Enhanced Gauge Symmetry and Braid Group Actions

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Abstract: Enhanced gauge symmetry appears in Type II string theory (as well as F-
and M-theory) compactified on Calabi–Yau manifolds containing exceptional divisors
meeting in Dynkin configurations. It is shown that in many such cases, at enhanced
symmetry points in moduli a braid group acts on the derived category of sheaves of
the variety. This braid group covers the Weyl group of the enhanced symmetry algebra,
which itself acts on the deformation space of the variety in a compatible way. Exten-
sions of this result are given for nontrivial B-fields on K3 surfaces, explaining physical
restrictions on the B-field, as well as for elliptic fibrations. The present point of view
also gives new evidence for the enhanced gauge symmetry content in the case of a local
A_{2n}-configuration in a threefold having global $\mathbb{Z}/2$ monodromy.

Introduction

The phenomenon that Type II string theory compactified on a Calabi–Yau manifold can
exhibit enhanced gauge symmetry was first observed in the physics literature in the
context of K3 surfaces [23, 1]. The existence of non-perturbatively enhanced symmetry
algebras is forced by the duality between heterotic string theory on T^4 and the Type
IIA string on K3, since the former obviously has enhanced symmetry at special points
in moduli. It was found that a K3 surface can have enhanced gauge symmetry if it has
rational double point (ADE) singularities, and the type of the (simply-laced) non-abelian
Lie algebra that appears precisely matches that of the singularity. The argument for non-
abelian gauge symmetry was later extended to Calabi–Yau threefolds in [2] and [16], for
threefolds with a curve of ADE singularities. In the presence of monodromy, non-sim-
ply laced Lie algebras can also appear. These symmetries and the arising representations
have also been analyzed in the context of M- and F-theory (see [13, 3] and references
therein).

   The purpose of the present paper is to give a mathematical interpretation of a
"holomorphic shadow" of this symmetry; namely, of the parameters needed to specify a

string vacuum, I will only concentrate on the complex structure and B-field parameters, ignoring the Kähler structure. In particular, by moving in the Kähler moduli space, I can resolve the singularities mentioned in the previous paragraph, and work with smooth K3 surfaces and Calabi–Yau threefolds, containing ADE configurations of rational curves and configurations of ruled surfaces respectively. The phenomenon that I will illustrate by several theorems is that enhanced gauge symmetry can occur at points in complex moduli when the derived category of the corresponding Calabi–Yau manifold has a large set of autoequivalences. Moreover, these derived equivalences always satisfy the relations of a generalized braid group, which covers the Weyl group of the enhanced gauge symmetry Lie algebra. When one deforms the complex parameters, these autoequivalences deform away to equivalences of derived categories between different manifolds; this is always governed by a Weyl group action on the deformation space. In particular, one can phrase the results of this paper as saying that the category of topological D-branes on a Calabi–Yau compactification (cf. [8]) has an extra braid group worth of symmetries at enhanced gauge symmetry points, not present at generic points in moduli.

Braid group actions for groups of Type A (and DE) on derived categories were first constructed in [19]. In Sect. 3 of the present paper I will show how to extend these actions in two dimensions (K3 surfaces) to cover deformations, and how this fits into the framework of enhanced gauge symmetry. The autoequivalences will be generalized to cover deformations with nonzero B-field; in particular I will derive the restrictions on the B-field found in [1] by a duality argument.

Calabi–Yau threefolds, as mentioned before, can exhibit gauge symmetries of all A ... G_2 types. Corresponding braid group actions are constructed in [22]. I explain in Sect. 4 the main points of the construction, referring back to the (easier) surface case. I also give some examples, including an amusing projective example exhibiting non-trivial monodromy, and make some comments related to the interpretation of the actions as enhanced gauge symmetry.

The threefolds appearing in this paper represent the simplest case of enhanced gauge symmetry, that of "uniform singularities" or geometrically ruled surfaces (no hypermultiplets in physics-speak). In case there are extra rational curves in fibers, the mathematics is more complicated (compare for example [13]); dissident curves can be flopped, there are many more autoequivalences and derived equivalences around, and it appears to be difficult to formulate a clean statement. However, for one highly singular situation studied for example in [3, Sect. 4], the ideas of the present paper are strong enough to provide supporting evidence (though alas not a proof) for the gauge symmetry content. The argument is spelled out in Remark 10.

The paper begins with two introductory sections: Section 1 recalls reflection groups and (generalized) braid groups, whereas Sect. 2 deals with (families of) equivalences of derived categories. The latter section contains a statement which may be of independent interest, connecting deformations of a Fourier–Mukai functor of a Calabi–Yau variety with its action on cohomology. Section 5 points out an extension of the results to elliptic fibrations and braid groups of affine type which may be interesting from the point of view of F-theory, whereas Sect. 6 poses a challenge for symplectic geometry via mirror symmetry.

1. Reflection Groups and Generalized Braid Groups

A Dynkin diagram Δ in this paper means an irreducible finite type diagram corresponding to a finite root system $\Sigma \subset \mathfrak{h}_{\mathbb{R}}$ in a real Euclidean inner product space $\mathfrak{h}_{\mathbb{R}}$. It is well

known that such diagrams can be of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2. The root system Σ defines a finite reflection group $W_\Delta = \langle r_i \rangle$ acting on $\mathfrak{h}_\mathbb{R}$, generated by a set of reflections $r_1, \dots, r_n$ indexed by nodes of Δ, equivalently by a set of simple roots. As an abstract group,

$$W_\Delta \cong \Big\langle r_i : i \in \text{Nodes}(\Delta) \Big\rangle \Big/ \Big\langle r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \Big\rangle$$

with one relation for every node i and one for every pair of different nodes (i, j) with label m_{ij}. The group W_Δ also acts on the complex vector space $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes \mathbb{C}$.

Define the *(generalized) braid group* (also called Artin group) B_Δ by generators and relations as

$$B_\Delta = \Big\langle R_i : i \in \text{Nodes}(\Delta) \Big\rangle \Big/ \Big\langle \underbrace{R_i R_j \dots}_{m_{ij}} = \underbrace{R_j R_i \dots}_{m_{ij}} \Big\rangle \tag{1}$$

with one relation for every pair of different nodes (i, j) of Δ, the *braid relation*. There is a group homomorphism $B_\Delta \to W_\Delta$ sending R_i to r_i. As an example, in the familiar case of type A_n the group W_Δ is the symmetric group on $(n + 1)$ letters, whereas B_Δ is the classical braid group on $(n + 1)$ strings.

2. Families of Derived Equivalences

If X is a smooth projective variety, let $D^b(X)$ denote the bounded derived category of coherent sheaves on X. A *kernel* (derived correspondence) between smooth projective varieties X_1, X_2 is an object $U \in D^b(X_1 \times X_2)$. Such an object defines a functor

$$\Psi^U : D^b(X_2) \to D^b(X_1)$$

by

$$\Psi^U(-) = \mathbf{R} p_{1*}(U \overset{\mathbf{L}}{\otimes} p_2^*(-)),$$

with $p_i : X_1 \times X_2 \to X_i$ the projections. If Ψ^U is an equivalence of triangulated categories, then it is called a *Fourier–Mukai functor* and U is said to be *invertible*.

