When Algebra met Topology

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Fridays@2
3. May 2019

Emmy Noether (1882–1935)

Habilitation 1919

”How can a women be allowed to be a Privatdozent? Once she was a Privatedozent, after all, she could become a professor and member of the University Senate. Can we allow a woman into the Senate?” This protest provoked Hilbert’s well-known reply: ”Meine Herren, der Senat is ja keine Badeanstalt, warum darf eine Frau nicht darin?” *Alexandroff 1935*

Emmy Noether (1882–1935)

1882 born in Erlangen

1915 Goettingen on invitation of Hilbert and Klein

1919 Habilitation

1932 ICM Zuerich

1933 dismissed by Nazis

1935 died after an operation at Bryn Mawr College

Where do new ideas come from?

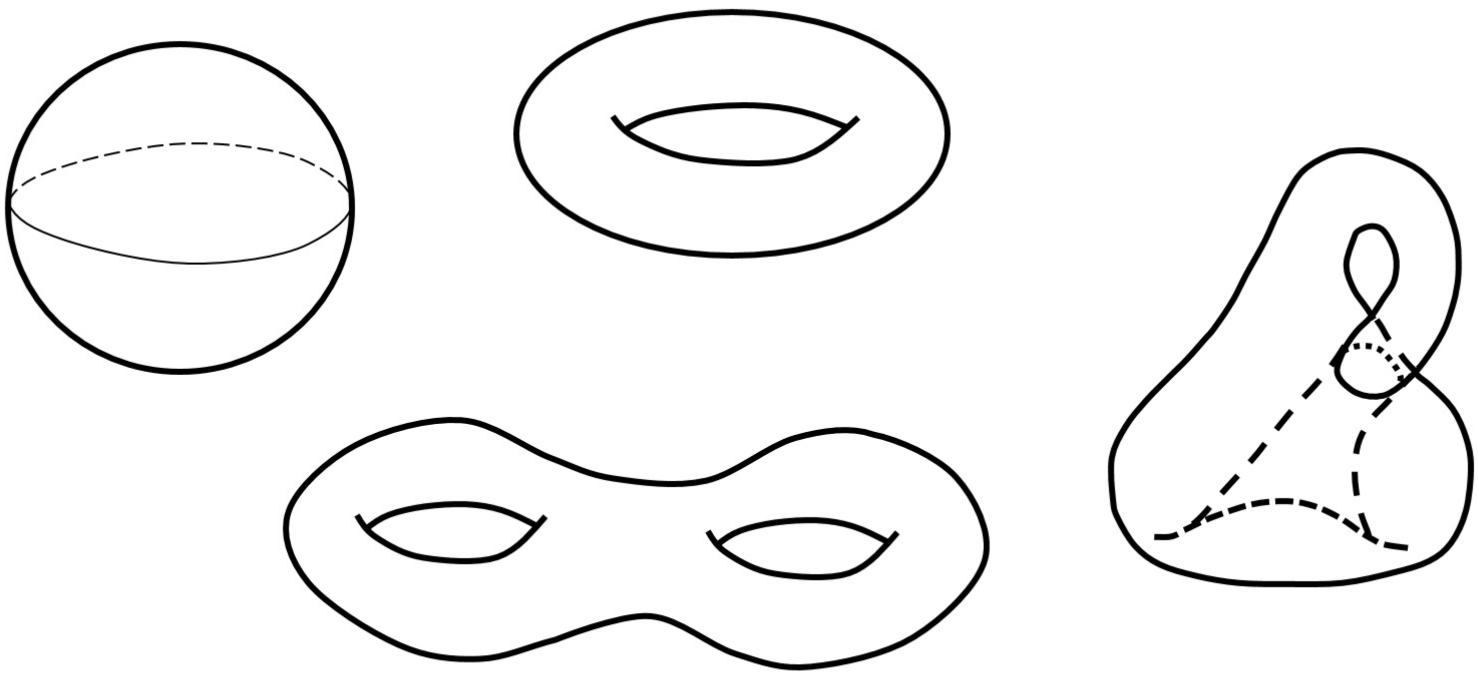
$\triangle$-complexes

and

Euler characteristic

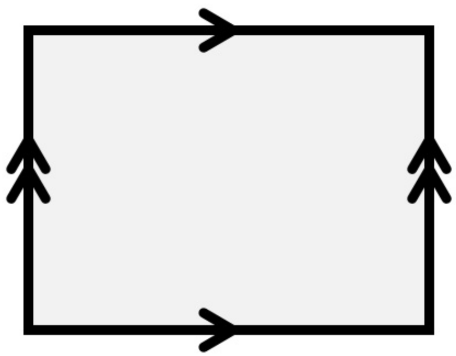
Surfaces

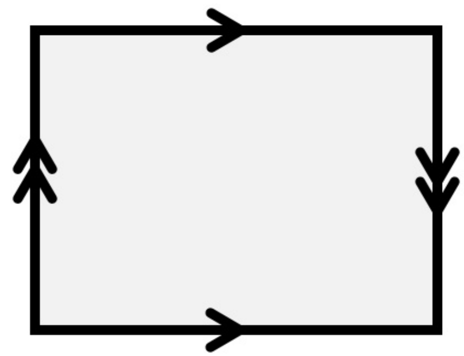
two-sided (orientable) and one-sided (non-orientable)



Fact

All surfaces can be made from polygons by gluing along boundary edges.





