

On a conjecture of Cox and Katz

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Abstract. This note shows that a certain toric quotient of the quintic Calabi–Yau threefold in \mathbb{P}^4 provides a counterexample to a recent conjecture of Cox and Katz concerning nef cones of toric hypersurfaces.

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Introduction

Anticanonical hypersurfaces in toric Fano varieties provide a large supply of Calabi–Yau varieties. Many explicit computations and constructions on these varieties rely on the strength of toric geometry. In particular, ever since the paper of Batyrev [2], toric constructions have played an important role in the study of mirror symmetry.

According to string theory, one of the basic moduli spaces involved in mirror symmetry is the so-called Kähler moduli space of a Calabi–Yau variety Z . This in turn is intimately related to the nef cone $\mathcal{N}(Z)$ of Z , i.e. the closed cone of divisors in the Picard group over \mathbb{R} spanned by nef classes. The nef cone of a variety also appears in birational geometry; faces of this cone give information about possible birational contractions and fibre space structures on the variety. The explicit computation of this cone is therefore often of interest.

For toric hypersurfaces, one could hope that this cone, or at least its intersection with the subspace of toric divisors on Z (divisors that lie in the image of the restriction map from the Picard group of the ambient space),

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can be computed explicitly in terms of the toric data. In a recent work [3], Cox and Katz give an combinatorial description of a certain cone \mathcal{N}_0 in the real vector space W of toric divisors on Z . This cone is constructed from the ambient toric variety and some related varieties birational to it; for details turn to [3, Sect. 6.2] or Sect. 3 of this note. Cox and Katz conjecture [3, Conjecture 6.2.8] that the toric nef cone of the Calabi–Yau variety Z is exactly \mathcal{N}_0 .

In this note, I consider a certain toric quotient of a quintic hypersurface in \mathbb{P}^4 and its toric Calabi–Yau resolution Z . The variety Z is a hypersurface in a toric Fano variety and so the conjecture of Cox and Katz applies to it. However, I obtain

Theorem 0.1 *The hypersurface Z in its ambient toric variety provides a counterexample to [3, Conjecture 6.2.8] of Cox and Katz: its (toric) nef cone is strictly larger than the cone \mathcal{N}_0 predicted by the conjecture.*

My attempts at formulating an alternative conjecture or computing the nef cone of Z have not been successful.

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Notation and conventions All varieties are defined over \mathbb{C} . If X is a projective variety, its *nef cone* is the closed cone $\mathcal{N}(X)$ in $\text{Pic}_{\mathbb{R}}(X)$ generated by nef divisor classes, i.e. classes $D \in \text{Pic}(X)$ satisfying $D \cdot C \geq 0$ for all effective curves $C \subset X$. I use the language of *toric geometry*, in particular the ideas of *linear Gale transform* and *Gelfand–Kapranov–Zelevinsky cones*; the main references are Fulton [4], Oda–Park [5] and Cox–Katz [3, Chapters 3 and 6].

1 Some toric varieties

Fix ξ , a primitive fifth root of unity. Consider the image D of the group

$$\left\{ (z_i) \mapsto (\xi^{a_i} z_i) : \sum_{i=0}^4 a_i = 0 \pmod{5} \right\}$$

in $PGL(5, \mathbb{C})$ and its subgroup $H = \langle g_1, g_2 \rangle$ generated by

$$g_1 : (z_0, z_1, z_2, z_3, z_4) \mapsto (z_0, \xi z_1, \xi^2 z_2, \xi^3 z_3, \xi^4 z_4)$$

and

$$g_2 : (z_0, z_1, z_2, z_3, z_4) \mapsto (z_0, \xi z_1, \xi^3 z_2, \xi z_3, z_4).$$

I will be interested in the quotient variety \mathbb{P}^4/H and its (partial) resolutions. To describe these torically, let $\widetilde{N} \cong \mathbb{Z}^4$, $\widetilde{M} = \text{Hom}(\widetilde{N}, \mathbb{Z})$ and consider the polyhedron

$$\widetilde{\Delta} = \left\{ \sum_{i=0}^4 m_i \leq 1, m_i \geq -1 \right\} \subset \widetilde{M}_{\mathbb{R}}$$

together with its dual polyhedron

$$\begin{aligned} \widetilde{\Delta}^* = \text{span}\{ & (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \\ & (0, 0, 0, 1), (-1, -1, -1, -1) \} \subset \widetilde{N}_{\mathbb{R}}. \end{aligned}$$

