

Dimer models and local non-commutative algebraic geometry

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- Fix a partition of vertices of Γ into black and white
- Assume Γ admits perfect matchings (and other conditions...)

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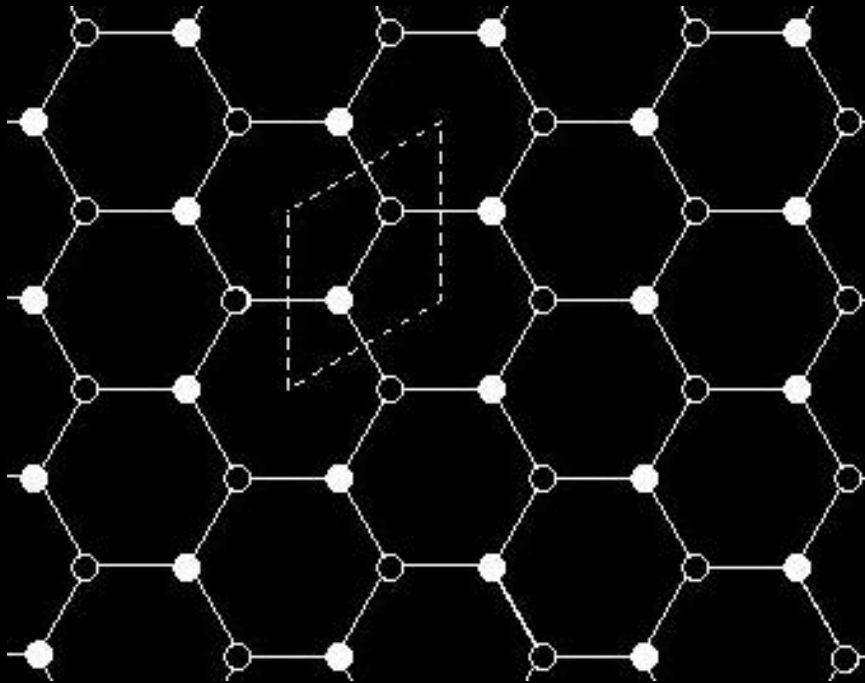
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$\tilde{\Gamma} \subset \mathbb{R}^2$: the pullback to the universal cover

Dimer configuration Δ : one-regular subgraph of $\tilde{\Gamma}$.

