Computational Algebraic Topology

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Computational Algebraic Topology

for

Topological Data Analysis

– A Primer –

1. Simplicial complexes

1.1. Definitions

An **(abstract)simplicial complex** $K_\bullet$ is a collection of non-empty subsets of a set of vertices K_0 that satisfy the condition

$$\alpha \in K_\bullet,\, \beta \subset \alpha \Longrightarrow \beta \in K_\bullet.$$

$\beta \subset \alpha$ is called a **face** of α. Without loss of generality, we will always assume that the singleton set for each element of K_0 is in $K_\bullet$.

The **dimension** $K_\bullet$ is one less than the cardinality of the largest element of $K_\bullet$:

$$\mathrm{dim}(K_\bullet) := \max_{\alpha \in K} \{\#\alpha - 1\}.$$

The subset $K_p \subset K_\bullet$ of sets of size $p+1$ are the **p-simplices**. The union $K_\bullet^p$ of all subsets of size $p+1$ or less is a subcomplex of $K_\bullet$ called the **p-skeleton**.

We will mainly be interested in finite abstract simplicial complexes. For such we can define the **Euler characteristic** of $K_\bullet$ as

$$\chi(K_\bullet) := \Sigma_p (-1)^p \# K_p.$$

Example: sphere $\simeq$ boundary of a tetrahedra

$K_0 = \{0, 1, 2, 3\}$,
$K_1 = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$
$K_2 = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}$

$\chi(K_\bullet) = 4 - 6 + 4 = 2$
$\dim(K_\bullet) = 2$

1.2. Geometric realization

The **simplex spanned** by $\{v_0, \ldots, v_p\} \subset \mathbb{R}^n$ is the set of points

$$\{\Sigma_{i=0}^p t_i v_i \,|\, 0 \le t_i \in \mathbb{R},\ \Sigma_i t_i = 1\}.$$

$\{v_0, \ldots, v_p\}$ are **affinely independent** if they span a p-simplex, or equivalently, if
$\{v_1 - v_0, \ldots, v_n - v_0\}$ are linearly independent.

Assume we have an assignment of the finite vertex set K_0 to points in $\mathbb{R}^n$ such that the images are affinely independent. The **geometric realization** of $K_\bullet$ (determined by the assignment $K_0 \to \mathbb{R}^n$) is the topological space

$$|K_\bullet| := \bigcup_{\sigma \in K_\bullet} |\sigma|.$$

Lemma: Let $K_\bullet$ be a simplicial complex with N vertices. Then $K_\bullet$ has a geometric realization in $\mathbb{R}^N$.

Proof: Map the elements in K_0 to the standard basis elements in $\mathbb{R}^N$. $|K_\bullet|$ is the union of all simplices spanned by the images of the elements in $K_\bullet$.

A **simplicial map** $f : K_\bullet \to L_\bullet$ is a map of vertices $f : K_0 \to L_0$ such that $f(\sigma) \in L_\bullet$ for all $\sigma \in K_\bullet$.

It induces a continuous map $|f|$ on realizations by setting

$$|f|(\Sigma_{i=0}^{p} t_i v_i) = \Sigma_{i=0}^{p} t_i f(v_i),$$

where v_i is the realization of the i-th vertex of a p-simplex in $K_\bullet$ and $f(v_i)$ is the realization of the image of this vertex in $|L_\bullet|$.

Note: Any two geometric realizations are canonically homeomorphic via piecewise linear maps.

1.3. Subdivision and approximation

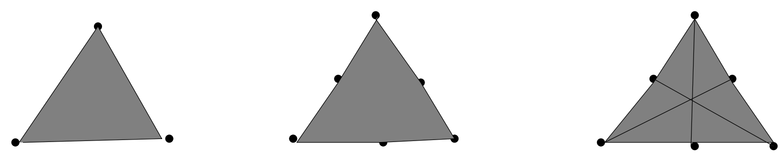
The **barycentric subdivision** of a simplicial set $K_\bullet$ is the simplicial set with p-simplices formed by 'flags' of length $p+1$ of strict inclusion:

$$Sd(K_\bullet)_p = \{\{\sigma_0 \subset \cdots \subset \sigma_p\} | \, \sigma_i \in K_\bullet, p \geq 1\}.$$

Thus vertices of $Sd(K_\bullet)$ are the simplices of $K_\bullet$.

The **barycenter** of the realization of a p-simplex is the point corresponding to $t_0 = \cdots = t_p = 1/(p+1)$.

Given a realization of $K_\bullet$, we may construct a realization of the barycentric subdivision $Sd(K_\bullet)$ by induction: at the p-th stage, for every p-simplex σ i $|k_\bullet|$, add the barycenter of σ and all the $(p-1)$-simplicies (and their faces) containing the barycenter and any other vertices and barycenters of any face of σ.



Simplicial Approximation Theorem: For any continuous map $g : |K_\bullet| \to |L_\bullet|$ there is an n and a simplicial map $f : Sd^n(K_\bullet) \to L_\bullet$ such that $|f|$ and g are homotopic.

Here Sd^n denotes the n-times repeated application of Sd.

Proof: [Hatcher 2002, p.177]

Recall: $f, g : X \to Y$ are **homotopic** if there exists $H : X \times [0, 1] \to Y$ with $H(-, 0) = f$ and $H(-, 1) = g$. For example, if Y is the n-disk then every map is homotopic to the constant map to 0.

Two spaces are **homotopy equivalent** if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity on Y and X.

A space is **contractible** if it is homotopic to a point.

Example: The n-dimensional ball B^n is homotopic to a point and hence contractible. In Euclidean space $\mathbb{R}^n$ the intersection of any number of round balls of fixed radius ϵ is contractible.

1.4 Čech complex

Let X be a nice * topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover, i.e. a collection of subset of X such that the union equals X. We define the p-simplices of the **Čech complex** of $\mathcal{U}$ (also called 'nerve complex') to be

Proof: See [Hatcher p.459, 2002].

*For example, paracompact; all realizationa of finite simplicial complexes are paracompact.

The Nerve Theorem is a fundamental fact for Topological Data Aanalysis.

In data analysis one often starts with a point cloud set S contained in some Euclidean space. Sometimes we view this set S as the sample set from a space X whose topological characteristics we are then interested to recover. Other times we simply associate a space to S whose topology will then tell us something about the set S itself.

Main example: Let $S \subset \mathbb{R}^d$ be finite and $\mathcal{U} = \{B_\epsilon(s)\}_{s \in S}$ the collection of open ϵ-balls around the points of S. Then the realization of the Čech complex $\check{C}ech_\epsilon(S)$ has the homotopy type of the union of these balls.