The data $(\widetilde{M}, \widetilde{\Delta})$ defines $\mathbb{P}_{\widetilde{M}, \widetilde{\Delta}} \cong \mathbb{P}^4$ in the contravariant description of toric varieties. The obvious map of lattices and polyhedra $(\widetilde{N}, \widetilde{\Delta}^*) \rightarrow (\widetilde{M}, \widetilde{\Delta})$ corresponds to the quotient map

$$\mathbb{P}_{\widetilde{M}, \widetilde{\Delta}} \cong \mathbb{P}^4 \longrightarrow \mathbb{P}_{\widetilde{N}, \widetilde{\Delta}^*} \cong \mathbb{P}^4/D.$$

Proposition 1.1 *In the contravariant description, $\mathbb{P}^4/H \cong \mathbb{P}_{M, \Delta}$, where $M \cong \mathbb{Z}^4$ and Δ is the polyhedron*

$$\begin{aligned} \Delta = \text{span}\{ & (1, 0, 0, 0), (-3, 5, -4, -2), \\ & (0, 0, 1, 0), (0, 0, 0, 1), (2, -5, 3, 1) \} \subset M_{\mathbb{R}}. \end{aligned}$$

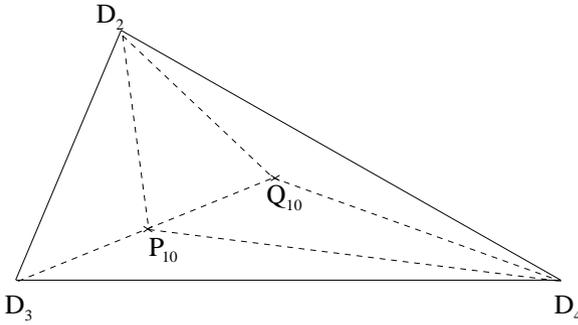
The dual polyhedron $\Delta^* \subset N_{\mathbb{R}}$ of Δ is

$$\begin{aligned} \Delta^* = \text{span}\{ & (-1, -2, -1, -1), (4, 1, -1, -1), \\ & (-1, -1, -1, -1), (-1, 2, 4, -1), (-1, 0, -1, 4) \}, \end{aligned}$$

where $N = \text{Hom}(M, \mathbb{Z})$. Moreover,

- i. there are no lattice points in the interior of Δ^* except for the origin;
- ii. there are no lattice points in the interiors of three- or one-dimensional faces;
- iii. there are precisely two lattice points $P_i, Q_i, i = 1, \dots, 10$ in the interiors of each of the ten two-dimensional faces; the combinatorics of the faces is shown on top of the next page.

Proof. The inclusion $\widetilde{N} \hookrightarrow \widetilde{M}$ corresponds to the inclusion in \widetilde{M} of the lattice of invariant monomials under the D action. The sublattice M of \widetilde{M} is the lattice of invariant monomials under the action of H . The points $(4, -1, -1, -1), (1, -1, 2, 0), (-1, -1, 4, -1), (-1, -1, -1, 4)$ of \widetilde{M} give a choice of basis for M . An easy computation gives Δ , its dual polyhedron Δ^* and the lattice points contained in it. For further reference, the lattice points of Δ^* are listed in the Appendix. □



Denote $\mathbb{P}_{M,\Delta}$ simply by \mathbb{P}_Δ . Let Σ be the fan consisting of cones over faces of Δ^* in $N_{\mathbb{R}}$. This fan defines the toric variety $\mathbb{X}_{N,\Sigma} \cong \mathbb{P}_\Delta$.

Proposition 1.2 *\mathbb{P}_Δ is a \mathbb{Q} -factorial Gorenstein variety, with ten curves of canonical singularities. Every permutation η of the lattice points $\{P_i, Q_i\}$ gives rise to a partial resolution*

$$\mathbb{X}_{\Sigma_\eta} \rightarrow \mathbb{P}_\Delta.$$

The varieties \mathbb{X}_{Σ_η} have isolated singularities only.

Proof. All this is basic toric geometry. The curves of singularities correspond to the ten two-dimensional faces of Δ^* . The singularities can be partially resolved by subdividing the fan Σ using the lattice points $\{P_i, Q_i\}$. Any permutation η of these points gives a fan Σ_η in the space $N_{\mathbb{R}}$ and a corresponding toric partial resolution \mathbb{X}_{Σ_η} with isolated singularities. \square

2 Some hypersurfaces

Let Q be a smooth anticanonical hypersurface in \mathbb{P}^4 invariant under the action of H ; for example the Fermat quintic will do. Let $\bar{Z} = Q/H$ be the corresponding non-degenerate anticanonical hypersurface of \mathbb{P}_Δ . \bar{Z} is singular at the ten intersection points with the curves of singularities of \mathbb{P}_Δ which are $\frac{1}{5}(1, 1, 3)$ quotient singularities. Every map $\mathbb{X}_{\Sigma_\eta} \rightarrow \mathbb{P}_\Delta$ gives rise to a morphism $Z_\eta \rightarrow \bar{Z}$. The hypersurface $Z_\eta \subset \mathbb{X}_{\Sigma_\eta}$ is a nonsingular Calabi–Yau variety because every \mathbb{X}_{Σ_η} is nonsingular in codimension three.

