CAT 2017-18

Problem Sheet (1)

Simplicial complexes and fundamentals of homology theory.

*It is not expected that all problems are solved. Depending on background, some of these are either too easy or too hard. What is expected is that students engage with the problems, in particular understand the statements and if possible have a go at them. A selection should be worked out in detail and handed in. The more testing ones are marked with an asterisk *. As a guide, please attempt all of problems (1)-(5), (14), and (16) - (19).*

(1) Let $K_\bullet$ be a simplicial complex. Assume (σ, τ) is a pair of its simplices such that σ is the only codimension one coface of τ. Prove that $L_\bullet := K_\bullet \setminus \{\sigma, \tau\}$ is a simplicial complex of the same homotopy type.

(2) * Consider the circle S^1 and cover it by three open arcs such that any pair intersects in a subarc. Construct its Čech complex and verify the Nerve Theorem in this context directly. Give an example of a cover by open arcs of the circle such that the corresponding Čech complex does not have the homotopy type of the circle.

(3) Let $S \subset \mathbb{R}^d$ be finite and consider real weights w_s for each $s \in S$. The weighted Voronoi cells are defined by

$$V(s) := \{x \in \mathbb{R}^d \parallel ||x - s|| - w_s \leq ||x - v|| - w_v,\, v \in S\}$$

Let $s_- = (-1, 0), s_+ = (1, 0) \in \mathbb{R}^2$ and sketch the bisectors for
(i) $w_{s_-} = 0, w_{s_+} = 0$
(ii) $w_{s_-} = 0, w_{s_+} = 1$
(iii) $w_{s_-} = 0, w_{s_+} = 2$
(iv) $w_{s_-} = 0, w_{s_+} = -1$.

(4) Show that $\mathrm{Alpha}_\epsilon(S) \subset \check{\mathrm{C}}\mathrm{ech}_\epsilon(S)$ for all finite $S \subset \mathbb{R}^d$ and any $\epsilon > 0$. Show that $\mathrm{Alpha}_\epsilon(S)$ has the homotopy type of $\bigcup_{s \in S} B_\epsilon(s) \cap V(s)$.

(5) * Find out about 'witness complexes'!

(6) Let $f : C_* \to D_*$ be a map of chain complexes. Prove that the kernel, image and cokernel of f are chain complexes in a natural way.

(7) Prove that for a finite simplicial complex K, the Euler characteristic can be computed in terms of its homology with field coefficients in a field $\mathbb{F}$ as

$$\chi(K) = \Sigma_{i \geq 0} (-1)^i \dim H_i(K; \mathbb{F})$$

(8) A simplicial complex is connected if there is a path of 1-simplices between any two vertices. Show that such a complex has 0th homology of rank 1.

(9) Prove that a map of simplicial complexes induces a map of homology groups.

(10) Compute the homology of a sphere modelled as the boundary of a standard 3-simplex.

(11) Compute the relative homology $H_*(D^n, \partial D^n)$ and show that it is isomorphic to the reduced homology of the sphere $\tilde{H}_*(S^n)$.

(12) Find a simplicial complex such that its geometric realisation is a cylinder $S^1 \times [0,1]$. Find its homology.

(13) Do the same as in the above question for the Klein bottle.

(14) * Compute the homology of the image of the Klein bottle immersed (in the standard way) in $\mathbb{R}^3$. Compare it with the homology of the Klein bottle.

(15) Identify a vertex v from X with a vertex w from Y to form a new simplicial complex $X \vee Y$. Prove $H_p(X \vee Y) = H_p(X) \oplus H_p(Y)$ for all $p > 0$.

(16) Let K be a simplicial complex the realisation of which is the real projective plane. Consider the filtration by skeleton $K^0 \subset K^1 \subset K^2 = K$. Compute the persistent homology groups of this filtration and draw the corresponding bar codes. Discuss briefly how the persistent homology groups depend on your choice of K.

(17) More generally, let K be a simplicial complex filtered by skeleta. Describe the associated persistent homology.

(18) Prove that (i,j) persistent p-th homology of a filtered complex K can be computed as

$$H_p^{i,j}(K) = \frac{Z_p^i}{B_p^j \cap Z_p^i}.$$

(See section 3.4 of the lecture notes.)

(19) Consider the standard 2-simplex with vertices $\{0,1,2\}$ as a filtered complex with filtrations:

K(0) = { {0 } , { 1 } }
K(1) = { {0 } , { 1 } , { 2 } { 0, 1 } }
K(2) = { {0 } , { 1 } , { 2 } { 0, 1 } {1,2} }
K(3) = { {0 } , { 1 } , { 2 } { 0, 1 } {1,2} { 0,2 }}
K(4)= $\triangle^2$
Use the standard algorithm to compute the persistent homology.