Spaces of graphs and surfaces - on the work of Søren Galatius

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New Orleans, January 2011

$\Sigma_n :=$ permutation group on n letters

$F_n :=$ free, non-abelian group on n letters

$\mathrm{Aut}F_n :=$ automorphism group of F_n

$$\Sigma_n \subset \mathrm{Aut}F_n$$

Galatius' main result:

For $0 < * < (n-1)/2$,

$$H_*(\Sigma_n) = H_*(\mathrm{Aut}F_n)$$

and in particular

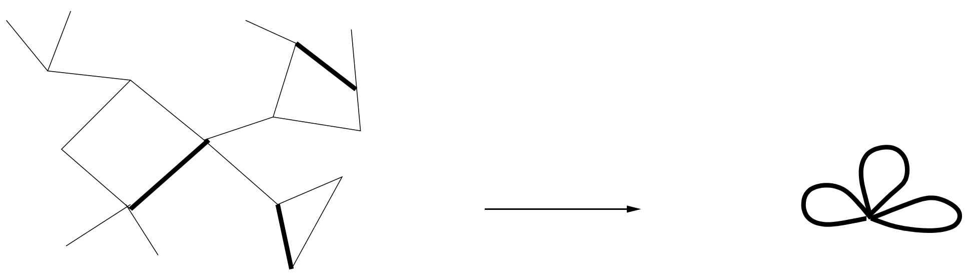
$$H_*(\mathrm{Aut}F_n) \otimes \mathbf{Q} = 0$$

$F_n \longrightarrow \mathbf{Z}^n$ induces a surjection

$$\mathrm{Aut}F_n \longrightarrow \mathrm{GL}_n\mathbf{Z}$$

$G_n :=$ a connected, compact graph with Euler characteristic $1 - n$. Then

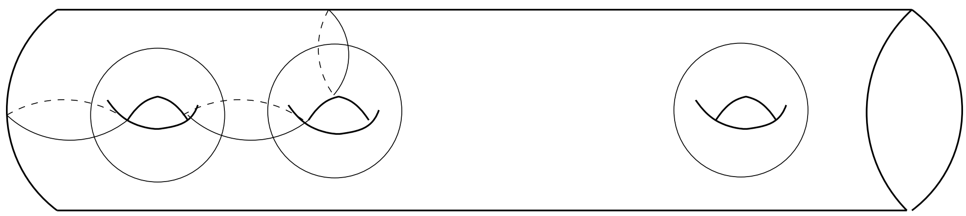
$$\pi_1(G_n) = F_n$$



$\mathrm{HE}(G_n; *) :=$ monoid of homotopy equivalences of G_n that fix a base point

$$\mathrm{Aut}F_n = \pi_0(\mathrm{HE}(G_n; *)) \simeq \mathrm{HE}(G_n; *)$$

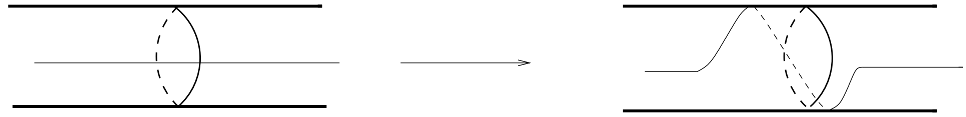
$S_{g,1}$:= oriented surface of genus g with 1 boundary component



$\mathrm{Diff}^+(S_{g,1};\partial)$:= group of diffeomorphisms that preserve the orientation and fix the boundary point wise

Associated *mapping class group*
$\Gamma_{g,1} := \pi_0(\mathrm{Diff}^+(S_{g,1};\partial)) \simeq \mathrm{Diff}^+(S_{g,1};\partial)$

generated by Dehn twists:



The action on

$H_1(S_{g,1}) = \mathbf{Z}^{2g}$ induces a surjection

$$\Gamma_{g,1} \longrightarrow \mathrm{Sp}_{2g}\mathbf{Z}$$

The action on

$\pi_1(S_{g,1}) = F_{2g}$ induces an injection

$$\Gamma_{g,1} \hookrightarrow \mathrm{Aut}F_{2g}$$

which corresponds to the natural map

$$\mathrm{Diff}^{+}(S_{g,1};\partial) \hookrightarrow \mathrm{HE}(S_{g,1};*)$$

Homology

Algebraic description:

$H_0(G) = \mathbf{Z}$ and $H_1(G) = G/[G, G]$

For $n > 4$ and $g > 4$,

$H_1(\Sigma_n) = H_2(\Sigma_n) = \mathbf{Z}/2\mathbf{Z}$
$H_1(\mathrm{Aut}F_n) = H_2(\mathrm{Aut}F_n) = \mathbf{Z}/2\mathbf{Z}$
$H_1(\Gamma_{g,1}) = 0$; $\quad H_2(\Gamma_{g,1}) = \mathbf{Z}$

Topological description:

$H_*(G) := H_*(BG)$ where $BG = EG/G$ is the orbit space of a contractible space EG with a good, free G action.

Example: $G = \mathbf{Z}$, $EG = \mathbf{R}$
$\Longrightarrow BG = \mathbf{R}/\mathbf{Z} \simeq S^1$
$\Longrightarrow H_*(\mathbf{Z}) = H_*(S^1)$

Homology stability

Let $\{G_n\}_{n>0}$ be a family of nested groups with $G_n \subset G_{n+1}$. Define the *stable group*

$$G_\infty := \lim_{n\to\infty} G_n$$

How is the homology of G_n related to that of G_{n+1} and G_∞?

$H_*(\Sigma_n) \simeq H_*(\Sigma_{n+1})$ for $* < (n+1)/2$

$H_*(\Gamma_{g,1}) \simeq H_*(\Gamma_{g+1,1})$ for $* < 2g/3$

$H_*(\mathrm{Aut}F_n) \simeq H_*(\mathrm{Aut}F_{n+1})$ for $* < (n-1)/2$

These are difficult theorems due to Nakaoka, Harer (Ivanov, Boldsen, Randal-Williams), and Hatcher-Vogtmann-Wahl.

Strategy: determine the homology of G_∞.

Miller/Morita:

$$H^*(B\Gamma_\infty) \otimes \mathbf{Q} \supset \mathbf{Q}[\kappa_i], \quad \deg(\kappa_i) = 2i$$

Mumford conj./Madsen-Weiss th.:

$$H^*(B\Gamma_\infty) \otimes \mathbf{Q} = \mathbf{Q}[\kappa_i], \quad \deg(\kappa_i) = 2i$$

Hatcher:

$$H^*(\mathrm{Aut}F_\infty) \supset H^*(\Sigma_\infty)$$

Hatcher-Vogtmann conj./Galatius th.:

$$H^*(\mathrm{Aut}F_\infty) = H^*(\Sigma_\infty)$$

and

$$H^*(\mathrm{Aut}F_\infty) \otimes \mathbf{Q} = \mathbf{Q}$$

Product maps

$$G_n \times G_m \longrightarrow G_{n+m}$$

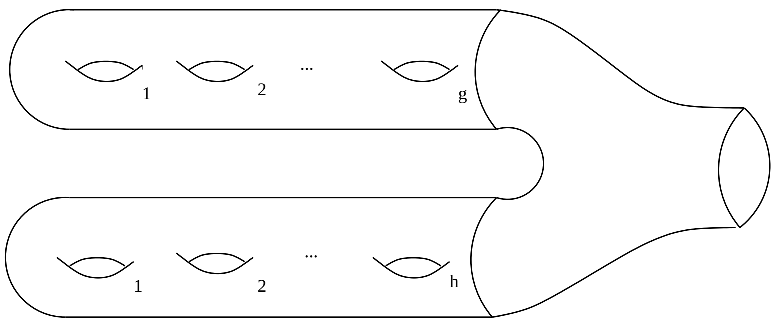
induce a monoid structure on

$$M := \coprod_{n \geq 0} BG_n$$

$\Sigma_n \times \Sigma_m \longrightarrow \Sigma_{n+m}$

$\mathrm{Aut}F_n \times \mathrm{Aut}F_m \longrightarrow \mathrm{Aut}F_{n+m}$

$\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$



all three products are commutative upto conjugation by an element in G_{n+m}
$\Longrightarrow$ products are commutative in H_*

Group completion

Algebraic:

