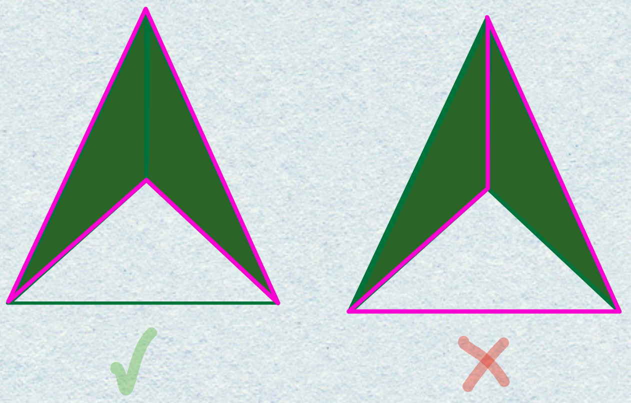
- n vertices $\{1,\dots,n\}$
- All $\binom{n}{2}$ edges
- Each of $\binom{n}{3}$ 2-dimensional faces appears ind. with probability $p=p(n)$.

Homological Connectivity

Theorem [Linial–Meshulam]: $\forall \epsilon > 0$,

$$\mathbb{P}\Big(\underbrace{H_1(Y_2(n,p);\mathbb{F}_2)=0}_{(*)}\Big) \xrightarrow[n\to\infty]{} \begin{cases} 0 & p<(1-\epsilon)\cdot\frac{2\log n}{n} \\ 1 & p>(1+\epsilon)\frac{2\log n}{n} \end{cases}.$$

(*) Every cycle is a linear combination of boundaries of the 2-faces of $Y_2(n,p)$



- Linial–Meshulam Theorem was extended to:
  - Other rings of coefficients
  - Higher dimensional complexes
  - Hitting-time results

Meshulam–Wallach, Hoffman–Kahle–Paquette, Kahle–Pittel, Łuczak–P., Newman–Paquette.

Simple Connectivity

Theorem [Babson - Hoffman - Kahle]: $\forall \varepsilon > 0$

$$\mathbb{P}\big(\underbrace{Y_2(n,p) \text{ is simply-connected}}_{(*)}\big) \longrightarrow 0 \quad \text{if } p < n^{-\frac{1}{2}-\varepsilon}.$$

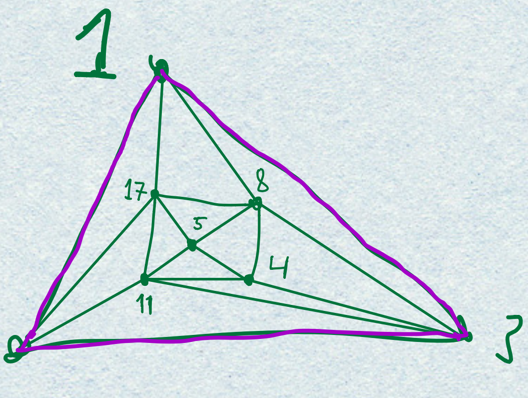
(*) Every cycle is null-homotopic:

$\exists$ Δ-ion of the disk:

(–) Vertices labelled in $\{1, \dots, n\}$

(–) Boundary is given cycle.