Suppose that $\pi : \mathcal{X} \to S$ is a smooth family of projective varieties over a complex base S. A *relative kernel* is a pair (U, φ), where

- $\varphi : S \to S$ is an automorphism, giving rise to the fibre product diagram

$$
\begin{array}{ccc}
\mathcal{X} \times_\varphi \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{X} & \xrightarrow{\varphi \circ \pi} & S
\end{array}
$$

and
- $U \in D^b(\mathcal{X} \times_\varphi \mathcal{X})$ is an object in the derived category of the product.

There is a map $\mathcal{X} \times_\varphi \mathcal{X} \to S$ with fibre $X_s \times X_{\varphi(s)}$ over $s \in S$. The (derived) restriction of U to this fibre gives a kernel

$$U_s = \mathbf{L}y_s^*(U) \in D^b(X_s \times X_{\varphi(s)}),$$

where $y_s \colon X_s \times X_{\varphi(s)} \hookrightarrow \mathcal{X} \times_\varphi \mathcal{X}$ is the inclusion. Hence a relative kernel defines a family of functors

$$\Psi_s = \Psi^{U_s} \colon D^b(X_{\varphi(s)}) \to D^b(X_s).$$

In the present paper, a relative kernel (U, φ) will be called *invertible*, if for all $s \in S$ the functor Ψ_s is a Fourier–Mukai functor. Every invertible relative kernel gives a family of Fourier–Mukai transforms over the base S.

The next statement is in some sense auxiliary, but it encompasses the point of view of the present article. Let X be a projective K3 surface or Calabi–Yau three-fold. Let $\pi \colon \mathcal{X} \to S$ be a family of projective deformations of X over a polydisc S, with $\pi^{-1}(0) \cong X$ for $0 \in S$. Assume that the Kodaira–Spencer map

$$\psi \colon T_0 S \to H^1(X, \Theta_X)$$

of the family is injective.

Let $U_0 \in D^b(X \times X)$ be an invertible kernel on X giving rise to a Fourier–Mukai functor $\Psi = \Psi^{U_0}$ on X. Using the Mukai map from the derived category to cohomology (see for example [7, Sect. 3.1]), there is an induced isomorphism

$$\psi \colon H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$$

preserving Hodge structures (in the sense of Mukai for the K3 case). In particular, $H^{n,0}$ is preserved where n is the dimension of X; so if $\Omega \in H^0(X, \Omega_X^n)$ is a holomorphic top-form then its image $\psi(\Omega)$ is also a holomorphic top-form (a constant multiple of Ω).

Theorem 1. *Assume that there is an invertible relative kernel (U, φ) on $\mathcal{X} \to S$ with $\varphi(0) = 0$ extending U_0. Then there is a commutative diagram*

$$
\begin{array}{ccc}
T_0(S) & \xrightarrow{\;d\varphi|_0\;} & T_0(S) \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\
H^1(X, \Theta_X) & & H^1(X, \Theta_X) \\
\downarrow{\scriptstyle\wedge\Omega} & & \downarrow{\scriptstyle\wedge\psi(\Omega)} \\
H^1(X, \Omega_X^{n-1}) & & H^1(X, \Omega_X^{n-1}) \\
\downarrow & & \downarrow \\
H^*(X, \mathbb{C}) & \xrightarrow{\;\psi\;} & H^*(X, \mathbb{C})
\end{array}
$$

where the last vertical maps are the inclusions coming from Hodge theory.

This statement may look complicated, but it says something very simple. Suppose you have a Fourier–Mukai functor Ψ on X. The action of Ψ on cohomology gives rise, via Hodge theory, to a map on the base of the local deformation space of X. Then the only way to extend Ψ over a deformation family of X is to a relative kernel whose action φ on the base is compatible with the map defined by Ψ. In particular, unless Ψ acts trivially on the local deformation space, it will never extend to a family of autoequivalences ($\varphi = \mathrm{id}_S$) in a family of deformations of X.

Proof of Theorem 1. Once the statement is properly formulated, the proof is not very difficult. Note that the family Ψ^{U_s} of Fourier–Mukai functors gives rise to an isomorphism of local systems $\oplus_n R^n (\varphi \circ \pi)_* (\mathbb{C}_\mathcal{X}) \cong \oplus_n R^n \pi_* (\mathbb{C}_\mathcal{X})$ on S (basically just a continuous family of cohomology isomorphisms), which preserves Hodge filtrations. Now use the fact that the period map of the family is injective (since the Kodaira–Spencer map of π is, and X is Calabi–Yau), and unwind the definition of the derivative of the period map at $0 \in S$.   □

3. K3 Surfaces with ADE Configurations

Let $\bar{Y}$ be a projective K3 surface with a du Val (rational double point) singularity at a point $p \in \bar{Y}$ and no other singularities. Let $g \colon Y \to \bar{Y}$ be its smooth K3 resolution with exceptional locus $E = E_1 \cup \ldots \cup E_r$. It is well known that each component E_i is a smooth rational curve of self-intersection -2, hence it defines a reflection

$$r_i \colon \omega \mapsto \omega + (E_i \cdot \omega)\, E_i \tag{2}$$

on $H^2(Y, \mathbb{C})$. The intersection graph of the curves $\{E_i\}$ is a Dynkin diagram Δ of type ADE, and as the notation suggests, the maps r_j generate an action of the reflection group W_Δ on $H^2(Y, \mathbb{C})$.

Proposition 2. *There exists a family $e \colon \mathcal{Y} \to Z$ of projective deformations of $e^{-1}(0) \cong Y$ over a complex polydisc $0 \in Z$, with an action of the finite group W_Δ on the base Z, such that the following properties hold:*

  (i) there is a proper subset $Z_i \subset Z$ such that $s \in Z_i$ if and only if the fibre Y_s contains a smooth rational curve which is a deformation of $E_i \in Y$;
  (ii) for every $s \in Z$, there is a contraction morphism $Y_s \to \bar{Y}_{i,s}$, which contracts the deformation of E_i in Y_s if $s \in Z_i$ and is an isomorphism otherwise;
  (iii) the fixed locus of r_i on Z equals Z_i; and
  (iv) for $w \in W_\Delta$ and $s \in Z$, the fibres Y_s, $Y_{w(s)}$ are isomorphic.

