CAT 2017-18

Problem Sheet (2)

Discrete Morse theory, Categorification, stability and spectral sequences

(1) Model the torus as a simplicial complex given by 18 triangles arranged in 9 squares forming a square with opposite sides identified. Find a discrete Morse function and compute the homology of the corresponding Morse complex.

(2) Let $X(k,n)$ be the k-skeleton of the n-simplex on vertices $v_0, v_1, \ldots, v_n$. The task is to determine its homotopy type of $X(k,n)$ using discrete Morse theory.

(i) Sketch $X(k,n)$ when $n = 3$ and $k = 0, 1, 2$. Show that each is homotopic to a wedge of spheres of dimension k.

(ii) Inductively construct a Morse matching ω as follows: leaving out v_0, match v_i for $i > 0$ with $[v_0, v_i]$; consider the remaining 1-simplices and match them with suitable 2-simplices; continue until all but some k-simplices are matched.

(iii) Use the above to completely determine the homotopy type of $X(k,n)$ for all k, n.

(3) Prove that the Dunce hat is homtopic but not simple homotopic to a point.

(4) Let $X = X_N \supset \cdots \supset X_0 = \emptyset$ be a finite filtered complex and $f : X \to \mathbb{R}$ be a discrete Morse function compatible with the filtration, i.e. $f(\sigma) < f(\tau)$ for all $\sigma \in X_s$ and $\tau \in X \setminus X_s$.

Prove that the associated Morse complex has a natural filtration such that the associated persistent homology is the same as that of the original filtration on X.

(5) For two $(\mathbb{R}, \leq)$-diagrams F and G in $\mathcal{D}$ and any functor $H : \mathcal{D} \to \mathcal{E}$ show that the interleaving distance satisfies:

$$d(HF, HG) \leq d(F, G).$$

(6) For the interval $I \subset \mathbb{R}$ let χ_I be the functor from $(\mathbb{R}, \leq)$ to the category of finite vector spaces, that is the characteristic diagram defined in lectures.

(i) Show that if χ_I has no critical value than it is constant. What is I?

(ii) Let I and J be two intervals. Find the interleaving distance from χ_I to χ_J. Treat the cases when I or J are empty, or when one of them is infinite separately.

(7) For partial matchings θ' between A and B, and θ'' between B and C, show that the penalty of the induced partial matching $\theta = \theta'' \circ \theta'$ is less or equal to the sum of the penalties for θ' and θ'':

$$P(\theta) \leq P(\theta'') + P(\theta').$$

(8) Given an exact sequence of vector spaces

$$V_{-2} \xrightarrow{i} V_{-1} \xrightarrow{p} E \xrightarrow{\partial} V_1 \xrightarrow{i'} V_2$$

prove that

$$\dim(E) = [\dim(V_{-1}) - \dim(\mathrm{Im}(i))] + [\dim(V_1) - \dim(\mathrm{Im}(i'))].$$

(9) Let (E, D, i, p, ∂) be an exact couple. Show that the derived couple is again exact.

(10) A pair of simplicial complexes (X, A) may be considered as a filtered simplicial complex. Compute the spectral sequence $\{E^{(r)}, d^{(r)}\}_{r>0}$.

Explicitly compute the spectral sequence of the pair $(\mathbb{R}P^2, \mathbb{R}P^1)$.