Proposition 2.1 *The resolutions Z_η are all canonically isomorphic to a Calabi–Yau resolution Z of \bar{Z} .*

Proof. Let η_1, η_2 be two permutations of the interior lattice points. There is a corresponding birational map

$$\mathbb{X}_{\Sigma_{\eta_1}} \dashrightarrow \mathbb{X}_{\Sigma_{\eta_2}}.$$

It is easy to check that the exceptional sets of this birational map are disjoint from the hypersurfaces Z_{t,η_i} . The statement follows. \square

The next statement shows that in the case at hand the space of toric divisors is in fact the whole Picard group.

Proposition 2.2 *The restriction homomorphisms*

$$\text{Pic}_{\mathbb{R}}(\mathbb{X}_{\Sigma_{\eta}}) \longrightarrow \text{Pic}_{\mathbb{R}}(Z)$$

are all isomorphisms.

Proof. This follows from [6, Sect. 6, Theorem 2]. The point is that every curve of singularities of \mathbb{P}_{Δ} meets the general hypersurface in one point, so the exceptional divisors in $\mathbb{X}_{\Sigma_{\eta}}$ restrict to irreducible divisors on the hypersurface Z . \square

3 Some cones

Let W denote the vector space $\text{Pic}_{\mathbb{R}}(Z)$. By Proposition 2.2, W can be canonically identified with the *linear Gale transform* [5] of the set of points $\{D_i, P_j, Q_r\}$ in N . The fans Σ_{η} for different permutations η give convex polyhedral cones, the so-called *Gelfand–Kapranov–Zelevinsky cones* $\text{cpl}(\Sigma_{\eta})$ in the vector space W , such that there are canonical maps and identifications

$$\begin{aligned} \text{Pic}_{\mathbb{R}}(\mathbb{X}_{\Sigma_{\eta}}) &\xrightarrow{\sim} \text{Pic}_{\mathbb{R}}(Z) = & W \\ \cup & & \cup \\ \mathcal{N}(\mathbb{X}_{\Sigma_{\eta}}) &= & \text{cpl}(\Sigma_{\eta}) \end{aligned}$$

Lemma 3.1 *Under these identifications, the cones $\text{cpl}(\Sigma_{\eta})$ are all contained in the nef cone $\mathcal{N}(Z) \subset W$.*

Proof. The anticanonical hypersurface in $\mathbb{X}_{\Sigma_{\eta}}$ is Z . Nef divisor classes on the ambient space clearly restrict to nef classes on the hypersurface. \square

Thus

$$\bigcup_{\eta} \text{cpl}(\Sigma_{\eta}) \subset \mathcal{N}(Z).$$

This is however not the full story. It is certainly possible that there are other subdivisions Σ_0 of Σ satisfying the property used above; namely, that the anticanonical hypersurface in \mathbb{X}_{Σ_0} is isomorphic to Z . To treat these fans, I recall some definitions following [3, 6.2].

Suppose Σ is a fan in $N_{\mathbb{R}}$, $\Sigma^{(1)}$ the set of its one-dimensional cones. A *linear circuit* is a linearly dependent set $S \subset \Sigma^{(1)}$, no subset of which is

linearly dependent. There is a decomposition $S = S_+ \cup S_-$ (depending on a choice) where S_+ , respectively S_- are the vectors appearing with positive, respectively negative coefficients in a linear relation. Correspondingly, there is a fan $\Sigma_+(S)$ given by cones spanned by $S \setminus n_i$ for $n_i \in S_+$ and their subcones, and a similar fan $\Sigma_-(S)$.

A linear circuit S is said to be *supported by* Σ , if the following two conditions are satisfied:

- $\Sigma_+(S)$ is a subfan of Σ .
- Let σ be a top-dimensional cone of $\Sigma_+(S)$. If there exists a subset $S' \subset \Sigma^{(1)}$ such that $\sigma \cup S'$ generates a top-dimensional cone of Σ , then for all other top-dimensional cones σ' of $\Sigma_+(S)$, $\sigma' \cup S'$ also generates a top-dimensional cone of Σ .