(–) All faces from $Y_2(n,p)$



$$\boxed{\text{OPEN: Improve this bound}}$$

Simple Connectivity

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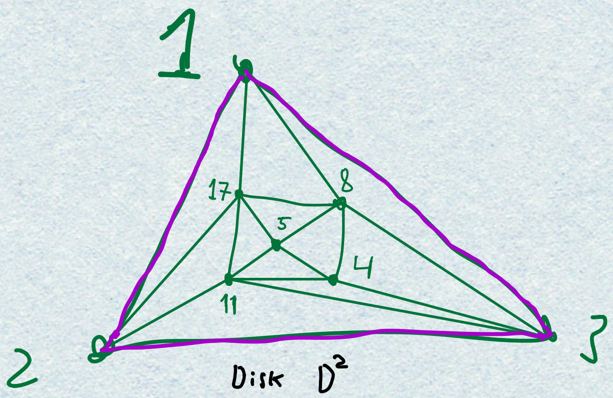
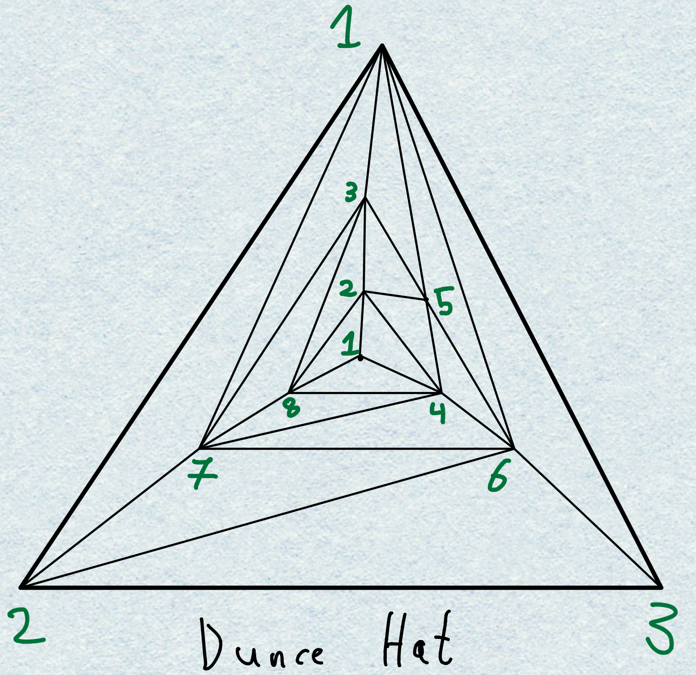
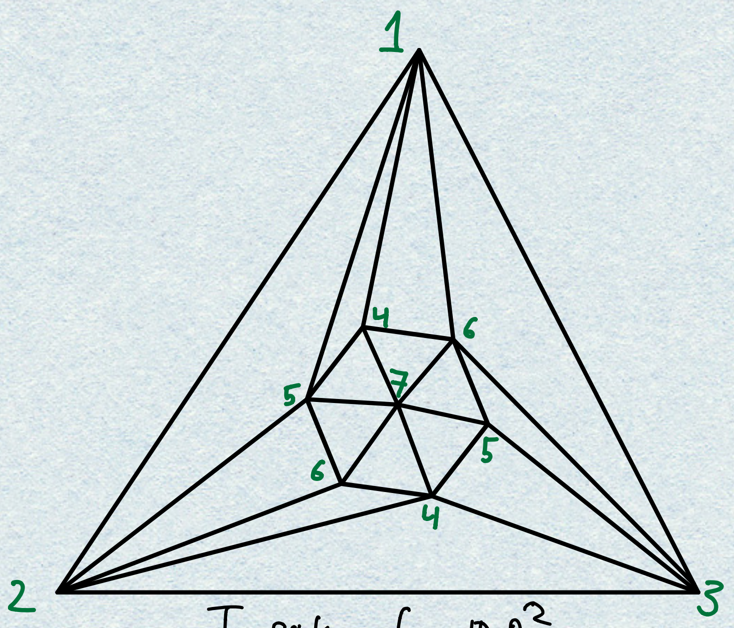


Disk D^2

Vertex labels in a null-homotopy may repeat.