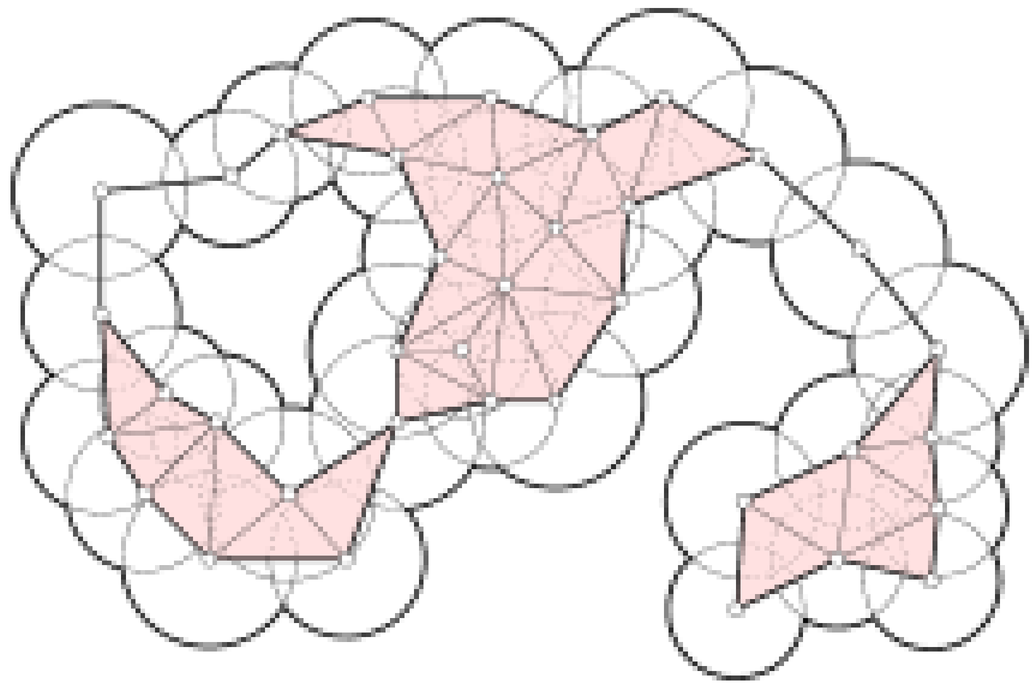


Figure: The Čech complex for a point cloud data set S and a collection of balls around its elements.

1.5 Vietoris-Rips complex

The Čech complex is 'expensive' as all intersection have to be computed. Instead one considers an approximation.

Let $S \subset \mathbb{R}^d$ (or some metric space X) be finite and consider the **ϵ-Vietoris-Rips complex** with p-simplices

$$VR_\epsilon(S)_p := \{\sigma \subset S \,|\; \#\sigma = p+1 \text{ and } \mathrm{diam}(\sigma) < 2\epsilon\}$$

We 'save' computational time as only distances between all pairs of points have to be computed.
Note:

$$\check{C}ech_\epsilon(S) \subset VR_\epsilon(S) \subset \check{C}ech_{\sqrt{2}\epsilon}(S).$$

The first inclusion is immediate and true for any metric space. To show the second is an inclusion one uses the fact that S is a subset of Euclidean space. Exercise!

1.6 Voronoi diagram, Delaunay and alpha complex

We note here that it is very easy to come up with examples where the dimension of the Čech or VR-complex exceeds the dimension of the background spaces (e.g. the dimension d of the ambient Euclidean space), and ultimately neither of these complexes gives an efficient representation of the underlying spaces.

Let $S \subset \mathbb{R}^d$ be finite and consider for $s \in S$ the set

$$V(s) := \{x \in \mathbb{R}^d \mid \ ||x - s|| \leq ||x - v|| \text{ for all } v \in S\}$$

Each $V(s)$ is a closed polyhedra (possibly with one vertex at infinity) and their union is all of $\mathbb{R}^d$. The **Voronoi diagram** of S is the collection of Voronoi cells of its points.

The **Delaunay complex** of S is the Čech complex associated to the collection $\mathcal{U} = \{V(s)\}_{s\in S}$ of Voronoi cells. Hence its p-simplices are

$$\text{Delaunay}(S)_p := \{\sigma \subset S \mid \#\sigma = p+1 \text{ and } \bigcap_{s\in\sigma} V(s) \neq \emptyset\}$$

Exercise*: If the points of S are in general position, i.e. no $d+2$ points lie on the same sphere, then the Delaunay complex has dimension at most d.

The **ϵ-alpha complex** of S is the Čech complex associated to the collection $\mathcal{U} = \{B_\epsilon(s) \cap V(s)\}_{s\in S}$. Its p-simplices are the sets

$$\text{Alpha}_\epsilon(S) := \{\sigma \subset S \mid \#\sigma = p+1 \text{ and } \bigcap_{s\in\sigma} B_\epsilon(s) \cap V(s) \neq \emptyset\}$$

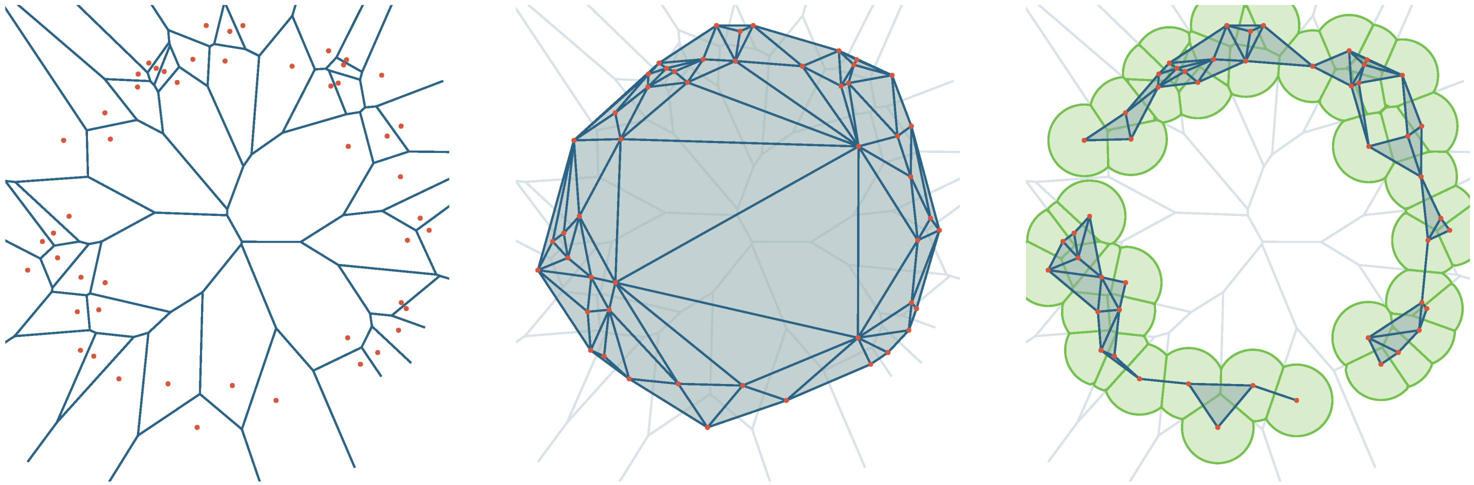


Figure: A Voronoi diagram with its Delaunay triangulation superimposed and an associated alpha complex for some fixed radius.

Note: The Delaunay complex by itself does not model the underlying data set.