$M \longrightarrow \mathcal{G}(M)$, the Grothendieck group of M

Example: $\mathbf{N} \longrightarrow \mathcal{G}(\mathbf{N}) = \mathbf{Z}$

Homotopy theoretic:

group $\sim$ *loop space*: $\Omega Y = \mathrm{m}ap_*(S^1, Y)$

$$M \longrightarrow \Omega BM$$

Examples: • M a group $\Longrightarrow \Omega BM \simeq M$
• M discrete $\Longrightarrow \Omega BM \simeq \mathcal{G}(M)$
• M arbitrary $\Longrightarrow \pi_0(\Omega BM) = \mathcal{G}(\pi_0(M))$
$\Longrightarrow$ for $M = \coprod_{n \geq 0} M_n$

$$\Omega BM = \mathbf{Z} \times \Omega_0 BM.$$

Group Completion Theorem:
Let $M = \bigsqcup_{n \geq 0} M_n$ be a topological monoid such that the multiplication on $H_*(M)$ is commutative. Then

$$H_*(\Omega_0 BM) = \lim_{n \to \infty} H_*(M_n) = H_*(M_\infty)$$

Barratt-Priddy-Quillen 1972

$$\Omega B(\bigsqcup_{n \geq 0} B\Sigma_n) \simeq \lim_{n \to \infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$$

Madsen-Weiss 2007

$$\Omega B(\bigsqcup_{g \geq 0} B\Gamma_{g,1}) \simeq \Omega^\infty \mathbf{MTSO}(2)$$

Galatius 201?

$$\Omega B(\bigsqcup_{n \geq 0} B\mathrm{Aut}F_n) \simeq \lim_{n \to \infty} \mathrm{map}_*(S^n, S^n) =: \Omega^\infty S^\infty$$

Moduli space

G acts on a compact geometric object W
$\Longrightarrow BG = \mathcal{M}^{top}(W)$, a *topological moduli space*:

(i) $W' \in \mathcal{M}^{top}(W)$ with $W' \simeq W$
(ii) W-bundle $E \to X \quad \longleftrightarrow \quad f_E : X \to \mathcal{M}^{top}(W)$

W a manifold:

$\mathcal{M}^{top}(W) :=$ space of embedded $W' \subset \mathbf{R}^N$ for some $N >> 0$

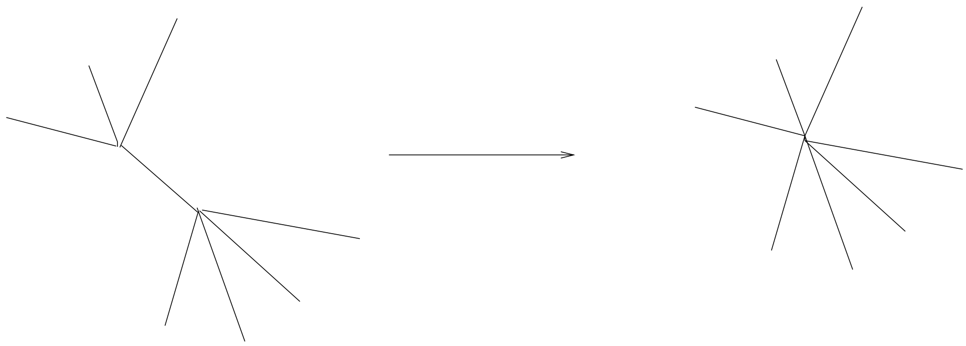
... for $W = n$ points, this is the configuration space $\mathcal{C}_n(\mathbf{R}^\infty)$ of n unordered points in $\mathbf{R}^N$, $N >> 0$

... for $W = S_{g,1}$, this is homotopic to the corresponding moduli space of Riemann surfaces

$$\mathcal{M}_{g,1} \simeq \mathcal{M}^{top}(S_{g,1}) \simeq B\Gamma_{g,1}$$

W a graph:

$\mathcal{M}^{top}(W)$:= space of embedded finite graphs W' (with base point) in $\mathbf{R}^N$, $N >> 0$, topologized such that edge collapses are continuous:



... for $W = G_n$,

$$\mathcal{M}^{top}(G_n) = B\mathrm{Aut}F_n$$

Galatius uses Culler-Vogtmann's Outer space and ideas of Igusa

W-characteristic classes of E:

W-bundle $E \to X \quad \longleftrightarrow \quad f_E : X \to \mathcal{M}^{top}(W)$