Proof. This can be proved using the language of lattice-polarized K3 surfaces [10]. Let M be the orthogonal complement of $\langle E_1, \ldots, E_n \rangle$ in the Picard group of Y, or any sublattice thereof containing the cohomology class of an ample divisor on Y; since $\bar{Y}$ was assumed projective, such M exist. Consider the local moduli space $\mathcal{Y} \to Z$ of M-polarized K3 surfaces [10] with central fibre $Y = e^{-1}(0)$ for $0 \in Z$, a smooth family of projective K3 surfaces. Since Z is small, the second cohomology $H^2(Y_s, \mathbb{Z})$ can be identified across the family. The base Z is isomorphic, using the Kodaira–Spencer map, to a small disc around the origin in $N \otimes \mathbb{C}$, where N is the orthogonal complement of M in Pic (Y). Since M does not include the class E_i, $E_i \in H^2(Y_s, \mathbb{Z})$ is algebraic (and represented by a rational curve) if and only if $s \in Z_i$ for a subvariety $Z_i \subset Z$. It is easy to see that the W_Δ-action on $H^2(Y, \mathbb{C})$ preserves $N \otimes \mathbb{C}$, and hence W_Δ can be made to act on Z. The isomorphisms $Y_s \cong Y_{w(s)}$ come from the Torelli theorem, since these surfaces have isomorphic Hodge structure. Finally the fact that Z_i is exactly the fixed locus of r_i is just chasing definitions.   □

Next I want to define relative kernels on $\mathcal{Y} \to Z$, indexed by nodes of the diagram Δ. By (ii) above, for a node i of Δ and $s \in Z$ there is a contraction $Y_s \to \bar{Y}_{i,s}$

which contracts E_i if $s \in Z_i$ and is an isomorphism otherwise. There is a diagram

$$
\begin{array}{ccc}
 & \widetilde{Y}_{i,s} & \\
\swarrow & & \searrow \\
Y_s & & Y_{r_i(s)} \\
\searrow & & \swarrow \\
 & \bar{Y}_{i,s} &
\end{array}
$$

where $\widetilde{Y}_{i,s}$ is the fibre product of the two contractions. This fibre product can be thought of as a subscheme of the product $Y_s \times Y_{r_i(s)}$; it is the "correspondence variety" on the product (pairs of points mapping to the same image). If $s \in Z \setminus Z_i$, then $\widetilde{Y}_{i,s}$ is simply the diagonal in $Y_s \times Y_{r_i(s)}$ with respect to the isomorphism $Y_s \cong Y_{r_i(s)}$. On the other hand, if $s \in Z_i$, then $E_i \subset Y_s$ is a rational curve, and $\widetilde{Y}_{i,s}$ has two components: one is the diagonal, and the other one is $E_i \times E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$. The components intersect along the diagonal Δ_{E_i}.

In any case, set $U_{i,s} = \mathcal{O}_{\widetilde{Y}_{i,s}} \in D^b(Y_s \times Y_{r_i(s)})$ to be the (pushforward of the) structure sheaf of this correspondence subscheme. It is possible to show (see [22, Theorem 4.1] for the case of threefolds) that the kernels $U_{i,s}$ are restrictions to the fibres of a relative kernel (U_i, r_i) on $\mathcal{Y} \to Z$.

Theorem 3. *For every node i of Δ, the relative kernel (U_i, r_i) is invertible: for $s \in Z$, the kernel $U_{i,s}$ defines a Fourier–Mukai functor*

$$
\Psi_{i,s} = \Psi^{U_{i,s}} \colon D^b(Y_{r_i(s)}) \xrightarrow{\sim} D^b(Y_s)
$$

such that for a pair of nodes (i, j) of Δ, there is a isomorphism of functors

$$
\underbrace{\Psi_{i,s} \circ \Psi_{j,r_i(s)} \circ \dots}_{m_{ij}} \cong \underbrace{\Psi_{j,s} \circ \Psi_{i,r_j(s)} \circ \dots}_{m_{ij}} \colon D^b(Y_{r_{ij}(s)}) \longrightarrow D^b(Y_s), \tag{3}
$$

where

$$
r_{ij} = \underbrace{r_i \circ r_j \circ \dots}_{m_{ij}} \cong \underbrace{r_j \circ r_i \circ \dots}_{m_{ij}} \in W_\Delta.
$$

Hence the derived category $D^b(Y)$ carries an action of the braid group B_Δ, and this action deforms to an action of B_Δ by a family of derived equivalences over the deformation space $\mathcal{Y} \to Z$ of Y.

Proof. The point $s = 0 \in Z$ is fixed by all r_i, and in this case the theorem is a re-statement of a special case of [19, Theorem 1.2]. In more detail, as proved in [22, Lemma 4.6], for $s = 0 \in Z$ the functors $\Psi_{i,0}$ are just the twist functors of [19] with respect to the spherical sheaves $\mathcal{O}_{E_i}(-1)$ on $Y = Y_0$. The relations (3) were proved in [19]. Hence mapping the braid group generator R_i to the autoequivalence $\Psi_{i,0}$ defines an action of B_Δ on $D^b(Y)$.

For arbitrary $s \in Z$, the fact that $\Psi_{i,s}$ is invertible is easy: if $s \in Z_i$ then it is still a twist functor; otherwise it is the structure sheaf of the diagonal in $Y_s \times Y_{r_i(s)}$ under the isomorphism $Y_s \cong Y_{r_i(s)}$, and hence clearly invertible. The relation (3) can be proved using the method of [22], which does the more complicated case of threefolds. The point is that the kernels for the composites on both sides of the relation (3) can be proved to

be structure sheaves; for a general point $s \in Z$ they are both isomorphic to the structure sheaf of the diagonal in $Y_s \times Y_{r_{ij}(s)}$ under the isomorphism $Y_s \cong Y_{r_{ij}(s)}$, and from this a specialization argument concludes that the two kernels are isomorphic everywhere. In particular, this gives an independent proof in this case of the braid relations on the central fibre Y. $\quad\square$

It is known from [1] that (for appropriate values of the Kähler form) Type II string theory on the surface Y exhibits enhanced gauge symmetry. The braid group action in Theorem 3 is a holomorphic shadow of this enhanced gauge symmetry: the derived category of Y has a braid group worth of autoequivalences covering the Weyl group of the nonperturbative gauge symmetry algebra, which deform to equivalences between different varieties under a deformation of its complex structure. In other words, at the enhanced gauge symmetry points the derived automorphism group of Y (the group of symmetries of the category of topological D-branes) is larger than that of its deformations.

I next extend Theorem 3 and its interpretation as enhanced gauge symmetry to gerbe deformations, also known as nonzero B-fields. I take the most simple-minded definition, according to which the B-field is a class $B \in H^2(Y, \mathbb{R}/\mathbb{Z})$. A B-field can be used to twist the derived category of coherent sheaves of the K3 surface Y as follows. Consider the natural map

$$\delta \colon H^2(Y, \mathbb{R}/\mathbb{Z}) \to H^2(Y, \mathcal{O}_X^*) \tag{4}$$

coming from the exponential sequence. The class $\beta = \delta(B) \in H^2(Y, \mathcal{O}_X^*)$ gives a gerbe on X, and there is a notion of a sheaf over this gerbe (also called β-twisted sheaf on Y). One wants to define the "derived category of β-twisted sheaves on Y" with some finiteness condition. If the class B is torsion in $H^2(Y, \mathbb{R}/\mathbb{Z})$, then the usual notion of coherence generalizes, and one obtains [7] a triangulated category $D^b(Y, B)$ with properties very similar to those of $D^b(Y)$. In the general case there does not seem to be an accepted definition, though see [14, Remark 2.6] for discussion. The following statement is therefore formulated for the case of torsion B-fields; I certainly expect it to hold in general.