2. Homology

2.1. Basic definitions

A **chain complex** (C, d) over a field $\mathbb{F}$ is a sequence of $\mathbb{F}$-vector spaces and maps

$$\ldots C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots C_0 \longrightarrow 0$$

such the $d_n \circ d_{n+1} = 0$.

n-th *boundary map*: d_n
n-*chains*: C_n
n-*cycles*: $Z_n := Ker(d_n)$
n-*boundaries*: $B_n := Im(d_{n+1})$
n-th *homology group* of C:

$$H_n(C) := Z_n/B_n = Ker(d_n)/Im(d_{n+1})$$

A **map of chain complexes** $F : (C, d) \to (C', d')$ is a collection of $\mathbb{F}$-linear maps $F_n : C_n \to C'_n$ such that $F_{n-1} \circ d_n = d'_n \circ F_n$.

Exercise: A map of chain complexes induces a map of homology groups.

A **chain homotopy** between two chain maps $F, G : (C, d) \to (C', d')$ is a collection of linear maps $h_p : C_p \to C'_{p+1}$ such that

$$h_{p-1} \circ d_p + d'_{p+1} \circ h_p = F_p - G_p.$$

Exercise: If F and G are chain homotopic then they induce the same map on homology.

2.2. $\mathbb{F}_2$-homology of a simplicial complex

Let $K_\bullet$ be a simplicial complex and let

$$C_n(K_\bullet) := \mathbb{F}_2[K_n]$$

be the $\mathbb{F}_2$-vector space with basis K_n. Define on basis elements

$$d_n(\alpha) := \Sigma_{\beta \subset \alpha, \, \#\beta = n} \, \beta$$

and extend d_n linearly to all chains.

Fundamental Lemma: $d_n \circ d_{n+1} = 0$.

Proof: It is enough to check this on basis elements. Let $\alpha = \{v_0, \ldots, v_{n+1}\}$ be an $(n+1)$-simplex. Then

$$d_n(d_{n+1}(\alpha)) = d_n(\Sigma_{i=0}^{n+1}\{v_0, \ldots, \widehat{v}_i, \ldots v_{n+1}\})$$

$$= 2\,\Sigma_{j \neq i}\{v_0, \ldots, \widehat{v}_j, \ldots, \widehat{v}_i, \ldots v_{n+1}\} = 0.$$

Caveat: When the characteristic of $\mathbb{F}$ is not 2, one has to take in the definition of the boundary map d_n the alternating sum over the faces. Otherwise the Fundamental Lemma does not hold.

Geometric interpretation: The boundary of a boundary is always empty.

The n-th **homology group** of a simplicial complex $K_\bullet$ is

$$H_n(K_\bullet)(= H_n(K_\bullet, \mathbb{F}_2)) := H_n(C(K_\bullet))$$

Exercise: A map $f : K_\bullet \to L_\bullet$ of simplicial complexes induces a map on chain complexes and hence on homology

$$H_n(f) : H_n(K_\bullet) \to H_n(L_\bullet).$$

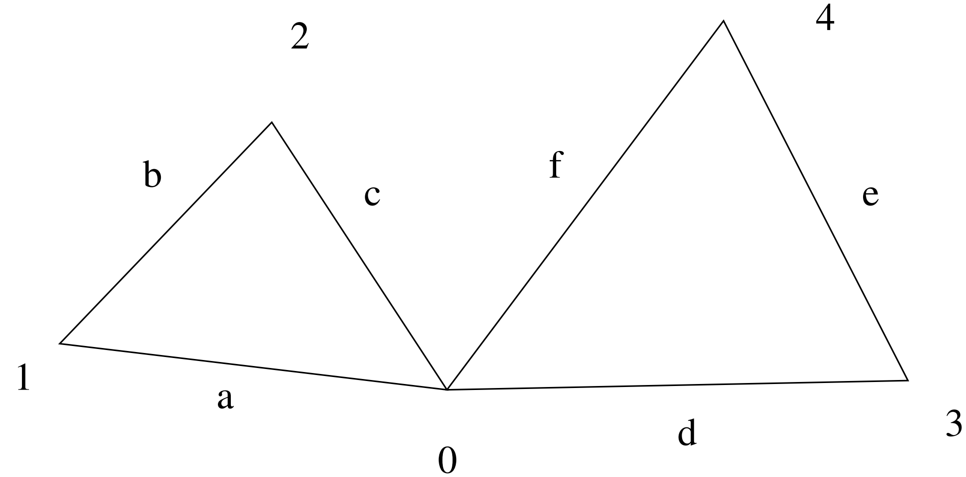
Example: $K_\bullet$ a one-point union of the boundary of two triangles with vertices $\{v_0, v_1, v_2\}$ and $\{v_0, v_3, v_4\}$. Then:

$C_0(K_\bullet) =< v_0, v_1, v_2, v_3, v_4 >$,

$C_1(K_\bullet) =< \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_0, v_3\}, \{v_3, v_4\} \{v_0, v_4\} >$

and d_1 has matrix

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{matrix}$$


2
4
b
c
f
e
1
a
0
d
3

Using column and row operations this can be transformed to Smith normal form

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Hence by the rank-nullity theorem $H_0(K_\bullet) = \mathbb{F}_2$ and $H_1(K_\bullet) = \mathbb{F}_2 \oplus \mathbb{F}_2$. All other homology groups are zero.

Exercise: Compute the map in homology induced by $f : K_\bullet \to K_\bullet$ which takes the vertices v_0, v_1, v_2, v_3, v_4 to v_0, v_1, v_2, v_0, v_0.

The p-th **Betti number** of $K_\bullet$ is

$$b_p := \dim H_p(K_\bullet).$$

Euler-Poincaré formula: For finite simplicial complexes

$$\chi(K_\bullet) = \Sigma_{p\geq 0}\,(-1)^p\, b_p$$

Proof: This follows form the rank-nullity formula.

Exercise: $K_\bullet$ is an n-simplex (i.e. n-ball B^n). Then $H_p(K_\bullet) = \mathbb{F}_2$ when $p = 0$ and is zero otherwise.

Exercise: $K_\bullet$ is the boundary of an n-simplex (i.e. $n-1$ sphere S^{n-1}). Then $H_p(K_\bullet) = \mathbb{F}_2$ when $p = n-1, 0$ and is zero otherwise.

2.3. Homotopy invariance

Theorem If $f, g : K_\bullet \to L_\bullet$ induce homotopic maps $|f|, |g|$ on the realizations then the induced maps on $H_n(K_\bullet) \to H_n(L_\bullet)$ are identical for all $n \geq 0$.

Proof: See [Hatcher p.111 and p.128, 2002]. The idea is that from the homotopy one can construct a chain homotopy.

2.4. Relative homology

Let $A \subset C$ be a sub chain complex of C (i.e. $A_n \subset C_n$ and $d_n(A_n) \subset A_{n-1}$ for all n). Then C/A is the quotient chain complex with n-chains C_n/A_n and boundary map induced by d_n.

Theorem: A short exact sequence of chain complexes $0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$ induces a long exact sequence on homology groups

$$\dots H_{n+1}(C/A) \xrightarrow{\delta} H_n(A) \to H_n(C) \to$$

$$H_n(C/A) \xrightarrow{\delta} H_{n-1}(A) \dots H_0(C/A) \to 0$$

The connecting homomorphims δ takes a cycle in C_n/A_n which is represented by a chain $c \in C_n$ to $d_n(c) \in A_{n-1}$.