Example: $E = X \times W \quad \longleftrightarrow \quad f_E \simeq \mathrm{const}_W$

$$c \in H^*(\mathcal{M}^{top}(W)) \longrightarrow f_E^*(c) \in H^*(X)$$

Example: $W = \mathbf{C}^n, \quad G = \mathrm{GL}_n\mathbf{C}$
$\Longrightarrow \mathcal{M}^{top}(\mathbf{C}^n) = \mathrm{Gr}^{\mathbf{C}}(n)$ $\mathbf{C}$-Grassmannian

$$H^*(\mathrm{Gr}^{\mathbf{C}}(n)) = \mathbf{Z}[c_i],\ \deg(c_i) = 2i$$

Example: $W = S^1$, $G = \mathrm{Diff}^+(S^1) \simeq S^1$
$\Longrightarrow \mathcal{M}^{top}(S^1) = \mathrm{Gr}^{\mathbf{C}}(1) = \mathbf{C}P^\infty$

$$H^*(\mathbf{C}P^\infty) = \mathbf{Z}[c_1]$$

What are the characteristic classes for more general W?

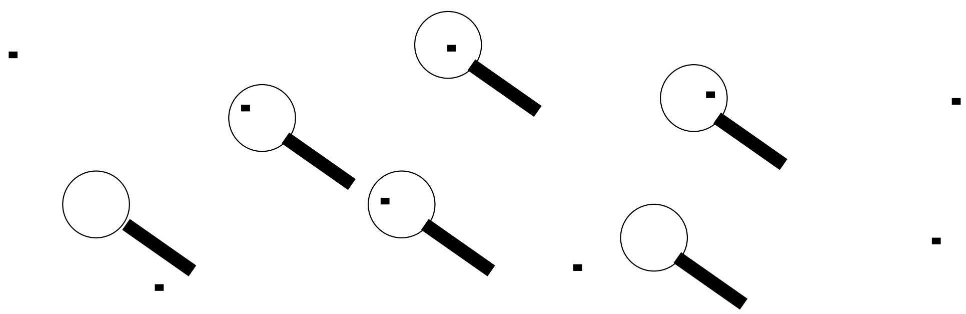
Madsen-Weiss: stable classes for surfaces
Galatius: stable classes for graphs

Scanning map

$$\alpha : \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N(L^N) := \mathsf{map}_*((\mathbf{R}^N)^c, L^N)$$

$\mathcal{M}^{top}(W)^N :=$ space of W's in $\mathbf{R}^N$
$L^N :=$ spaces of local data

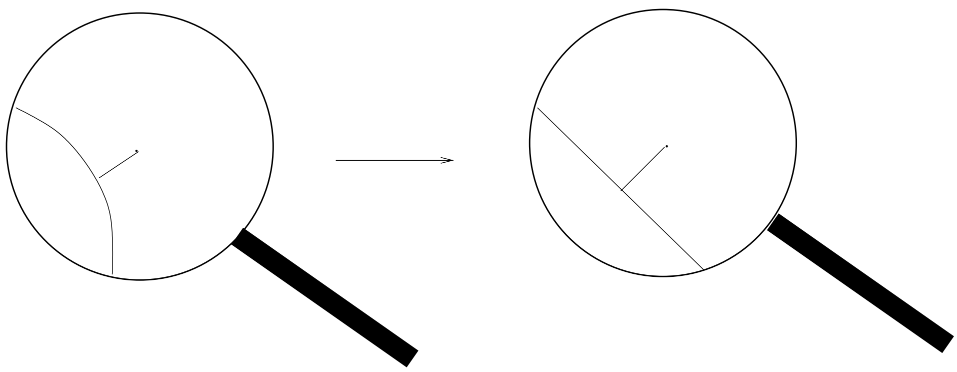
$W = n$ points:



$$L^N = (\mathbf{R}^N)^c = S^N$$

$$\alpha : \coprod_{n \geq 0} \mathcal{C}_n(\mathbf{R}^N) \longrightarrow \Omega^N S^N$$