Suppose that S is a linear circuit supported by Σ and both S_- and S_+ are non-empty. Then there exists a new fan $\text{Flip}_S(\Sigma)$ from Σ obtained by replacing the simplices of Σ spanned by $\sigma \cup S'$, where σ is a cone of $\Sigma_-(S)$ and $S' \subset \Sigma^{(1)}$, by the simplices spanned by $\sigma' \cup S'$ where σ' is a cone of $\Sigma_+(S)$. Then the fan $\text{Flip}_S(\Sigma)$ is simplicial, the toric variety defined by it is projective, and the cones $\text{cpl}(\Sigma)$ and $\text{cpl}(\text{Flip}_S(\Sigma))$ in the linear Gale transform W touch along a common face.

Corresponding to the two fans in $N_{\mathbb{R}}$, there is a birational map

$$\varphi : \mathbb{X}_{\Sigma} \dashrightarrow \mathbb{X}_{\text{Flip}_S(\Sigma)}.$$

It is easy to check that in case S_- contains only one element, this map is in fact a morphism contracting a divisor. If however both S_- and S_+ contain more than one element, the birational map is a *generalized flop*, a small contraction followed by a small resolution. If the flop φ has exceptional locus disjoint from the anticanonical hypersurface of \mathbb{X}_{Σ} , it is called a *trivial flop* and in this case the flip attached to S is referred to by [3] as a *trivial flip*.

Return to the lattice N containing the polyhedron Δ^* . Define a fan Σ_0 in $N_{\mathbb{R}}$ to be *good*, if it satisfies the following

Condition. There is a permutation η such that the fan Σ_0 can be obtained from the fan Σ_{η} by a sequence of trivial flips.

The Condition implies that the set of one-dimensional cones of Σ_0 is precisely $\{D_i, P_i, Q_i\}$. So the cones $\text{cpl}(\Sigma_0)$ defined by the good fans Σ_0 embed canonically into W . As the flips involved are trivial, the proper transform of Z in \mathbb{X}_{Σ_0} is isomorphic to Z . Setting

$$\mathcal{N}_0 = \bigcup_{\Sigma_0 \text{ good}} \text{cpl}(\Sigma_0),$$

there is an inclusion

$$\mathcal{N}_0 \subset \mathcal{N}(Z).$$

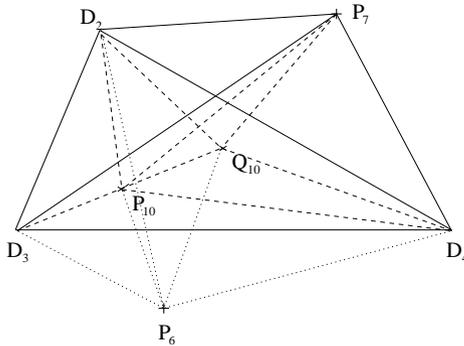
[3, Conjecture 6.2.8] expects this inclusion to be an equality. However, the situation is more complicated. The following is Theorem 0.1 stated in the Introduction:

Theorem 3.2 *In the case discussed, the inclusion above is strict. The anticanonical hypersurface Z in the toric variety \mathbb{X}_{Σ_η} provides a counterexample to [3, Conjecture 6.2.8] of Cox and Katz.*

Proof. Let Σ_0 be a good fan satisfying the condition that the cones over the tetrahedra

$$D_2P_{10}Q_{10}P_6, D_2D_4Q_{10}P_6, D_4P_{10}Q_{10}P_6, \\ D_2P_{10}Q_{10}P_7, D_2D_4Q_{10}P_7, D_4P_{10}Q_{10}P_7$$

are top-dimensional cones in Σ_0 (see the figure; remember it is a three-dimensional image of a four-dimensional setup).



Under this assumption, there are two interesting circuits supported on Σ_0 . The first one is

$$S = \{D_2, D_4, P_{10}, Q_{10}\}.$$

The linear relation is

$$-3\mathbf{q}_{10} + \mathbf{d}_2 + \mathbf{d}_4 + \mathbf{p}_{10} = 0$$

in obvious notation. Setting $S_+ = \{D_2, D_4, P_{10}\}$, it is easy to check that the assumptions imply that S is supported on Σ_0 . The corresponding birational map φ_1 is a contraction of the divisor E_1 given by the one-dimensional cone spanned by Q_{10} . It is easy to check that $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$ contracting to \mathbb{P}^1 .

The map φ_1 restricts to the threefold Z as the contraction of an irreducible exceptional divisor $F_1 \cong \mathbb{P}^2$ to a singular point.