Note: As the k-simplex B^k is homotopy equivalent to a point it follows that it has the homology of a point. (Exercise 1 in 2.2.)

Example: n-simplex B^n and its boundary S^{n-1}. Then $C(S^{n-1})$ is a sub-complex of $C(B^n)$, and the long exact sequence yields $H_p(B^n, S^{n-1}) = \mathbb{F}_2$ if $p = n$ and zero otherwise.

2.5. Homology of $Sd(K_\bullet)$

Let $K_\bullet$ be a simplicial complex of dimension d and $Sd(K_\bullet)$ be its barycentric subdivision. On chain complexes we have a map

$$C(K_\bullet) \longrightarrow C(Sd(K_\bullet)), \quad \sigma \mapsto \Sigma_{\mu=\{\sigma_0\subset\cdots\subset\sigma_p=\sigma\}}\, \mu$$

that sends a p-simplex $\sigma \in K_\bullet$ to the sum of all p-simplices $\mu \in Sd(K_\bullet)$ which are part of σ. This is a map of chain complexes as

$$d\sigma = \Sigma_\mu\, d\mu$$

Block Lemma: The induced map on homology induces an isomorphism

$$H(K_\bullet) \xrightarrow{\simeq} H(Sd(K_\bullet)).$$

2.6. Mayer-Vietoris sequence

Theorem: Let $K_\bullet = A_\bullet \cup B_\bullet$ be the union of two simplicial subcomplexes. Then there is a long exact sequence

$$\dots H_n(A_\bullet \cap B_\bullet) \to H_n(A_\bullet) \oplus H_n(B_\bullet) \to H_n(K_\bullet) \xrightarrow{\delta}$$

$$H_{n-1}(A_\bullet \cap B_\bullet) \to \dots \to H_0(K_\bullet) \to 0.$$

The first two maps are induced by the inclusions of complexes with $c \mapsto (c, c)$ and $(a, b) \mapsto a - b$. To define the connecting homomorphism δ, write a cycle $c \in Z_n(K_\bullet)$ as a sum of chains $a \in C_n(A_\bullet)$ and $b \in C_n(B_\bullet)$. Then c is taken to the chain $d_n(a) = d_n(b)$ in $A_\bullet \cap B_\bullet$.

Proof: Apply Theorem 2.3 to the exact sequence of chain complexes induced by the given maps

$$0 \to C(A_\bullet \cap B_\bullet) \to C(A_\bullet) \oplus C(B_\bullet) \to C(K_\bullet) \to 0$$

2.7. Cohomology

Consider a finite dimensional k-vector spaces V and its dual

$$V^* := Hom(V, k).$$

If $T : V \to W$ is a map of vector spaces then the adjoint

$$T^* : W^* \to V^*$$

is defined by $T^*(f)(v) := f(T(v))$. If the matrix presentation of T with respect to some bases of V and W is A then the matrix presentation of T^* with respect to the dual bases is A^T, the transpose of A.

A cochain complex (C, d) has an associated dual cochain complex (C^*, d^*)

$$\cdots \xleftarrow{d^*_{n+1}} C^n \xleftarrow{d^*_n} C^{n-1} \xleftarrow{d^*_{n-1}} \ldots C^0 \longleftarrow 0$$

where $C^n = (C_n)^* = Hom(C_n,\, k)$.

n-*cocycles*: $Z^n := Ker(d^*_{n+1})$
n-*coboundaries*: $B^n := Im(d^*_n)$
n-th *cohomology group*:

$$H^n(C) := Z^n/B^n$$

For a simplicial complex $K_\bullet$,

$$H^n(K_\bullet) := H^n(C(K_\bullet))$$

Example: Let $K_\bullet$ be connected. Then the only non-trivial 0-cocycle is the map that assigns 1 to every vertex of $K_\bullet$. Hence, $H^0(K_\bullet) = \mathbb{F}_2$.

Example: $K_\bullet$ the one-point union of the boundary of two triangles as in section 2.2. Then $C^0 = (\mathbb{F}_2)^5$ and $C^1 = (\mathbb{F}_2)^6$. As d^* is the transpose of d, it has rank 4. So $H^0(K_\bullet) = \mathbb{F}_2$ and $H^1(K_\bullet) = (\mathbb{F}_2)^2$.

Universal Coefficient Theorem:

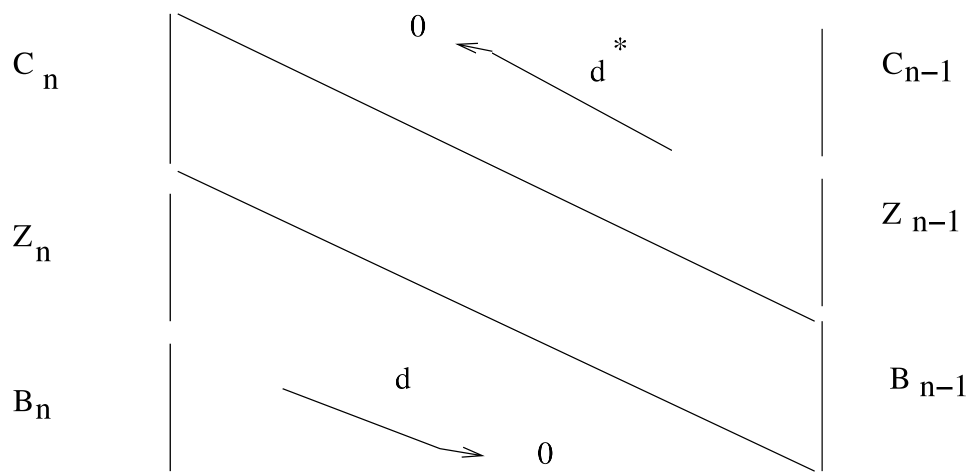
$$dim(H_n(C)) = dim(H^n(C)).$$

Proof: $dim(H_n) = dim(Z_n) - dim(B_n)$ and $dim(H^n) = dim(Z^n) - dim(B^n)$.

So the theorem follows from

$$dim Z_n + dim B^n = dim C_n$$

$$= dim C^n = dim Z^n + dim B_n.$$


C_n
Z_n
B_n
0
d^*
C_{n-1}
Z_{n-1}
B_{n-1}
d
0

Remark: This proves that cohomology encodes the information given by the homology groups. It well-known that cohomology also has addition structure, a multiplication which makes it into a finer invariant than homology. For example this allows one to distinguish a torus from the one point union of two circles and a 2-sphere. They have the same cohomology but the product structure distinguishes them.

We will not develop the ring structure here. So far this extra information has proved difficult to take advantage of in TDA.

3. Persistent Homology

Motivation: data analysis.

More specifically, when given a point cloud S, we do not know in advance which choice of ϵ associates a meaningful topological space/ homology groups to S.