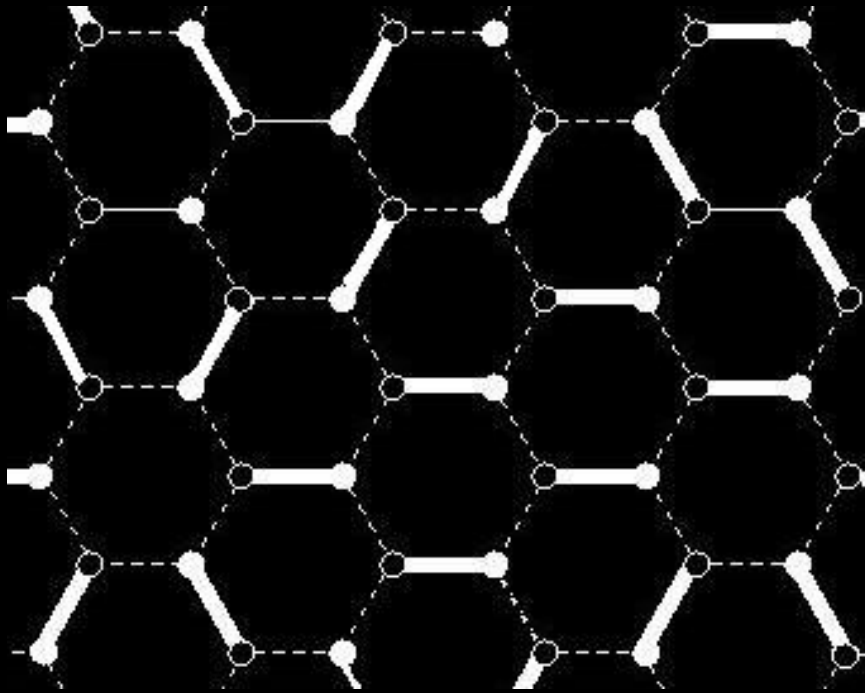
Hexagonal lattice

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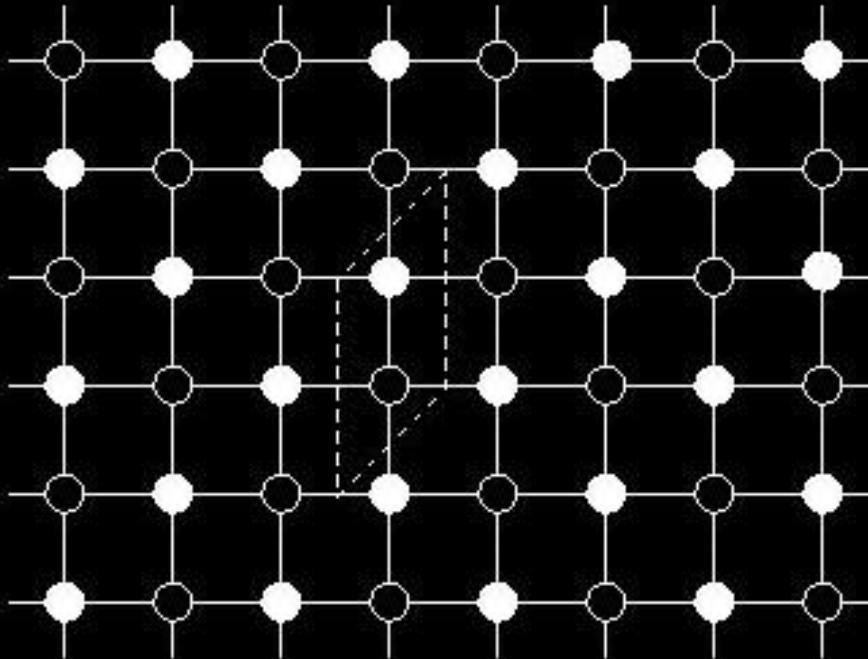
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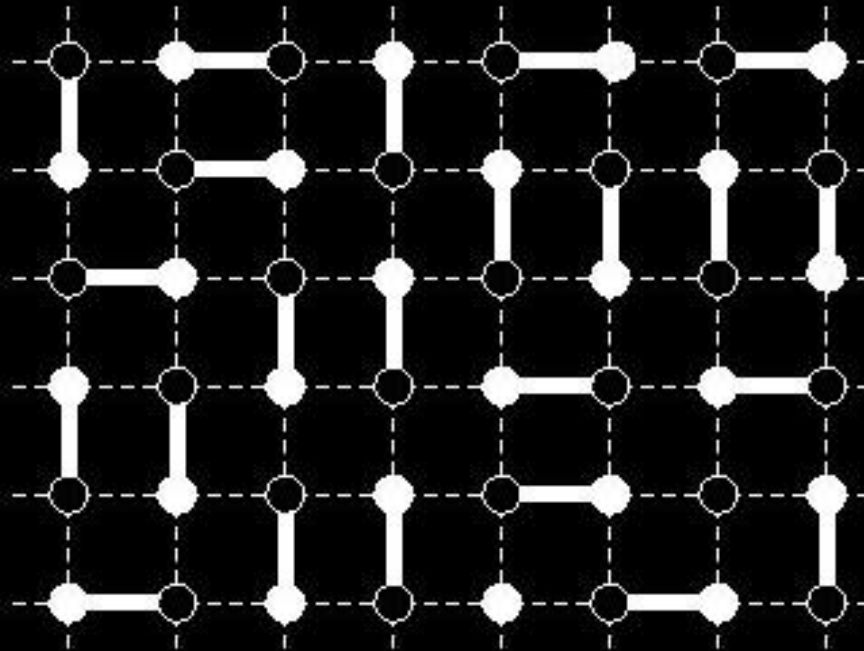
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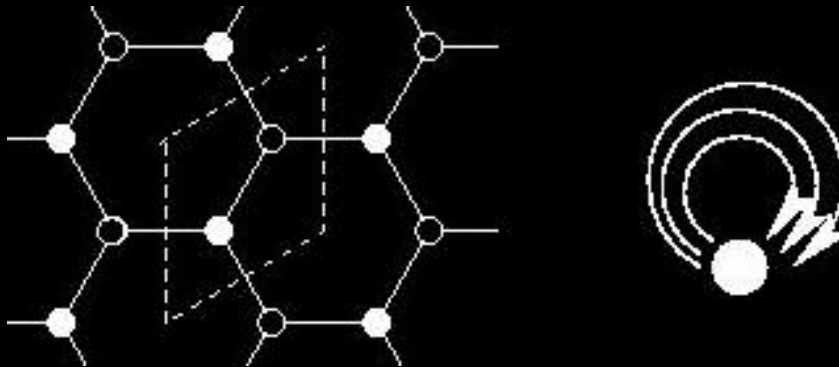
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From the dimer model to the quiver