Theorem 4. *Let $B \in H^2(Y, \mathbb{Q}/\mathbb{Z})$ be a torsion B-field. Then for every vertex i of Δ, there is a family of twisted Fourier–Mukai functors*

$$\Psi_{i,s,B} \colon D^b(Y_{r_i(s)}, r_i(B)) \xrightarrow{\sim} D^b(Y_s, B) \tag{5}$$

deforming the functor $\Psi_{i,s,0} = \Psi_{i,s}$. Here W_Δ acts on $H^2(Y, \mathbb{R}/\mathbb{Z})$ via its action on $H^2(Y, \mathbb{R})$.

Proof. Let p_1, p_2 denote the projections of $Y_s \times Y_{r_i(s)}$ onto its factors. A twisted functor (5) needs, by [7, Sect. 3.1], a kernel

$$V_{i,s} \in D^b(Y_s \times Y_{r_i(s)}, p_2^*(r_i(B)) - p_1^*(B))$$

(note that I am using additive notation for classes in cohomology with values in $\mathbb{Q}/\mathbb{Z}$).

Recall the correspondence variety $\widetilde{Y}_{i,s}$ in $Y_s \times Y_{r_i(s)}$ with respect to the ith contraction. The sheaf $U_{i,s}$ was defined as the structure sheaf of this correspondence; more precisely, if $k \colon \widetilde{Y}_{i,s} \hookrightarrow Y_s \times Y_{r_i(s)}$ is the inclusion, then $U_{i,s} = k_* \mathcal{O}_{\widetilde{Y}_{i,s}}$.

Let

$$\widetilde{B} = p_2^*(r_i(B)) - p_1^*(B).$$

Note that by [7, Theorem 2.2.6], there is a twisted pushforward functor

$$k_*\colon D^b(\widetilde{Y}_{i,s}, k^*(\widetilde{B})) \to D^b(Y_s \times Y_{r_i(s)}, \widetilde{B}).$$

I claim that the structure sheaf of the scheme $\widetilde{Y}_{i,s}$ is naturally a sheaf on $\widetilde{Y}_{i,s}$ over the gerbe defined by the class $k^*(\widetilde{B})$. This implies that the kernel $U_{i,s} = k_*\mathcal{O}_{\widetilde{Y}_{i,s}}$ can be thought of as a sheaf on $Y_s \times Y_{r_i(s)}$ over the gerbe corresponding to $\widetilde{B}$ and hence it can be used to define the twisted functor (5).

To prove the claim, I distinguish two cases. First assume $s \in Z_i$. Then E_i deforms to Y_s and as I said above, $\widetilde{Y}_{i,s}$ has two components: one is $E_i \times E_i$ and the other one is Δ_{Y_s}, the diagonal. It is enough to show that the structure sheaf of either component is a sheaf over the gerbe coming from $k^*(\widetilde{B})$ restricted to that component. But one component $E_i \times E_i$ is simply the quadric surface, which has a trivial Brauer group and hence there is nothing to prove. On the other component, $k^*(\widetilde{B})|_{\Delta_{Y_s}} = (B \cdot E_i)E_i$. Now the point is that since $s \in Z_i$, E_i is an algebraic class on Y_s, hence the class $k^*(\widetilde{B})|_{\Delta_{Y_s}}$ defines the trivial gerbe (see Remark 5 for the argument). Hence again, the structure sheaf is a sheaf over this gerbe!

Next assume that $s \in Z \setminus Z_i$. Then there is an isomorphism $Y_s \cong Y_{r_i(s)}$. It can be shown that this isomorphism induces the map r_i on second cohomology. On the other hand, $\widetilde{Y}_{i,s}$ is in this case irreducible and isomorphic to the diagonal; moreover, $\widetilde{B}$ pulls back to the trivial gerbe over this diagonal. Hence the structure sheaf is again a sheaf over the gerbe defined by $k^*(\widetilde{B})$.

The fact that the kernel $U_{i,s}$ defines an equivalence of categories can be proved using [7, Theorem 3.2.1], which generalizes the criterion of Bridgeland [5, Theorems 5.1 and 5.4]; I omit the details. □

Remark 5. The statement of Theorem 4 involves a subtlety concerning the W_Δ-action on gerbes. On the central fibre Y, all cohomology classes E_i are algebraic. On the other hand, the map

$$H^2(Y, \mathbb{R}) \to H^2(Y, \mathcal{O}_Y^*)$$

factors through $H^2(Y, \mathcal{O}_Y)$ and by Hodge theory, the image of $E_i \in H^2(Y, \mathbb{R})$ in $H^2(Y, \mathcal{O}_Y)$ is zero. This implies that B and $r_i(B)$ give the same gerbe on Y. However, for generic Y_s the classes E_i are transcendental, and B, $r_i(B)$ are different gerbes. In fact, Theorem 4 should be complemented by a statement that there is no family of equivalences

$$D^b(Y_{r_i(s)}, B) \to D^b(Y_s, B).$$

The family of sheaves $\{U_{i,s}\}$ is certainly not appropriate, since as the proof above shows, $\widetilde{B}$ gives a nontrivial gerbe for $s \in Z \setminus Z_i$ exactly because the class $[E_i]$ is transcendental on Y_s. Indeed I expect that the only possible way to deform the equivalence $\Psi_{i,s}$ in the B-field direction is that compatible with its cohomology action; in other words, there is an analogue of Theorem 1 for gerbe deformations. I have no idea how to prove this statement.

I wish to offer the following interpretation of Theorem 4: Type II string theory on (Y, B) has enhanced gauge symmetry (for appropriate values of the Kähler parameter) if and only if the derived category $D^b(Y, B)$ admits a set of twisted autoequivalences, which deform to twisted Fourier–Mukai functors between different points in moduli when the complex structure and B-field parameters are deformed. Theorem 4, together

with Remark 5, says that this is the case if and only if $r_i(B) = B$ for all i, in other words if and only if $E_i \cdot B = 0$ for all exceptional curves. Note that this condition on the B-field is identical to that of [1, p. 4], found by an analysis involving heterotic/Type II duality.