3.1. Definitions

Let $(K(\bullet), \phi_\bullet)$ be a sequence of finite simplicial complexes $K(i), i = 0, \ldots N$ and simplicial maps ϕ_i

$$K(0) \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{i-1}} K(i) \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_{N-1}} K(N).$$

For each degree $p \geq 0$, this gives rise to a sequence of maps in homology

$$H_pK(0) \xrightarrow{\phi_0} \cdots \xrightarrow{\phi_{i-1}} H_pK(i) \xrightarrow{\phi_i} \cdots \xrightarrow{\phi_{N-1}} H_pK(N).$$

The p-th *persistent homology groups* are the images

$$H_p^{i,j} := \mathrm{Im}\ (\phi_{j-1} \circ \cdots \circ \phi_i : H_pK(i) \to H_pK(j))$$

A non-zero homology class $\alpha \in H_pK(i)$ is said to be *born* in $H_pK(i)$ if it is not in the image of ϕ_{i-1} and it is said to *die* in $H_pK(j)$ if

$$\phi_{j-2} \circ \cdots \circ \phi_i(\alpha) \neq 0 \text{ and } \phi_{j-1} \circ \cdots \circ \phi_i(\alpha) = 0$$

For such a class α, we define

$$\textit{persistence}\ (\alpha) := j - i$$

and similarly, if α does not die for any j,

$$\textit{persistence}\ (\alpha) := \infty$$

Graphically we can represent the persistence of the class α in the two cases by the half open intervals $[i, j)$ and $[i, \infty)$, its *barcode* . This would lead to a complicated tree structure. Instead we adopt the following rule.

Elder rule: At a juncture, the older of the two merging paths continues and the younger path ends. When two different classes $x, y \in H_pK(i-1)$ are identified in $H_pK(i)$, i.e.

$$\phi_{i-1}(x) = \phi_{i-1}(y)$$

then the path for y ends at i if x is born before y.

Problem: This does not deal with the case when both x and y are born at the same time.

To come: It is a fundamental theorem of persistent homology that for finite sequences of finite simplicial complexes a basis of the persistent homology can be chosen such that the associated barcodes respect the elder rule. Furthermore, up to reordering of the 'bars', the resulting barcode is independent of the choice of such a basis.

3.2. Examples

Example 1: Consider the sphere S^2 embedded in $\mathbf{R}^3$ and height function $h : S^2 \to \mathbf{R}$ as in the figure below. Triangulate S^2 by K such that each $h^{-1}([0, a_i])$ is triangulated by a sub-complex $K(i) \subset K$. Consider the system $(K(\bullet), \phi_\bullet)$ where each ϕ_i is the inclusion map. Note the following homotopies

$K(0) = \emptyset$

$K(1) \simeq D^2 \simeq *$

$K(2) \simeq D^2 \sqcup D^2 \simeq * \sqcup *$

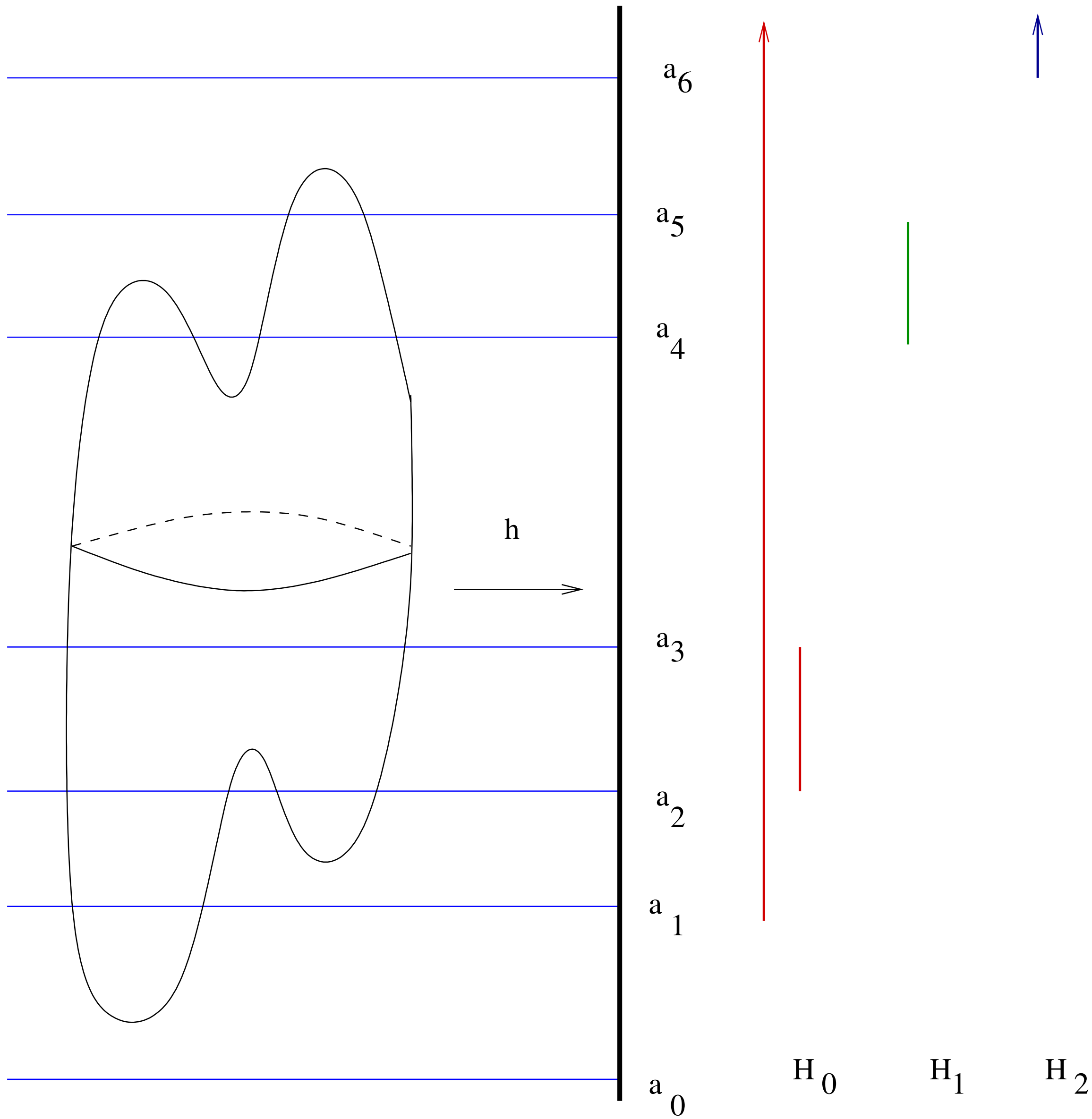
$K(3) \simeq D^2 \simeq *$

$K(4) \simeq S^2 \setminus (D^2 \sqcup D^2) \simeq S^1$

$K(5) \simeq D^2$

$K(6) = S^2$

We track the birth and death of classes for each $p = 0, 1, 2$ in barcodes as in the figure below.



a_6
a_5
a_4
h
a_3
a_2
a_1
a_0
H_0
H_1
H_2

Example 2: Let K be the 2-skeleton of a standard 3-simplex and filter it by skeletons:

$$K^0 = K_0 \subset K^1 \subset K^2 = K$$

$K(0)$ has 4 points: v_0, v_1, v_2, v_3

$K(1)$ is a graph with 4 vertices and 6 edges which is homotopic to the one-point union of 3 circles. To see this, contract the 4 edges containing v_0 to zero length. Then the 3 circles are formed by the edges $e_1 = [v_1, v_2], e_2 = [v_2, v_3]$ and $e_3 = [v_1, v_3]$.