Given a bipartite $\Gamma \in T^2$, define a **quiver** (oriented graph) as follows:

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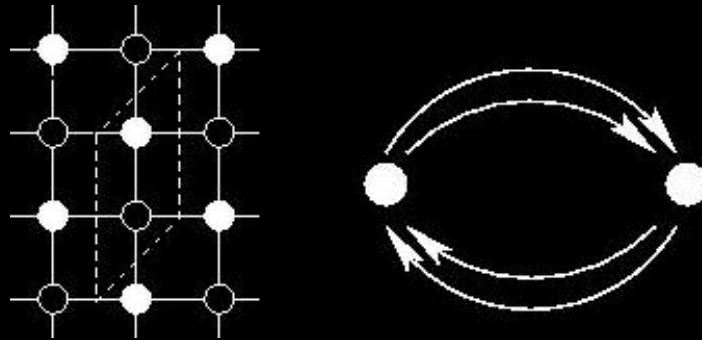


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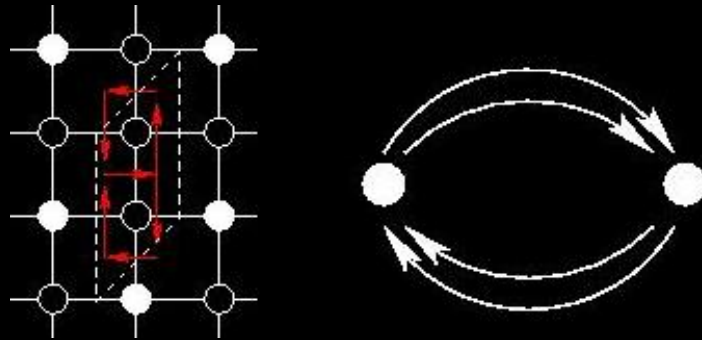
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Following [Hanany et al], define the **superpotential**

$$W = \sum_{\substack{i \in V(\Gamma) \\ i \text{ black}}} W_i - \sum_{\substack{i \in V(\Gamma) \\ i \text{ white}}} W_i$$

The non-commutative algebra

Let $\mathbb{C}Q$ be the **quiver algebra** of the quiver

- \mathbb{C} -algebra generated by oriented paths
- Product is given by concatenation of oriented paths (or zero)
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Define the **non-commutative quotient algebra**

$$A_{Q,W} = \mathbb{C}Q / \langle\langle \partial_e W : e \in E(Q) \rangle\rangle$$

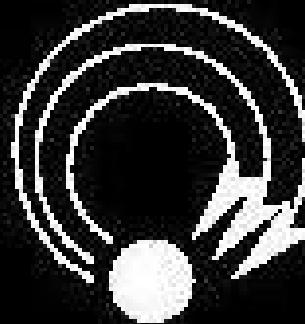
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Take the quiver of Example 1. We have a free non-commutative algebra

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hence

$$A_{Q,W} = \mathbb{C}[x, y, z]$$

commutative!