. torus Klein bottle

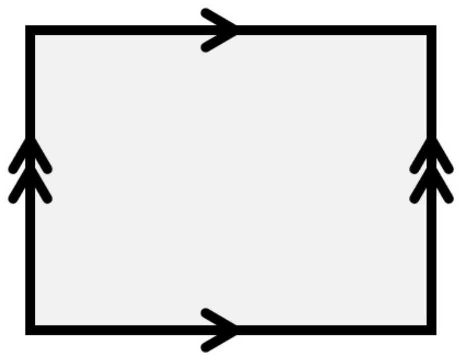
Definition

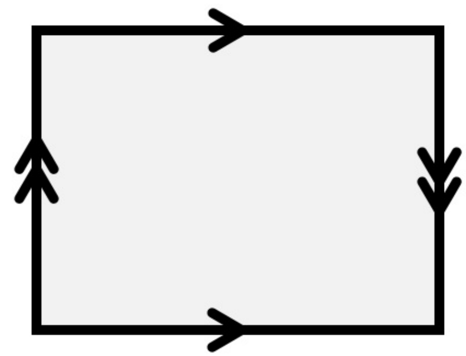
If S is a surface subdivided into polygons then its Euler characteristic is given by

$$\chi(S) = V - E + F$$

$$V = \sharp \text{ vertices}, \; E = \sharp \text{ edges}, \; F = \sharp \text{ faces}$$

Examples





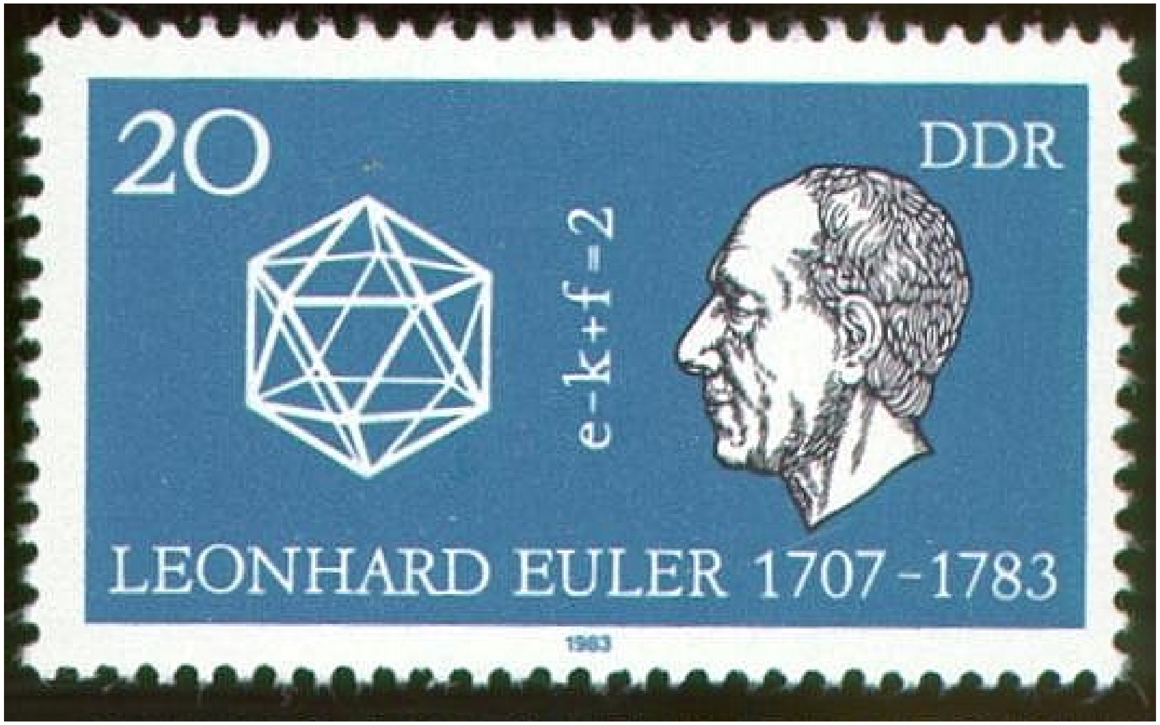
$\chi(\text{torus}) = 1-2+1 = 0$ $\chi(\text{Klein bottle}) = 1-2+1 = 0$

Exercise

Compute the Euler characteristic of a
(1) cube, (2) octahedron, and (3) icosahedron!