$K(2)$ is homotopic to the sphere.

We describe the persistent homology for each degree $p = 0, 1, 2$.

$H_0K(0) = (\mathbb{F}_2)^4 =< v_0, v_1, v_2, v_3 >$ is generated by the four vertices. Change the basis to

$$\alpha_0 = v_0,\ \alpha_1 = v_1 - v_0,\ \alpha_2 = v_2 - v_0,\ \alpha_3 = v_3 - v_0$$

which have persistence $\infty, 1, 1, 1$ respectively, with associated barcodes $[0, \infty), [0, 1), [0, 1), [0, 1)$.

$H_1K(1) = (\mathbb{F}_2)^3 =< e_1, e_2, e_3 >$ and all three die in $H_1K(2) = 0$. So e_1, e_2, e_3 have persistence $1, 1, 1$ with associated barcodes $[1, 2), [1, 2),\ [1, 2)$.

$H_2K(2) = \mathbb{F}_2$ and the non-zero class has persistence ∞ and associated barcode $[2, \infty)$.

3.3 Existence of barcodes

Module Structure

Consider the total p-th persistent homology group of a sequence $(K(\bullet), \phi_\bullet)$

$$PH_p = \bigoplus_{i=0}^{N} H_p(K(i))$$

PH_p has a natural, graded module structure over the ring of polynomials $\mathbb{F}_2[t]$ where t acts on a class $\alpha \in H_pK(i)$ by $t.\alpha := \phi_i(\alpha)$.

Theorem: (Carlsson, Zomorodian 2005)

$$PH_p \simeq \bigoplus_{i=0}^{N} \bigoplus_{j>i} U(i,j)^{\beta_{ij}}$$

where $i \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{\infty\}$ and

$$U(i,j) = \Sigma^i(\mathbb{F}_2[t]/(t^{j-i}))$$

Note that Σ^i denotes a shift in degree by i and $U(i,\infty) = \Sigma^i(\mathbb{F}_2[t])$.

Sketch: $\mathbb{F}_2[t]$ is a principal ideal domain and $\{PH_p\}_{\geq p}$ is a finitely generated, graded module over it as we only consider finite sequences of finite simplicial complexes. Furthermore, the module is graded and the theorem follows by the general structure theory for such modules. $\diamondsuit$

The isomorphism class of PH_p determines and is determined uniquely by the multiplicities β_{ij}. This is the rank of the subspace of elements in H_p that are born at $H_pK(i)$ and die at $H_pK(j)$.

Corollary: To each PH_p we can associate a well-defined barcode: the union of βij copies of half open intervals $[i, j)$ for each $i = 0, \ldots, N$.

Warning: This theorem fails when working over the integers or when a second parameter of filtration is introduced (mutivariate persistence) as $\mathbb{Z}$ and $\mathbb{F}[s,t]$ are not principal ideal domains.

3.4 Standard algorithm

Most examples arise as filtered complexes. In other words, the maps $\phi : K(i) \to K(i+1)$ are inclusions and $K(N) = K$ is the total complex.

Lemma: Assume $(K(\bullet), \phi_\bullet)$ is a filtration. Then

$$H_k^{i,j} = \frac{Z_k^i}{B_k^j \cap Z_k^i}$$

If a simplex σ first appears in $K(i)$ we say it has degree i. Let $\{e_j\}$ be a homogeneous basis for $C_k := C_k(K)$ and $\{b_i\}$ for C_{k-1}. Then d_k can be represented by a matrix M_k satisfying:

$$\mathsf{deg}(b_i) + \mathsf{deg}(M_k(ij)) = \mathsf{deg}(e_j)$$

Aim: represent d_k relative to the standard basis of C_k and a homogeneous basis for Z_{k-1}.

As $d_0 = 0$, $Z_0 = C_0$ and d_1 is the standard representation.

Assume d_k is of the desired form. Order the basis for Z_{k-1} is reverse degree order. Transform M_k into lower column-echelon form. Then

$$\mathrm{rank}(M_k) = \mathrm{rank}(B_{k-1}) = \sharp \text{ pivots}$$

The basis elements corresponding to non-pivot columns form a basis of Z_k.

Lemma: Let M_k' be the column-echelon form of d_k relative to bases for C_k and Z_{k-1}. If row i has pivot $M_k'(i,j) = t^n$, it contributes $\Sigma^{\deg b_i}(\mathbb{F}_2[t]/t^n)$ to H_{k-1}, and if the column is zero, it contributes $\Sigma^{\deg b_i}(\mathbb{F}_2[t])$.

Standard algorithm: Let K be a filtered complex and let $\sigma_1, \ldots, \sigma_n$ be a total order of all the simplices such that every face of σ goes before σ and all simplices introduced at the k filtration go before those introduced at later filtration steps.

Consider the upper triangular matrix M with $m_{ij} = 1$ if σ_i is a face of σ_j and zero otherwise. For every $j = 1, \ldots, n$, define low(j) to be the index of the lowest row that contains a 1 in column j, i.e. low(j) $= i$ if $m_{ij} = 1$ and $m_{kj} = 0$ for all $k > i$. If column j only contains 0 entries, then the value of low(j) is undefined. We say that the boundary matrix is *reduced* if the map low is injective on its domain of definition.

.**for** $i = 1$ **to** n **do**

. **while there exists** $i < j$ **with low**$(i) =$ **low**(j) **do**

. **add column** i **to column** j

. **end while**

.**end for**

Reading off the intervals: Once the boundary matrix is reduced, we can read off the intervals in a barcode by pairing the simplices.

- If $\mathrm{low}(j) = i$, then the simplex σ_j is paired with σ_i, and the entrance of σ_i in the filtration causes the birth of a feature that dies with the entrance of σ_j.

- If $\mathrm{low}(j)$ is undefined, then the entrance of the simplex σ_j in the filtration causes the birth of a feature. It there exists k such that $\mathrm{low}(k) = j$, then σ_j is paired with the simplex σ_k, whose entrance in the filtration causes the death of the feature. If no such k exists, then σ_j is unpaired.