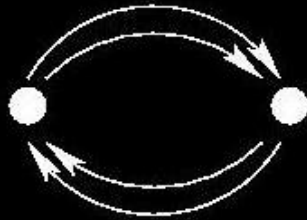
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which gives a non-commutative algebra $A_{Q,W}$. Its **center**

$$Z(A_{Q,W}) = \mathbb{C}[x, y, z, t]/(xy - zt)$$

with

$$x = ac + ca, \dots$$

is the coordinate ring of the **ordinary double point singularity**.

Properties of $A_{Q,W}$

Under suitable conditions (King/Broomhead) the following hold:

- $A_{Q,W}$ is a smooth non-commutative 3-Calabi–Yau algebra
 - vanishing of Ext’s on A -mod above degree 3
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- The algebra $A_{Q,W}$ is Van Den Bergh’s **non-commutative crepant resolution** of $X = \text{Spec}(Z(A))$

Moduli spaces of A -modules

Fix dimension vector $\mathbf{n} \in \mathbb{N}^V$ and a vertex $i \in V$.

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The **moduli space of cyclic representations** of the quiver Q with relations:

$$\mathcal{M}_{\mathbf{n},i} = \left\{ \begin{array}{l} (\phi_e \in \text{Hom}(U_{t(e)}, U_{h(e)}))_{e \in E}, v \in U_i \\ (\phi_e) \text{ satisfy relations } dW = 0 \\ v \text{ generates } \bigoplus_{i \in V} U_i \text{ under } (\phi_e) \end{array} \right\} / \prod \text{GL}(U_i).$$

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- Cyclicity: **stability condition**
- In Example 1:

$$\mathcal{M}_n \cong (\mathbb{C}^3)^{[n]},$$

the Hilbert scheme of n points on \mathbb{C}^3 .

Partition function

Proposition $\mathcal{M}_{n,i}$ is cut out in a smooth variety by zeros of one-form $d\text{Tr}W$.
Hence

- it carries a virtual moduli cycle of dimension 0
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Fixing the vertex $i \in V$, get **partition function** of cyclic $A_{Q,W}$ -modules based at vertex i :

$$Z(\mathbf{q}) = \sum_{\mathbf{n} \in \mathbb{N}^V} e_{\text{vir}}(\mathcal{M}_{\mathbf{n},i}) \mathbf{q}^{\mathbf{n}}$$

for a set of auxiliary variables $\mathbf{q} = \{q_1, \dots\}$.

Torus action

Recall: the center of $A_{Q,W}$ defines a toric Calabi–Yau X .

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Proposition

- The torus T acts as outer automorphisms of the algebra $A_{Q,W}$.
- This defines an action of T on all moduli spaces $\mathcal{M}_{\mathbf{v},i}$.
- The T -fixed points on $\mathcal{M}_{\mathbf{v},i}$ are isolated.

Hence the partition function can be computed by localizing to the fixed points.

Torus localization: dimer configurations

For simplicity, restrict now to Examples 1-2.

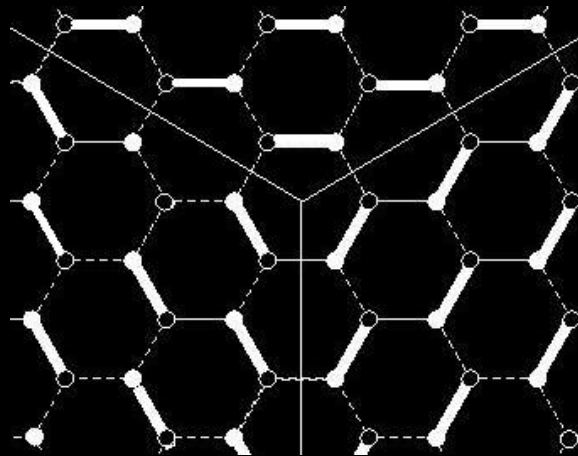
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Example 1: the “empty room” dimer configuration Δ_0 on the hexagonal lattice.