$W =$ a closed, oriented d dimensional manifold:



$$L^N \simeq (\gamma_{d,N}^{\perp})^c = Th(\gamma_{d,N}^{\perp})$$

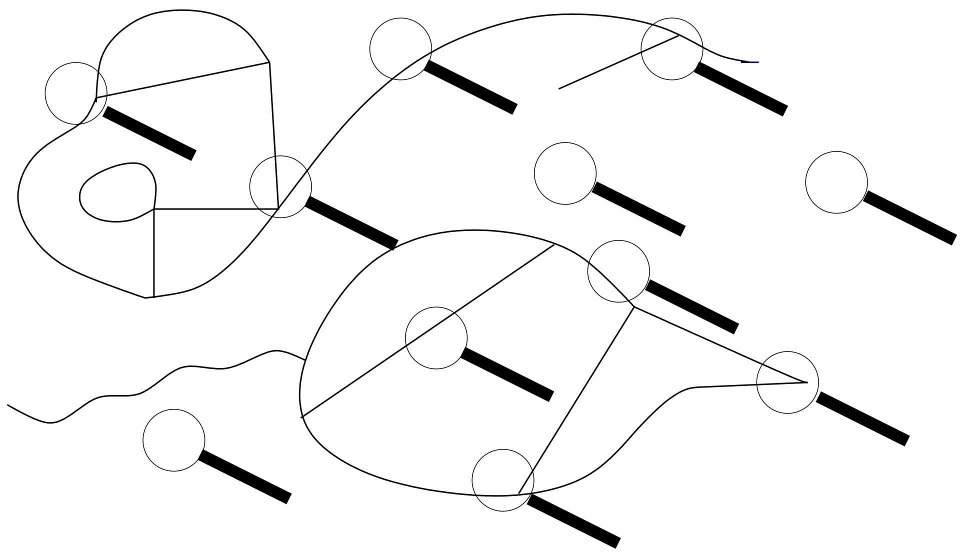
$\gamma_{d,N} \to \mathrm{Gr}(d,N)$:= canonical bundle over the Grassmannian of real, oriented d-planes in $\mathbf{R}^N$

$$\alpha : \bigsqcup_W \mathcal{M}^{top}(W)^N \longrightarrow \Omega^N Th(\gamma_{d,N}^{\perp})$$

$$\Omega^\infty \mathbf{MTSO(d)} := \lim_{N\to\infty} \Omega^N Th(\gamma_{d,N}^{\perp})$$

$$H^*(\Omega^\infty \mathbf{MTSO(d)}) \otimes \mathbf{Q} = \mathbf{Q}[H^{*>d}(\mathrm{Gr}(d))[-d] \otimes \mathbf{Q}]$$

$W =$ a finite graph:



$$\alpha : \coprod_{n \geq 0} \mathcal{M}^{top}(G_n)^N \longrightarrow \Omega^N L^N$$

$$L^N \sim S^N \quad (2N - c) - \text{connected}$$

$$\lim_{N\to\infty} \Omega^N L^N \sim \lim_{N\to\infty} \Omega^N S^N = \Omega^\infty S^\infty$$

Outline of proof (for graphs):

(the argument also works for d-manifolds
$\Longrightarrow$ main result of Galatius-Madsen-T-Weiss)

$$\Phi^{N,N} \supset \ldots \supset \Phi^{N,1} \supset \Phi^{N,0} \simeq \coprod_{n \geq 0} \mathcal{M}(G_n)^N$$

$\Phi^{N,i} :=$ all graphs in $\mathbf{R}^i \times (0,1)^{N-i}$,
possibly not connected or infinite;
the empty graph is the basepoint

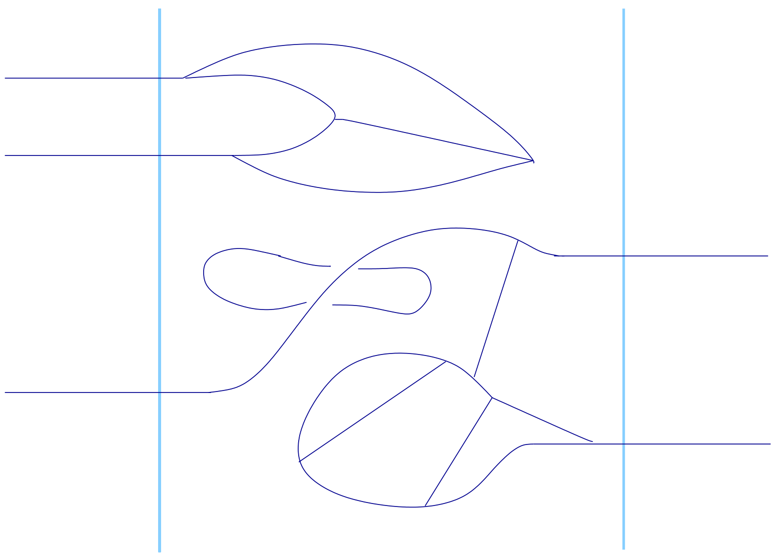
Step 1: $\Phi^{N,N} \simeq L^N$

Topology of $\Phi^{N,N}$ allows to push radially away from origin.