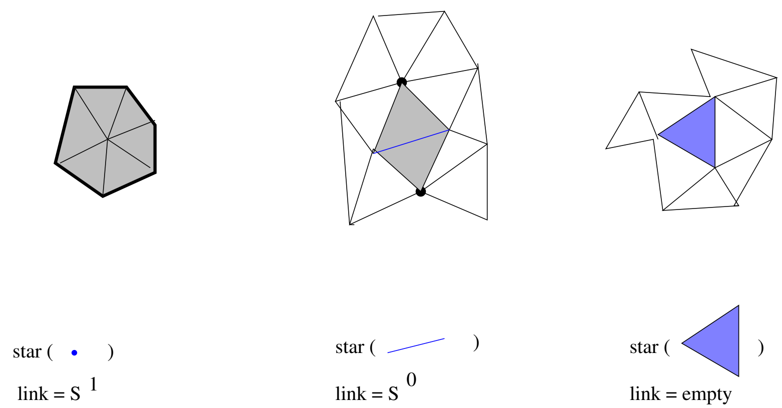
4. Manifolds and duality

4.1 Combinatorial d-manifolds

A **manifold** of dimension d is a topological space M for which every point lies in an open neighbourhood homeomorphic to the open d-dimensional unit disk D^d. M is **triangulated** by a simplicial complex $K_\bullet$ if M is a realization of $K_\bullet$. $K_\bullet$ is also called a **triangulation** of M. For a p simplex $\sigma \in K_\bullet$ define

$$star(\sigma) := \{\tau \in K_\bullet | \, \sigma \leq \tau\}$$

$$link(\sigma) := \{\tau \in \overline{star(\sigma)} | \, \tau \cap \sigma = \emptyset\}$$



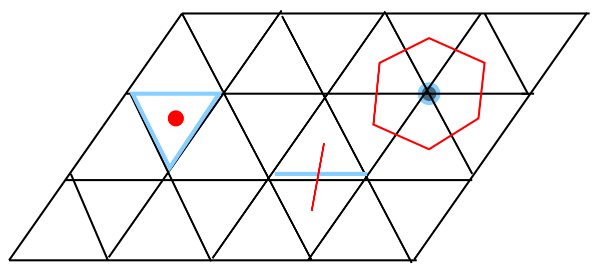
A **combinatorial d-manifold** is a d-dimensional manifold with a triangulation such that
(**) the link of every i-simplex triangulates a sphere of dimension $d - i - 1$.

Caveat: Not every triangulation of a manifold satisfies condition (**).

Exercise: If $K_\bullet$ satisfies condition (**) then so does $Sd(K_\bullet)$.

4.2. Dual bock complex

Let $K_\bullet$ be a triangulation of a combinatorial d-dimensional manifold. Label the vertices of $Sd(K_\bullet)$ by the dimension of the simplex in K that they belong to. For a p-simplex σ define its **dual block** $\widehat{\sigma}$ as the union of simplices in $Sd(K_\bullet)$ that contain the barycenter of σ as their lowest vertex.



simplices (in blue) and

their dual blocks (in red)

The **boundary** $\partial(\widehat{\sigma})$ of a block is the union $\bigcup_{\sigma\subset\tau} \widehat{\tau}$ of blocks $\hat{\tau}$; the block is homeomorphic to a $d-p-1$ sphere by the condition (**) on links in the triangulation K.

The **dual block chain complex** D is the chain complex in which the dual blocks of p-simplices form a basis for D_{d-p} and the boundary map is given by ∂.

Exercise: $\partial \circ \partial = 0$.

Block Lemma: The map that sends a block $\mu \in D_p$ to the sum $\Sigma_{\sigma \subset \mu} \sigma \in C_p(Sd(K_\bullet))$ induces an isomorphism in homology

$$H_p(D) = H_p(Sd(K_\bullet)) = H_p(K_\bullet).$$

Proof: The proof is the same as that in section 2.4. Here X_p is the subcomplex of $Sd(K_\bullet)$ which contains all simplices in a block of dimension p or less. Note that as before $H_q(X_p, X_{p-1}) = D_p$ if $p = q$ and 0 otherwise.

4.3. Poincaré Duality

Theorem: Let M be a combinatorial d-dimensional manifold. For all $0 \leq p \leq d$,

$$H_{d-p}(M) \simeq H^p(M)$$

Proof: Let M be triangulated by $K_\bullet$ and $\sigma \in K_p$. The linear map

$$\psi : D_{d-p} \longrightarrow C^p(K_\bullet), \quad \hat{\sigma} \mapsto \sigma^*$$

that sends the dual block $\hat{\sigma}$ to the dual basis element σ^* is a bijection on basis elements and hence an isomorphism of vector spaces. To prove the theorem, it remains to show that ψ commutes with the boundary maps:

$$\psi(\partial(\hat{\sigma})) = \psi(\Sigma_{\sigma\subset\tau}\,\hat{\tau}) = \Sigma_{\sigma\subset\tau}\,\tau^*$$

$$d^*(\psi(\hat{\sigma})) = d^*(\sigma^*) = \Sigma_{\sigma\subset\tau}\,\tau^*$$

Corollary: $H_{d-p}(M) \simeq H_p(M)$

Proof: This follows from Poincare Duality and the Universal Coefficient Theorem.

Example 1: If d is odd then $\chi(M) = 0$.

Example 2: If M is connected then

$$H_d(M) \simeq H_0(M) \simeq \mathbb{F}_2$$

and the only non-zero d-dimensional cycle is the **fundamental class** $[M] := \Sigma_{\sigma \in K_d} \sigma$.

4.4. Intersection theory

Let M be a d-dimensional combinatorial manifold with triangulation $K_\bullet$.
Goal: to get a better geometric picture of P.D.

For $\sigma, \tau \in K_p$ the intersection $\sigma \cap \hat{\tau}$ is the barycenter of σ if $\tau = \sigma$ and is empty if $\tau \neq \sigma$. Define

$$< \sigma, \hat{\tau} > := 1 \text{ if } \sigma = \tau, \text{ and } = 0 \text{ if } \sigma \neq \tau$$

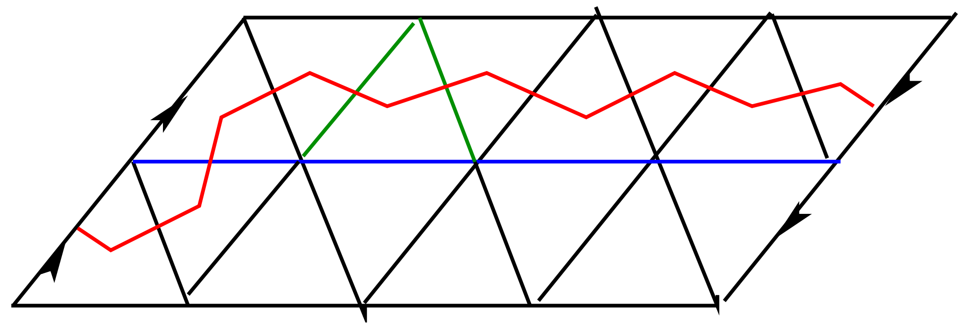
and extend this bilinearly to a pairing

$$C_p(K_\bullet) \times D_{d-p} \longrightarrow \mathbb{F}_2$$

Claim: The intersection number $< c, d >$ does not change if c or d is replaced by a homologous cycle.

Proof: If c is homologous to c_0 then there exists a $p+1$-chain e with $\partial(e) = c + c_0$.
Let τ be a summand of e.
Then for any $(d-p)$-simplex $\hat{\sigma}$ (of d), $\tau \cap \hat{\sigma}$ is either empty or, if σ is a face of τ, the 1-simplex from the barycenter of τ to the barycenter of σ.

As d is a cycle, the path of which $\hat{\sigma}$ is a part needs to continue. The possibilities are that the path goes in and out of τ through faces both belonging to c, or both belonging to c_0, or one belonging to c and one belonging to c_0. In all cases the intersection number with c is the same as that with c_0.