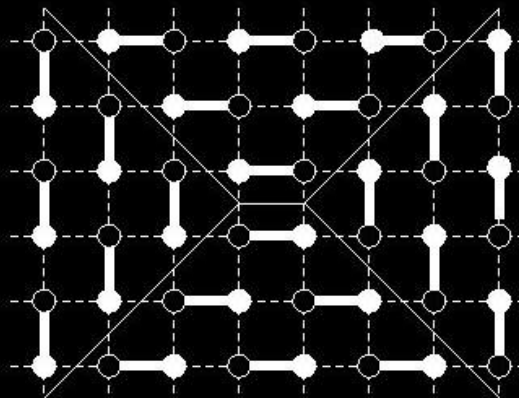


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Torus localization: crystal combinatorics

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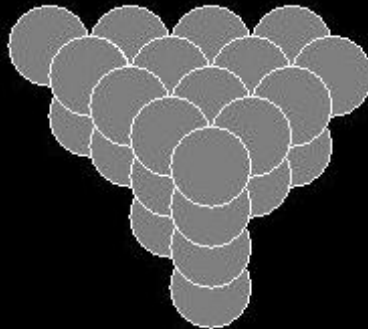
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Example 1: the crystal is the **triangular pyramid**, its suitable subsets are

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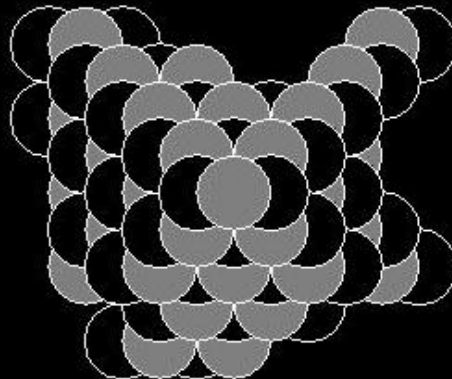


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Equivalent combinatorics for Examples 1-2

Proposition The torus-fixed points on the moduli spaces $\mathcal{M}_{\mathbf{v},i}$ are in one-to-one correspondence with suitable finite subsets of a crystal configuration.

Example 2: the crystal is the **square-based pyramid**, its suitable subsets are **finite 3-dimensional pyramid partitions**

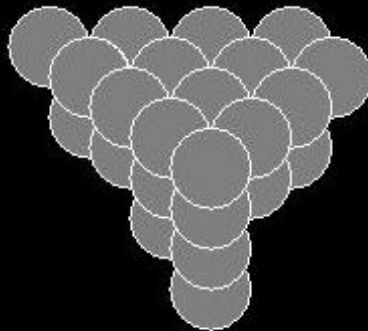


Exact results for the partition function

In Example 1, the set of T -fixed points on \mathcal{M}_v is the set \mathcal{P} of 3-dimensional partitions. Hence

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Theorem (MacMahon, 19th c.)

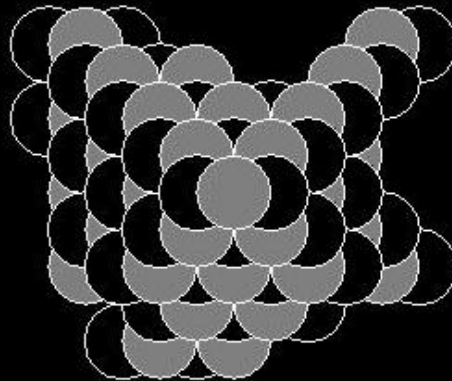
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In Example 2, the set of T -fixed points on \mathcal{M}_v is the set $\tilde{\mathcal{P}}$ of “pyramid partitions”. Hence

$$Z(q_1, q_2) = \sum_{\pi \in \tilde{\mathcal{P}}} q_1^{|\pi|_1} (-q_2)^{|\pi|_2},$$

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Theorem (Sz.-Young, 2007/8)

$$Z(q_1, -q_2) = \prod_{n \geq 1} (1 - q_1^n q_2^{n-1})^n (1 - q_1^n q_2^{n+1})^n (1 - q_1^n q_2^n)^{-2n}.$$

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One is the commutative crepant resolution Y , which has DT partition function

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An example of Kontsevich–Soibelman/Denef–Moore **wall crossing!**