4. Calabi–Yau Threefolds Containing Ruled Surfaces

Let $\bar{X}$ be a projective threefold with a curve of singularities

$$B = \mathrm{Sing}(X) \hookrightarrow \bar{X},$$

such that along the curve $\bar{X}$ has du Val singularities of uniform ADE type. The iterated blowup of the singular locus $f : X \to \bar{X}$ is a resolution of singularities. Locally over a point $p \in B \in \bar{X}$, the fibre of f is a set of rational curves as before, intersecting according to the appropriate ADE type Dynkin diagram. However, globally there may be monodromy (see Fig. 1): as p moves over the curve B, the configuration of curves may be permuted according to a diagram symmetry of the Dynkin diagram.

It is well known that quotients of ADE Dynkin diagrams by (subgroups of) their automorphism groups are non-simply laced Dynkin diagrams in a well-defined sense. Concretely, the action of $\mathbb{Z}/2$ on the diagrams A_{2n+1}, D_n and E_6 gives, respectively, the diagrams C_{n+1}, B_{n-1} and F_4, whereas the action of $\mathbb{Z}/3$ and the symmetric group on three letters leads to the diagram G_2. The group $\mathbb{Z}/2$ also acts on the diagram A_{2n}; this is a special case which I exclude from consideration, though see Remark 10 below.

Globally therefore, the exceptional locus of $f : X \to \bar{X}$ consists of a set of smooth geometrically ruled surfaces $\{\pi_j : D_j \to B_j\}$ intersecting in a Dynkin configuration Δ,
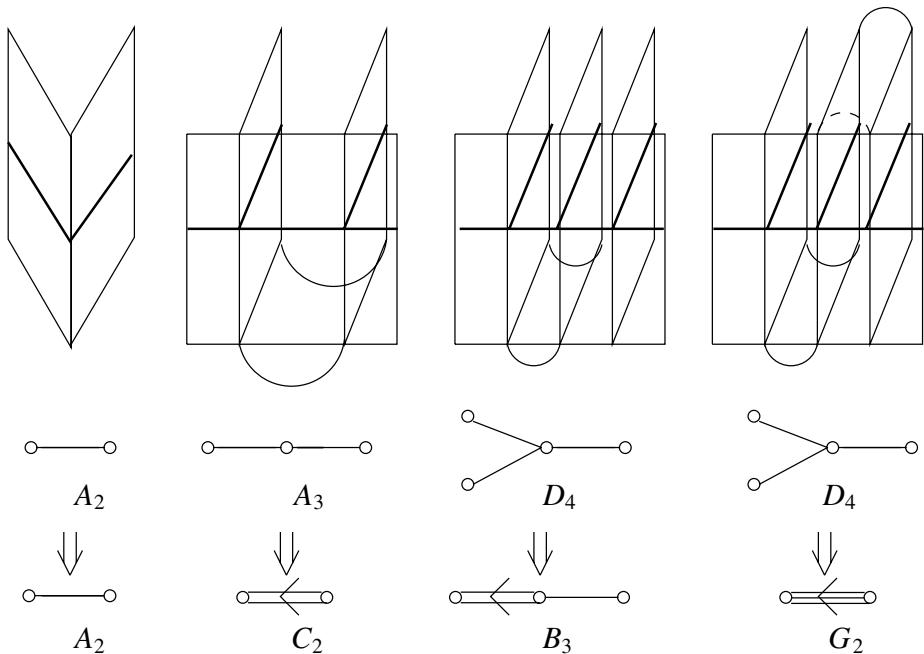

Fig. 1. Dynkin diagrams and configurations of surfaces

which may or may not be simply laced. If Δ is simply laced then each B_j is isomorphic to B, whereas in the general case each B_j is an unramified cover of B of the appropriate degree.

As in the case of surfaces, I want to describe some deformations of the threefold X. In the local case (when one restricts attention to a neighbourhood of the exceptional surfaces), this problem is studied in detail in [22, Sect. 2]. Globally there may be some obstructions to realizing all local deformations as actual projective deformations of X. In simple cases (see below) it can be checked that the deformations I describe actually exist. The next proposition therefore should be considered a kind of "ideal scenario" statement.

Proposition 6. *Let X be the Calabi–Yau threefold constructed above, with a set of exceptional surfaces $\{\pi_j : D_j \to B_j\}$ indexed by nodes of a Dynkin configuration Δ, which may or may not be simply laced. Assume that X has good deformation theory. Then the universal family of (projective Calabi–Yau) deformations $e : \mathcal{X} \to S$ of $X = e^{-1}(0)$ over a polydisc $0 \in S$ carries an action of the reflection group W_Δ on its base S; moreover, the following properties hold.*

   (i) *For every $s \in S$, there is a contraction $f_s : X_s \to \bar{X}_s$ deforming the contraction f.*
   (ii) *There is an analytic subset $S_j \subset S$ of codimension equal to the genus of B_j, such that $s \in S_j$ if and only if the fibre X_s contains a smooth ruled surface in the exceptional locus of f_s which is a deformation of $D_i \in X$.*
   (iii) *The fixed locus of r_j on S equals S_j.*
   (iv) *For $w \in W_\Delta$ and $s \in S$, the fibres X_s, $X_{w(s)}$ are birational.*

Assume moreover that the genus g of B is at least one, and $s \in S$ is a general point in the base. Then

   (v) *The exceptional locus of $X_s \to \bar{X}_s$ consists of rational $(-1, -1)$-curves, coming in sets of $l(2g - 2)$ naturally indexed by positive roots of Δ (l is the squared length of a root).*
   (vi) *For $w \in W_\Delta$ and $s \in S$, the birational map $X_s \dashrightarrow X_{w(s)}$ flops some of these curves.*

Note that in the central fibre, the exceptional locus of f_s consists of a set of surfaces indexed by *simple* roots (nodes) of Δ. In the general fibre (assuming genus at least two), the exceptional set of f_s is a set of curves indexed by *positive* roots of Δ. Figure 2 illustrates the case $\Delta = A_2$, $g = 2$.

Note also that the deformation theory of X is very different if the genus of B is zero. In that case, the W_Δ-action is trivial ($S_j = S$ for all j and hence every generator fixes S) and the exceptional locus is always two-dimensional. For higher genus the W_Δ-action is non-trivial and the general exceptional locus is one-dimensional. The case $g = 1$ is also somewhat special: in that case, for general $s \in S$, the contraction $f_s : X_s \to \bar{X}_s$ is an isomorphism, which is reminiscent of the surface case. This distinction is discussed further below.

The next statement is the exact analogue of Theorem 3.