$|K_\bullet| =$ Möbius band

$c =$ core of the Möbius band − blue

$c_0 =$ core: one blue simplex replaced by two green

$d =$ cycle in D_* homotopic to the core − red

Theorem: The bilinear map

$$H_p(M) \times H_{d-p}(M) \longrightarrow \mathbb{F}_2$$

$$([c],[d]) \mapsto < c, d >$$

defines a perfect pairing, i.e. the map

$$[d] \mapsto < \ , d >$$

defines an isomorphisms $H_{d-p}(M) \to (H_p(M))^*$.

Proof: By Poincaré Duality, the pairing

$$H_p(K_\bullet) \times H_{d-p}(D) \longrightarrow \mathbb{F}_2$$

$$(\Sigma\, \sigma_i, \Sigma\, \widehat{\tau}_j) \mapsto \Sigma < \sigma_i, \widehat{\tau}_j >$$

is the same as the pairing

$$H_p(K_\bullet) \times H^p(K_\bullet) \longrightarrow \mathbb{F}_2$$

$$(\Sigma\,\sigma_i, \Sigma\,\tau_j^*) \mapsto \Sigma\,\tau_j^*(\sigma_i)$$

as $<\sigma, \hat{\tau}> = \tau^*(\sigma)$ for all $\sigma, \tau \in K_p$.

The second pairing induces by the Universal Coefficient Theorem a perfect pairing
$H^p(K_\bullet) \simeq (H_p(K_\bullet))^*$.

4.5. Lefschetz Duality

A **manifold with boundary** of dimension d is a topological space M for which every point lies in an open neighbourhood homeomorphic to the open d-dimensional unit disk D^d or the half disk $D^d \cap (\mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1})$. The points with an open neighbourhood of the second kind form the **boundary** ∂M of M.

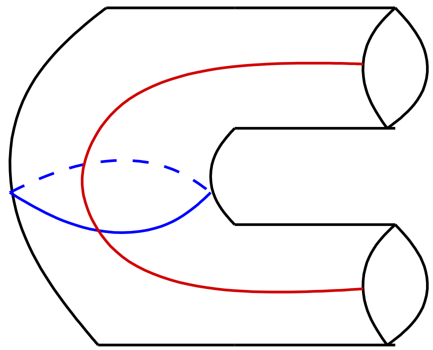
A **combinatorial d-manifold with boundary** is a d-dimensional manifold with boundary and a triangulation such that
(**) the link of every i-simplex triangulates a sphere or a half-sphere of dimension $d-i-1$.

Theorem: For a d-dimensional combinatorial manifold M with boundary, there are isomorphisms

$$H_{d-p}(M, \partial M) \simeq H^p(M) \text{ and } H_{d-p}(M) \simeq H^p(M, \partial M)$$

Equivalently, intersection defines a perfect pairing

$$H_p(M) \times H_{d-p}(M, \partial M) \longrightarrow \mathbb{F}_2$$



intersection of a cycle

with a relative cycle

in an annulus

Example: Annulus A

$H_0(A) = H_2(A, \partial A) = \mathbb{F}_2$

$H_1(A) = H_1(A, \partial A) = \mathbb{F}_2$

$H_2(A) = H_0(A, \partial A) = 0$

4.6. Alexander Duality

Let $X \subset S^d$ be the realization of a simplicial complex. Consider its ϵ-neighbourhood

$$N(X) := \{y \in B_\epsilon(x) \mid x \in X\}.$$

Its closure $\overline{N(X)}$ is a compact d-manifold with boundary. Its complement $Y := S^d \setminus N(X)$ is also a compact d-manifold with (the same) boundary.

Note: A simplicial complex version of $\overline{N(X)}$ and its complement can be constructed as follows. Let S^d be triangulated by $K_\bullet$ such that X is the realization of a subcomplex of $K_\bullet$. Consider $Sd^2(K_\bullet)$ and let $S(X)$ be its subcomplex representing X. Then $\overline{N(X)}$ can be replaced by the simplicial d-manifold (with boundary)

$$N := \bigsqcup_{u \in S(X)} \overline{star(u)}$$

Theorem: For all p, $\tilde{H}_p(X) = \tilde{H}^{d-p-1}(Y)$.

Notation: $\tilde{H}_p(X) := H_p(X, *)$.

Example 1: $X = \{0\} \subset S^d$
$N(X) = B^d \subset S^d$
$Y = S^d \setminus B^d$
$\tilde{H}_*(X) = 0 = \tilde{H}^*(Y)$

Example 2: $X = S^1 \subset S^3$
$\overline{N(X)}$ is a solid torus in S^3
$Y = S^3 \setminus N(X)$ is another solid torus in S^3

$$\tilde{H}_0(X) = 0 = \tilde{H}^2(Y)$$

$$\tilde{H}_1(X) = \mathbb{F}_2 = \tilde{H}^1(Y)$$

$$\tilde{H}_2(X) = 0 = \tilde{H}^0(Y)$$

Note: Alexander Duality implies that the homology of all knot complements are the same and can therefore not be used to distinguish knots.

Proof: Let $p < d-1$. Then

$$
\begin{array}{rcll}
\tilde{H}^{d-p-1}(Y) & = & H^{d-p-1}(Y) & \\
 & = & H_{p+1}(Y, \partial Y) & \text{by L.D.} \\
 & = & H_{p+1}(S^d, \overline{N(X)}) & \\
 & = & \tilde{H}_p(\overline{N(X)}) & \text{by l.e.s.} \\
 & = & \tilde{H}_p(X) &
\end{array}
$$

Let $p = d-1$. Then by a similar argument

$$
\begin{array}{rcl}
\tilde{H}^0(Y) \oplus \mathbb{F}_2 & = & H^0(Y) \\
 & = & H_d(Y, \partial Y) \\
 & = & H_d(S^d, \overline{N(X)}) \\
 & = & \tilde{H}_{d-1}(\overline{N}(X)) \oplus \mathbb{F}_2 \\
 & = & \tilde{H}_{d-1}(X) \oplus \mathbb{F}_2
\end{array}
$$

as the l.e.s. (in relative homology) gives

$$
0 \to H_d(S^d) \to H_d(S^d, \overline{N(X)}) \to \tilde{H}_{d-1}(\overline{N(X)}) \to 0
$$

We have used here that by L.D.

$$H_d(N(X)) = H^0(N(X), \partial N(X)) = 0$$

as every component of $\overline{N(X)}$ has non-empty boundary.

Finally, let $p \geq d$. Then $H^{d-p-1}(Y) = 0$ and so is

$$H_p(X) = H_p(\overline{N(X)}) = H^{d-p}(\overline{N(X)}, \partial\overline{N(X)}) = 0$$

by L.D.

Example 3: X a torus embedded in S^3
By Alexander Duality,
$\tilde{H}_0(X) = 0 = \tilde{H}^2(Y)$
$\tilde{H}_1(X) = \mathbb{F}_2 \times \mathbb{F}_2 = \tilde{H}^1(Y)$
$\tilde{H}_2(X) = \mathbb{F}_2 = \tilde{H}^0(Y)$

The last equation implies that Y has two connected components. Indeed, $Y \simeq S^1 \cup S^1$, one circle corresponding to the interior of the torus and one to the exterior.

Exercise: Prove the Jordan Curve Theorem that says that every closed embedded curve in the plane divides the plane in an outside and an inside.