Now consider the circuit $S = \{P_6, P_7, Q_{10}\}$. The relation is

$$-\mathbf{q}_{10} + \mathbf{p}_6 + \mathbf{p}_7 = 0.$$

With $S_+ = \{P_6, P_7\}$, the circuit is supported on Σ_0 . The corresponding contraction φ_2 on \mathbb{X}_{Σ_0} is again divisorial, having exceptional divisor defined by the one-dimensional cone spanned by D_{10} . So φ_2 has the same exceptional divisor $E_2 = E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$ as φ_1 . The image of the exceptional divisor is in this case two-dimensional: the contraction φ_2 restricts to E_2 as the projection to \mathbb{P}^2 .

The restriction of φ_2 to Z contracts the set $\text{Exc}(\varphi_2) \cap Z$ to \mathbb{P}^2 . However, in the first part of the discussion I have shown that $\text{Exc}(\varphi_2) \cap Z = \text{Exc}(\varphi_1) \cap Z = \mathbb{P}^2$. Under φ_2 , this maps isomorphically to \mathbb{P}^2 . Hence the contraction φ_2 restricts to Z as the identity.

The contraction φ_2 is divisorial, in particular not a flop; hence the corresponding face of $\text{cpl}(\Sigma_0)$ is a face of the cone \mathcal{N}_0 . However, divisors in this face (and beyond) are still ample on Z . This implies that the corresponding face is not in the boundary of $\mathcal{N}(Z)$. Thus the cone $\mathcal{N}(Z)$ is strictly larger than \mathcal{N}_0 as claimed. □

Remark 3.3 It is easy to see that possible counterexamples to [3, Conjecture 6.2.8] can only arise where the relevant face of \mathcal{N}_0 gives a contraction with fibre dimension one. In all other cases, the hypersurface Z contains at least one contracted curve, and so the face is indeed a face of the nef cone of Z .

From this point of view, it is instructive to consider the following, much simpler example. Let $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^3$ and let Z be an anticanonical Calabi–Yau threefold. \mathbb{P} has a nef cone with two faces, the faces corresponding to the contractions to the two factors. In particular, the nef cone of \mathbb{P} is also the effective cone, the cone of effective classes.

One of the contractions restricts to Z as a K3 fibration. However, the (Stein factorization of) the morphism to \mathbb{P}^3 is not a fibration, and not even a divisorial contraction: it is the contraction of a finite set of rational curves. In particular, it is a flopping face, there is another marked birational model for Z (which as an unmarked model is incidentally isomorphic to Z). What happens here is that the nef cones are the same, but the effective cone changes: the effective cone of Z is strictly larger than its nef cone. Note that the trouble came again from a contraction of the toric ambient space of fibre dimension one.

Appendix: Description of the polyhedron Δ^*

The vertices of Δ^* in N :

$$D_0 = (-1, -2, -1, -1), D_1 = (4, 1, -1, -1), D_2 = (-1, -1, -1, -1), \\ D_3 = (-1, 2, 4, -1), D_4 = (-1, 0, -1, 4)$$

The lattice points on the two-dimensional faces:

$$\begin{array}{lll} P_1 = (2, 0, -1, -1) & Q_1 = (0, -1, -1, -1) & \text{on } D_0D_1D_2 \\ P_2 = (0, 1, 2, -1) & Q_2 = (1, 0, 0, -1) & \text{on } D_0D_1D_3 \\ P_3 = (0, -1, -1, 0) & Q_3 = (1, 0, -1, 1) & \text{on } D_0D_1D_4 \\ P_4 = (-1, -1, 0, -1) & Q_4 = (-1, 0, 1, -1) & \text{on } D_0D_2D_3 \\ P_5 = (-1, -1, -1, 0) & Q_5 = (-1, -1, -1, 1) & \text{on } D_0D_2D_4 \\ P_6 = (-1, 0, 0, 2) & Q_6 = (-1, 0, 1, 0) & \text{on } D_0D_3D_4 \\ P_7 = (0, 0, 0, -1) & Q_7 = (1, 1, 1, -1) & \text{on } D_1D_2D_3 \\ P_8 = (0, 0, -1, 2) & Q_8 = (1, 0, -1, 0) & \text{on } D_1D_2D_4 \\ P_9 = (2, 1, 0, 0) & Q_9 = (0, 1, 1, 1) & \text{on } D_1D_3D_4 \\ P_{10} = (-1, 1, 2, 0) & Q_{10} = (-1, 0, 0, 1) & \text{on } D_2D_3D_4 \end{array}$$

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