Theorem 7. *For every node j of Δ, there is a family of Fourier–Mukai functors*

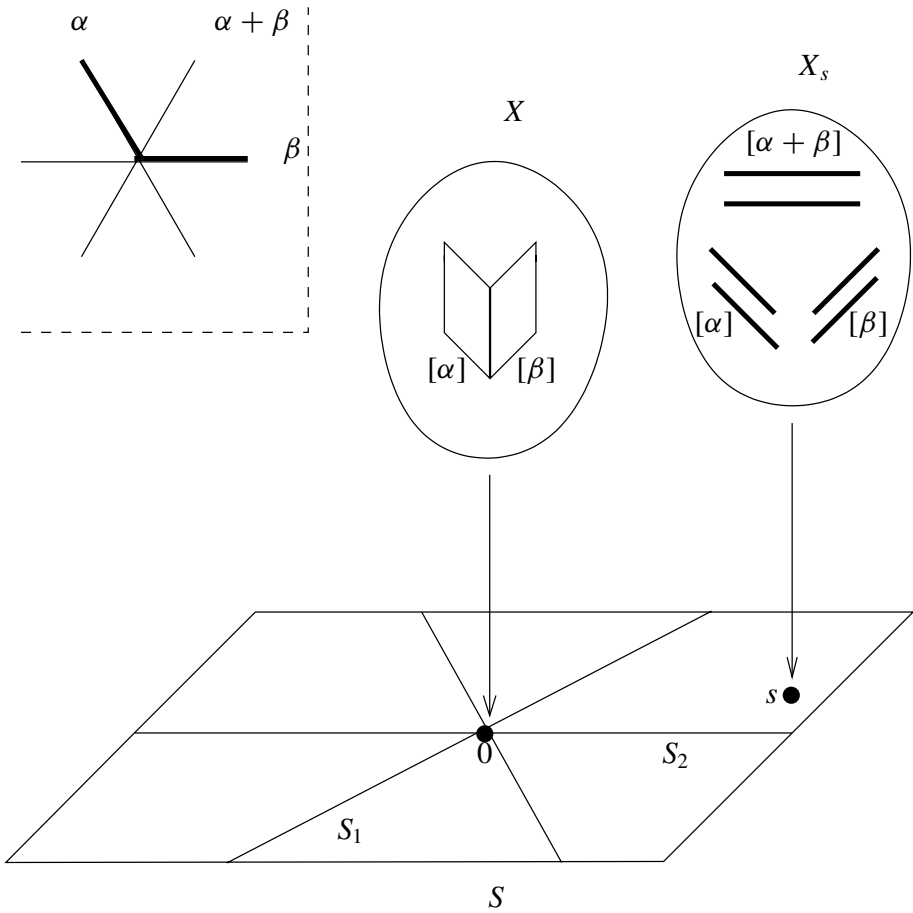$$\Psi_{j,s} : D^b(X_{r_j(s)}) \xrightarrow{\sim} D^b(X_s)$$

Fig. 2. The root system of A_2 and exceptional loci for $g = 2$

such that for a pair of nodes (i, j) of Δ, there is a isomorphism of functors

$$\underbrace{\Psi_{i,s} \circ \Psi_{j,r_i(s)} \circ \ldots}_{m_{ij}} \cong \underbrace{\Psi_{j,s} \circ \Psi_{i,r_j(s)} \circ \ldots}_{m_{ij}} : D^b(Y_{r_{ij}(s)}) \longrightarrow D^b(Y_s) \qquad (6)$$

where

$$r_{ij} = \underbrace{r_i \circ r_j \circ \ldots}_{m_{ij}} \cong \underbrace{r_j \circ r_i \circ \ldots}_{m_{ij}} \in W_\Delta.$$

Hence the derived category $D^b(X)$ carries an action of the braid group B_Δ, and this action deforms to an action of B_Δ by a family of equivalences over the deformation space $\mathcal{X} \to S$ of X.

Proof. The proof, given in detail in [22, Sect. 4], is similar to that of Theorem 3. The individual functors $U_{j,s}$ are defined using a diagram

$$X_s \quad \dashrightarrow \quad X_{r_j(s)}$$
$$\searrow \qquad \swarrow$$
$$\bar{X}_s.$$

For $s \in S_i$, the functor turns out to be a special case of a functor written down by Horja in [11, (4.31)], and proved invertible in [12]. The proof of the braid relations relies, as before, on a specialization argument. □

According to [2, 16, 3] and references cited in these works, threefolds X of the above type (for suitable values of the Kähler form) exhibit enhanced gauge symmetry. Theorem 7 is a holomorphic shadow of this symmetry: the derived category of X has a braid group worth of autoequivalences covering the Weyl group of the gauge algebra, which for genus at least one deforms away to a set of equivalences between different deformations. In particular, the derived automorphism group of X is larger than generic at these enhanced symmetry points.

It is interesting to consider the case when the curve B has genus zero. In this case, the projective threefold X has no deformations where the surfaces deform away. The braid group still acts on the derived category of X, but it also acts as a set of derived autoequivalences on all deformations. Hence nothing gets "enhanced". This phenomenon was also observed in the physics literature: as explained in [16, p.2], enhanced gauge symmetry needs that B is not rational; if $B \cong \mathbb{P}^1$ then the symmetry is only present in the limit when the area of B goes to infinity [2]. The lack of deformations is also an issue in the proof of the braid relations in [22]; the proof proceeds via decomposing X locally into a union of two pieces $X_1 \cup X_2$, so that both contain ruled surfaces over the affine line $\mathbb{A}^1$ and have enough deformations. Decomposing $\mathbb{P}^1$ into a union of two lines is here the mathematics equivalent to taking the area of the $\mathbb{P}^1$ to infinity.

Examples 8. Varieties $\bar{X}$ with a curve of singularities of uniform type A_n can be found among hypersurfaces or complete intersections in weighted projectice spaces; compare for example [16]. The resolution X is then embedded in a (partial) resolution of the ambient space, typically with n distinct divisors over the relevant singular locus; hence the configuration in X is still of type A_n. It can often be shown by concrete methods that the deformation theory of these threefolds is good in the sense needed for Proposition 6 to hold. Such varieties can be systematically searched for and in low codimension classified using the graded ring method pioneered by Reid; see the A_1 case in [20] and the general case in [6].

I proceed to give an example of a projective Calabi–Yau threefold X which contains a C_2 configuration of surfaces, inspired by [3, Sect. 3]; to the best of my knowledge, this is the first explicit example of this kind. Begin with an auxiliary variety

$$\bar{V} = \left\{ x_1^8 + x_2^8 + y_1^4 + y_2^4 + y_3^4 + z^2 = 0 \begin{array}{c} x_2^4 = y_1 y_2 \\ \end{array} \right\} \subset \mathbb{P}^5[1, 1, 2, 2, 2, 4].$$

$\bar{V}$ is a degenerate degree $(4, 8)$ complete intersection Calabi–Yau threefold in the indicated space. It can be checked by explicit computation that $\bar{V}$ has three curves of singularities, which are all elliptic. Along two of the curves at $\{x_1 = x_2 = y_1 = 0\}$ and $\{x_1 = x_2 = y_2 = 0\}$, $\bar{V}$ has generically A_1 singularities; this is a result of the

identifications on the weighted projective space. For a generic $(4, 8)$ complete intersection (which is simply an octic in $\mathbb{P}^4[1, 1, 2, 2, 2]$, since the degree four variable can be eliminated), there is one irreducible curve of A_1 singularities, but in the special $\bar{V}$ this part of the singular locus becomes reducible because of the first equation. The last curve is $\{x_2 = y_1 = y_2 = 0\}$, arising also because of the first equation; the singularity along the last curve is generically A_3. The three curves all meet at the two points $(0 : 0 : 0 : 0 : 1 : \pm i)$ of the weighted projective space. A patient calculation shows that these points are also quotient singularities, under the group $\mathbb{Z}/2 \times \mathbb{Z}/4$ acting on $\mathbb{C}^3$ by $(-1, -1, 1) \times (1, i, -i)$.