Dunce Hat



Tweak of $\mathbb{R}P^2$

Simple Connectivity: Upper Bounds

Upper Bounds: BHK, Gündert-Wagner, Korándi-P-Sudakov

Theorem [Luria-P.]: Let $\gamma := \frac{4^4}{3^3}$. $\forall \varepsilon > 0$

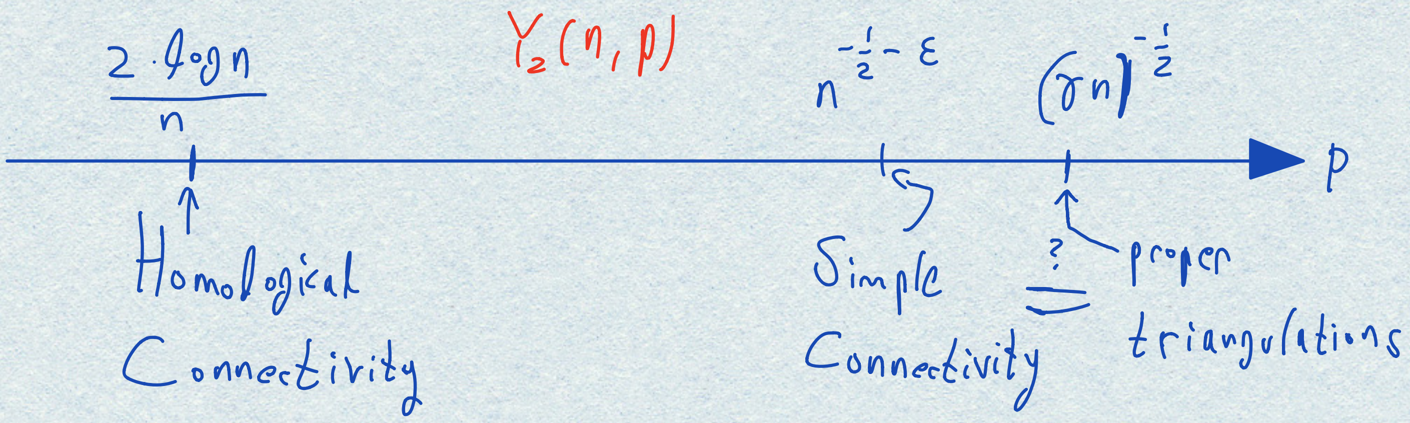
$p = (1+\varepsilon)(\gamma n)^{-\frac{1}{2}}$ $\Longrightarrow$ WHP every cycle of length 3 is null-homotopic in $Y_2(n,p)$ by a proper Δ-ion

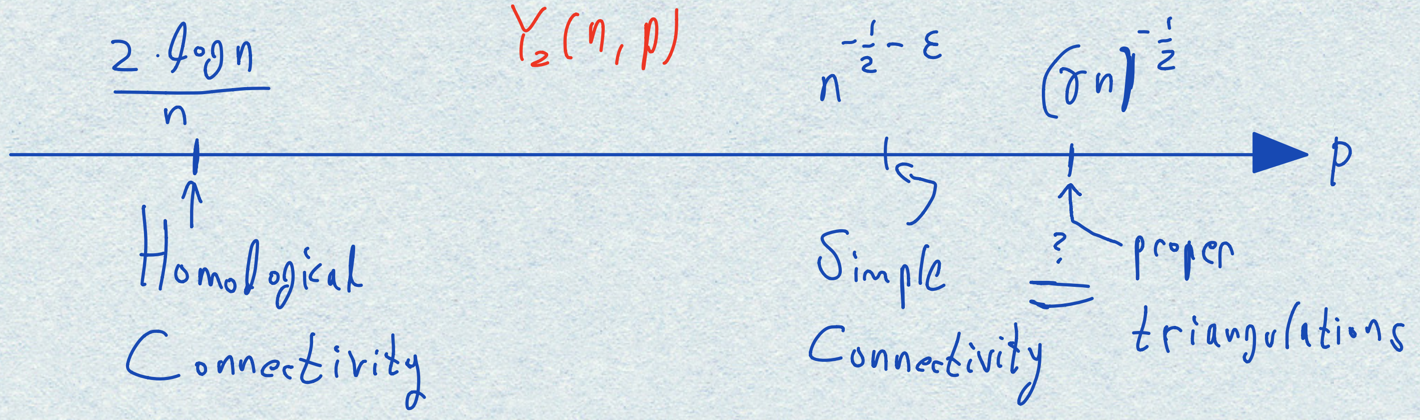
$\Longrightarrow$ $Y_2(n,p)$ is WHP simply-connected

Def: A labeled Δ-ion is called proper if the vertex labels are distinct.

Theorem [Tutte]: # rooted unlabeled planar triangulations K internal vertices $= (C+o(1)) K^{-\frac{5}{2}} \cdot \gamma^K$

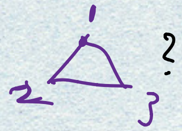
Conjecture: $(\gamma n)^{-\frac{1}{2}}$ is the sharp threshold prob. for simple-connectivity.