Step 2: $\Phi^{N,k} \simeq \Omega\Phi^{N,k+1}$ for $k > 0$

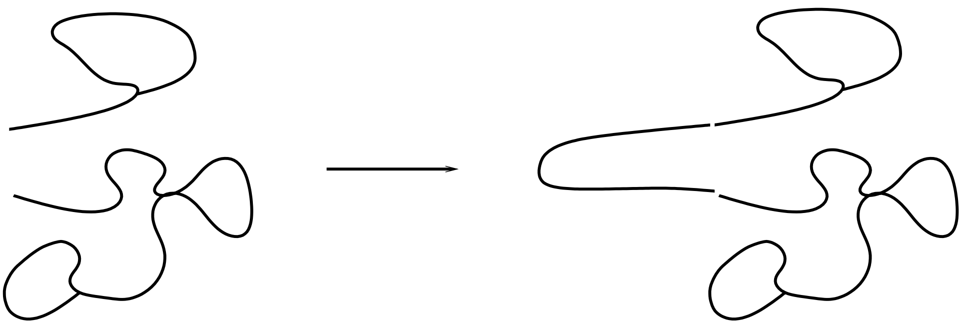
At time t, $W' \in \Phi^{N,k}$ is translated by t in the direction of e_{k+1}.

Step 3: $\lim_{N\to\infty} \Phi^{N,1} \simeq B(\mathsf{Cob})$ where Cob is the cobordism category of embedded graphs



Step 4: $B(\mathsf{Cob}) \simeq B(\coprod_{n\geq 0} \mathcal{M}(G_n))$

Needs surgery argument.



Generalizations

W_n	H_*-stab.	$\Omega B(\sqcup_n \mathcal{M}(W_n))$
n pts	(n+1)/2	$\Omega^\infty S^\infty$
$S_{g,1}$	(2g)/3	$\Omega^\infty \mathbf{MTSO(2)}$
$N_{g,1}$	(n-3)/3	$\Omega^\infty \mathbf{MTO(2)}$
$\#_n(S^k \times S^k)_1$	?	$\Omega^\infty \mathbf{MTSO(2k)}^{\langle k \rangle}$
G_n	(n-1)/2	$\Omega^\infty S^\infty$
$\#_g(S^1 \times D^2)_1$	(g-1)/2	$\Omega^\infty S^\infty BSO(3)_+$
$\#_n(S^1 \times S^2)_1$	?	$\Omega^\infty S^\infty BSO(4)_+$

Wahl: $N_{g,1}$ non-orientable surface of genus g

Galatius & Randal-Williams:
$\#_n(S^k \times S^k)_1$ a $(k-1)$-connected manifold

Hatcher:
$\#_g(S^1 \times D^2)_1$ a handlebody of genus g
$\#_n(S^1 \times S^2)_1$ a 3-manifold with one boundary sphere

Questions:

• Does homology stability also for n connected sums of

$$S^k \times S^k \quad \text{or} \quad S^1 \times S^2?$$

• Are there other manifolds for which similar results can be proved?

• Can we generalise from graphs to higher dimensional simplicial complexes?

Scanning: for any oriented d-dimensional W

$$\alpha : \mathcal{M}^{top}(W) \to \Omega^\infty \mathbf{MTSO\ (d)}$$

Ebert: When d is even,
for any $c \in H^*(\Omega^\infty \mathbf{MTSO\ (d)}\ ) \otimes \mathbf{Q}$
there is a W with $\alpha^*(c) \neq 0 \in H^*(W) \otimes \mathbf{Q}$.

When d is odd, this no longer holds.

• What is the natural space $X(d)$ for d odd whose cohomology contains all rational stable characteristic classes for d dimensional manifolds (and no more)?

$X(1) = \Omega^\infty S^\infty (BSO(2))_+$

$X(3) = \Omega^\infty S^\infty (BSO(4))_+$?