Construct a particular crepant partial resolution $V \to \bar{V}$ in two steps. First perform the blowup of both intersection points according to the right hand arrow of the toric diagram Fig. 3. This introduces two exceptional divisors over the two points, and leaves behind three disjoint curves of singularities of uniform type A_1, A_1 and A_3 respectively, with no dissident points. Then blow up the two disjoint A_1 curves to get a Calabi–Yau threefold V with a single elliptic curve of uniform A_3 singularities.

Consider the action

$$\iota: (x_1 : x_2 : y_1 : y_2 : y_3 : z) \mapsto (x_1 : (-x_2) : y_2 : y_1 : (-y_3) : (-z))$$

on the weighted projective space. This action fixes $\bar{V}$; since it interchanges the two A_1 singular curves, it extends to the partial resolution V. Further, ι acts by a free action on the elliptic curve of A_3 singularities of V; in the transverse coordinates x_2, y_1, y_2 to this curve satisfying the relation $x_2^4 = y_1 y_2$, the action interchanges y_1 and y_2, and maps x_2 to $-x_2$. A final check shows that ι acts freely on $\bar{V}$ and hence on V. Thus letting

$$\bar{X} = V/\langle \iota \rangle,$$

the projective Calabi–Yau threefold $\bar{X}$ has an elliptic curve of A_3 singularities and is smooth otherwise; moreover, the local coordinates along this curve undergo $\mathbb{Z}/2$ monodromy. Hence its Calabi–Yau resolution $X \to \bar{X}$ contains a C_2 configuration of exceptional surfaces ruled over elliptic curves.

Remark 9. The braid group action on the derived category gives rise to actions on even and odd cohomology, using the Mukai map. The action on odd cohomology $H^3(X, \mathbb{C})$ leads, as discussed in Proposition 1, to a Weyl group action on the tangent space to the deformation space, in a compatible fashion with the way the derived equivalences deform. There is also an induced Weyl group action on the Picard group. Some of these actions were described before; e.g. [16] has a symmetric group action in the case of Type A, both on the Picard group and the deformation space. The action of the braid group on the derived category explains all these actions in a uniform way.
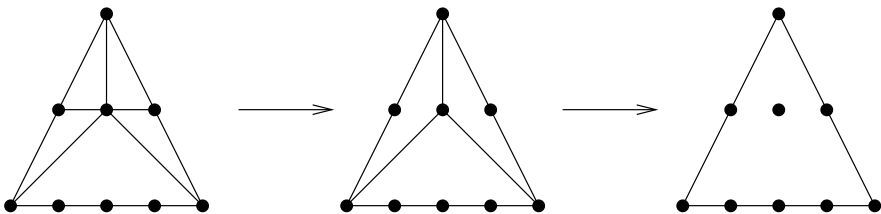

Fig. 3. The toric partial resolution of $\mathbb{C}^3/(\mathbb{Z}/2 \times \mathbb{Z}/4)$
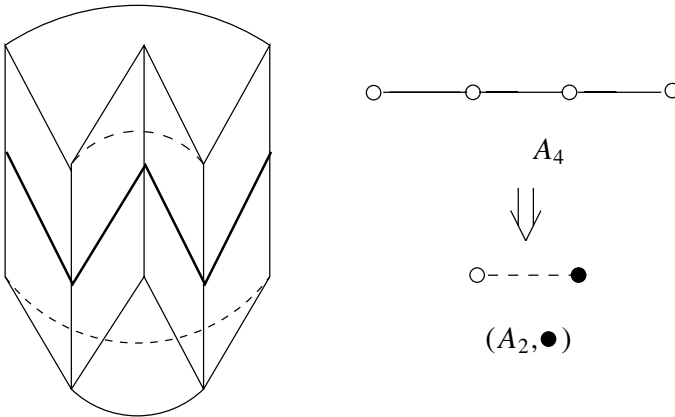
Fig. 4. The A_4 configuration with $\mathbb{Z}/2$ monodromy

Remark 10. The case of monodromy $\mathbb{Z}/2$ acting on the Dynkin diagram A_{2n} has been excluded from consideration all along. This case has caused considerable headache also in the physics literature [3, Sect. 4]. In this case, the exceptional divisors D_i of $f : X \to \bar{X}$ are still indexed by vertices of a kind of quotient quiver, the A_n-quiver with a marked vertex at one end corresponding to the adjacent $\mathbb{Z}/2$-orbit of vertices of A_{2n}. However, the marked node corresponds to a singular exceptional surface. It is an irreducible non-normal surface $\pi_n \colon D_n \to B$ whose double locus is a section and whose fibre over any point $b \in B$ is a line pair. I do not know whether there exists an autoequivalence Ψ_n of $D^b(X)$ corresponding to this surface, but I suspect that the answer is yes; this is a contracting EZ-configuration in the sense of Horja [12], with singular E.

The Main Assertion of [3, Sect. 4], supported by various arguments including the analysis of the matter spectrum, claims that the enhanced gauge symmetry is $\mathfrak{sp}(n)$, or in the language of the present paper, of type C_n. The point of view exposed in this paper gives additional support to this claim. Namely, the derived category of $D^b(X)$ is acted on by the autoequivalences $\Psi_1, \ldots, \Psi_{n-1}$ coming from the smooth ruled surfaces, as well as the hypothetical autoequivalence Ψ_n; the question is what are the relations. One can make an educated guess based on the following argument.