$Y_2(n,p)$
$\frac{2 \cdot \log n}{n}$
$n^{-\frac{1}{2}-\varepsilon}$
$(\gamma n)^{-\frac{1}{2}}$
p
Homological Connectivity
Simple Connectivity
$\stackrel{?}{=}$
proper triangulations



Minimum Weight Triangulations

Assign independent Exp(1) weights to the $\binom{n}{3}$ 2-dimensional simplices on $\{1,\dots,n\}$.

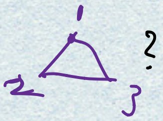
What is the total minimum weight of a triangulation whose boundary is fixed cycle  ?

$\mathbb{F}_2$-homological filling? null-homotopy? proper?

$$F_n \le W_n \le D_n$$

Bounds from $Y_2(n,p)$: $n^{-\frac{1}{2}-\varepsilon} \le W_n$, $D_n = O\left(\frac{\log n}{\sqrt{n}}\right)$

Minimum Weight Triangulations

What is the total minimum weight of a triangulation whose boundary is fixed cycle 

?

$\mathbb{F}_2$-homological filling? null-homotopy? proper?

$$F_n \leq W_n \leq D_n$$

Theorem [Benjamini - Lubetzky - P.]:

(1) $$D_n = (\gamma n)^{-\frac{1}{2}}\left(\frac{\log(n)}{2} + \frac{5\log\log(n)}{2} + O_p(1)\right).$$

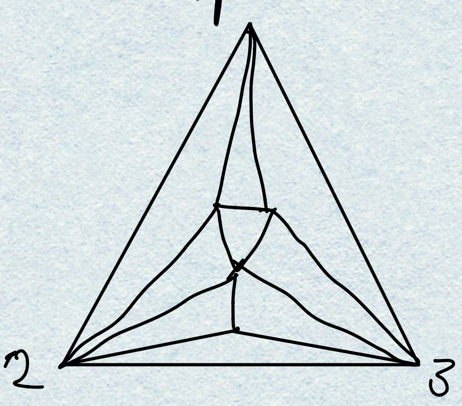
(2) WHP, $F_n = W_n = D_n$

Ideas from the Proof

1) To find D_n: 1st + 2nd moment:

(1st): $$\mathbb{E}\left[\#\left(\begin{array}{l}\text{proper } \Delta\text{-ions of total weight} \\ \le (\gamma n)^{-\frac{1}{2}}\left(\frac{\log(n)}{2} + \frac{5\log\log(n)}{2} + A\right)\end{array}\right)\right] \longrightarrow c' \cdot e^{\frac{A}{2}}$$

(2nd): consider Δ-ions with $k \approx \frac{\log n}{4}$ internal vertices and throw out:



Triangulation with $r = \Theta(1)$ vertices

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Inner triangulation of a face with $k-r$ vertices

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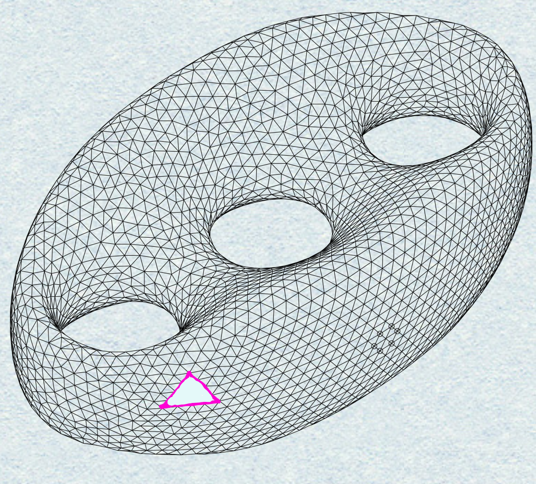
2) To show $F_n = D_n$:

(*) Every $\mathbb{F}_2$-homological filling contains a surface



(*) 1st moment on # surfaces below given weight $\Rightarrow$ "genus expansion"