In the singular threefold $B \subset \bar{X}$, take a small quasiprojective surface $\bar{Y} \subset \bar{X}$ intersecting B once transversally. Let $Y \to \bar{Y}$ be its resolution in X, with exceptional curves $E_1, \ldots, E_{2n} \subset Y$. Set $\mathcal{E}_i = \mathcal{O}_{E_i}(-1) \in D^b(Y)$ for $i = 1, \ldots, 2n$. The functors Ψ_i can be restricted to Fourier–Mukai functors on Y (compare [22, Proof of Theorem 4.5]). The functor Ψ_i for $1 \leq i \leq n-1$ restricts in fact to the composite of two twist functors $T_{\mathcal{E}_i}$ and $T_{\mathcal{E}_{2n+1-i}}$. On the other hand, by [19], the twist functors $\{T_{\mathcal{E}_i} : 1 \leq i \leq 2n\}$ generate the braid group $B_{A_{2n}}$ acting on the derived category of Y. Moreover, the monodromy $\mathbb{Z}/2$ acts on this braid group, mapping $T_{\mathcal{E}_i} \mapsto T_{\mathcal{E}_{2n+1-i}}$ for $i = 1, \ldots, n$. The guess I want to make is that the functors $\Psi_1, \ldots, \Psi_n$ satisfy the relations of the fixed subgroup $(B_{A_{2n}})^{\mathbb{Z}/2}$. By a result in algebra [17], this fixed subgroup is generated by the composites $T_{\mathcal{E}_i} \circ T_{\mathcal{E}_{2n+1-i}}$ (note these commute) for $i = 1, \ldots, n-1$ and a final element $T_{\mathcal{E}_n} \circ T_{E_{n+1}} \circ T_{\mathcal{E}_n}$ (note these braid), and the group generated by these elements is the braid group corresponding to the Weyl group $(W_{A_{2n}})^{\mathbb{Z}/2}$. This latter group can be checked by a direct argument to be isomorphic to the Weyl group of the diagram C_n.

Hence the conjectural answer is that X has a set of derived equivalences $\Psi_1, \ldots, \Psi_n$ satisfying the braid relations of the Dynkin diagram C_n. In other words, X has enhanced gauge symmetry of type C_n (or $\mathfrak{sp}(n)$).

Remark 11. To conclude this section, I remark that as opposed to the case of dimension two, the braid group actions of [19] can never be interpreted as enhanced gauge symmetry in dimension three. The reason is the following: it can easily be checked that if $\mathcal{E}$ is a spherical object in the sense of [19], then the corresponding twist functor acts on cohomology by $\alpha \mapsto \alpha + \langle \mathrm{ch}(\mathcal{E}), \alpha \rangle \mathrm{ch}(\mathcal{E})$, where $\langle , \rangle$ is a linear combination of intersection forms on cohomology. However, $\mathrm{ch}(\mathcal{E})$ only has even components, hence the action of the twist functor on odd cohomology and so on $H^1(X, \Theta_X)$ is trivial. In particular, by Theorem 1, a twist functor always deforms to all deformations as an autoequivalence in dimension three, and hence it can never be part of an "enhanced" action.

5. Elliptic Fibrations and Braid Groups of Affine Type

Let $\sigma \colon X \to S$ be an elliptic fibration of a projective threefold X. Assume that there is a smooth component $C \subset S$ of the discriminant locus of σ, over which the fibres of σ are of uniform Kodaira type I_n with $n > 2$, $I_n^\star$, II^*, III^* or IV^*. These are the fibre types corresponding to the affine diagrams $\widetilde{A}_{n-1}$ $(n > 2)$, $\widetilde{D}_{n+4}$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$. In X, the rational curves in the fibres over $p \in C$ undergo monodromy, and trace out ruled surfaces $\pi_j \colon D_j \to C_j$. Assume that in the type $\widetilde{A}_{n-1}$ case the monodromy is not transitive, and in the type $\widetilde{D}_4$ case it does not act transitively on the outer vertices. Then the global intersections of the exceptional surfaces are described by an affine Dynkin diagram $\widetilde{\Delta}$, which is the original $\widetilde{ADE}$ diagram for trivial monodromy and a quotient non-simply laced $\widetilde{BCGF}$ type diagram otherwise. The diagram $\widetilde{\Delta}$ gives rise to a braid

$\widetilde{A}_4$ $\widetilde{C}_3$
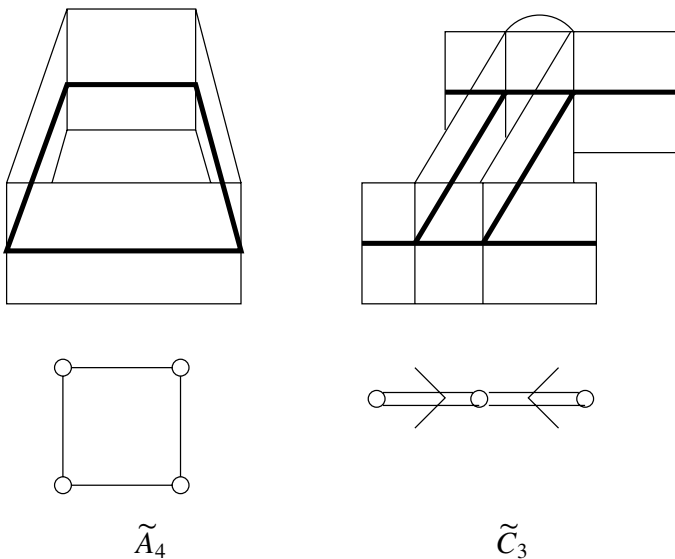
Fig. 5. Some ruled surface configurations in elliptic fibrations

group $B_{\widetilde{\Delta}}$, with one generator for every node of $\widetilde{\Delta}$ and one (braid) relation for every pair of nodes as dictated by the labels of the diagram $\widetilde{\Delta}$.

Theorem 12. *The affine type braid group $B_{\widetilde{\Delta}}$ acts on the derived category $D^b(X)$.*

Proof. The ruled surfaces $D_j \to C_j$ give rise to Fourier–Mukai functors Ψ_j on X as before. The proof of a single braid relation only concerns two surfaces and the functors defined by them. Under the assumptions made, every pair of surfaces forms an $A_1 \times A_1$, B_2 or G_2 configuration. Moreover, the computation of the composed functors can be restricted to a small neighbourhood of these two surfaces. Hence the proof of [22] applies. $\square$

Enhanced gauge symmetry for threefolds with elliptic fibrations has been discussed in the context of F-theory compactifications; see [18, 3, 9] and references therein.

6. Braiding Mirror Symplectomorphisms?

The paper [19], a direct predecessor of the present work, is directly motivated by mirror symmetry. Namely, the original motivation of that paper was to find the mirrors of certain symplectomorphisms of symplectic manifolds (M^{2n}, ω), *Dehn twists* in Lagrangian spheres $S^n \subset M$. The twist functors in spherical objects are natural candidates for the mirrors of Dehn twists.

As discussed in [11, 21 and 4], the derived equivalences studied in this paper, arising from ruled surfaces collapsing to curves in X, are mirror to certain diffeomorphisms of the mirror manifold, arising as monodromy transformations around certain boundary components of the complex moduli space of the mirror M. These diffeomorphisms are symplectomorphisms of (M^{2n}, ω) for special values of the symplectic form ω. It would be of interest to find a direct symplectic geometric construction of these diffeomorphisms. It is tempting to speculate that they are given by some kind of twisting with respect to a fibered submanifold of M, just as the Fourier–Mukai functors of X are constructed from the ruled surfaces. [15] begins the topological study of the mirrors of some explicit Calabi–Yau manifolds X containing a single ruled surface; the situation appears to be quite intricate. It would also be interesting to see whether in appropriate cases the braid relations (1) can be proved for these symplectomorphisms.

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