Answer: all have Euler characteristic equal to 2

(1) cube: 8 vertices, 12 edges, and 6 faces
(2) octahedron: 6 vertices, 12 edges, and 8 faces
(3) icosahedron: 12 vertices, 30 edges, and 20 faces

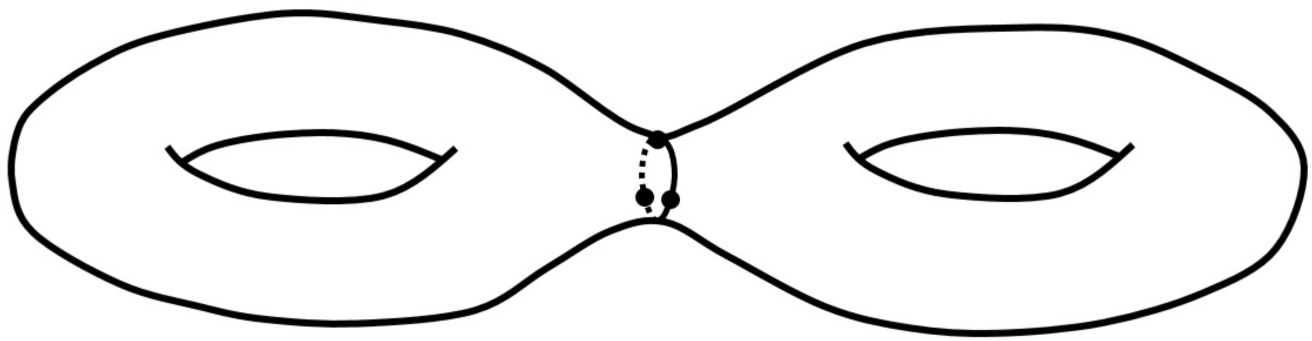

20
DDR
e-k+f=2
LEONHARD EULER 1707-1783
1983

Fact

The Euler characteristic of a surface is well defined:

It does not matter how we subdivide a surface into polyhedra, we always get the same number when we compute the Euler characteristic.

Gluing Formula



$$\chi(F_2) = \chi(F_1) + \chi(F_1) - 2 - (-3) - 3 = -2$$

Classification

Two-sided and one-sided surfaces
are characterised by their Euler characteristic.

$$\chi(F_g) = 2 - 2g \quad \text{and} \quad \chi(N_g) = 2 - g$$

However,

$$\chi(* \sqcup *) = 2 = \chi(\text{sphere})$$

or

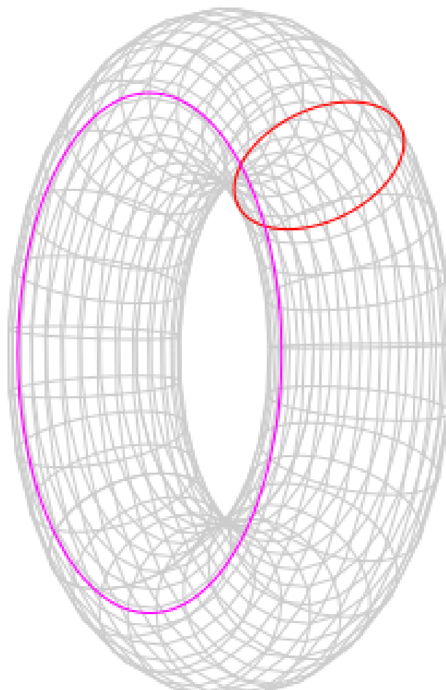
$$\chi(S^1) = 0 = \chi(\text{torus})$$

Goal:

Find a refinement of the Euler characteristic which exhibits the contribution form each dimension while remaining independent on how we represent the space as a $\triangle$-object.

Betti numbers and Homology

Counting holes



Torus: one connected component, two cycles, one void

Let K be a $\triangle$-complex with K_n the set of all n-dimensional cells.

The associated **Chain complex** is a sequence of $\mathbb{Z}$-abelian groups and maps

$$\mathbb{Z}[K_{n+1}] \xrightarrow{d_{n+1}} \mathbb{Z}[K_n] \xrightarrow{d_n} \mathbb{Z}[K_{n-1}] \ldots \longrightarrow \ldots \mathbb{Z}[K_1] \xrightarrow{d_1} \mathbb{Z}[K_0] \longrightarrow 0$$

$$d_n(\alpha) := \Sigma_{\beta \subset \alpha,} \text{ sign}(\beta, \alpha)\ \beta$$

Key-observation: The boundary of a boundary is empty

$$d_k \circ d_{k+1} = 0$$

and hence $Ker(d_k) \supset Im(d_{k+1})$.

k-th **Homology group**: $H_k(K) := Ker(d_k)/Im(d_{k+1})$
n-th **Betti number**:

$$\beta_k := \text{rank } H_k(K) = \text{rank } Ker(d_n) - \text{rank } Im(d_{n+1})$$

Example: $K =$ interval $[0, 1]$
$\beta_0 = 1$
$\beta_1 = 0$

Example: $K =$ circle S^1
$\beta_0 = 1$
$\beta_1 = 1$

Example: $K =$ torus
$\beta_0 = 1$
$\beta_1 = 2$
$\beta_2 = 1$

Example: $K =$ Klein bottle
$\beta_0 = 1$
$\beta_1 = 1$
$\beta_2 = 0$

Working with coefficients $\mathbb{F}_2 = \{\bar{0}, \bar{1}\}$ will distinguish between the circle and the Klein bottle.

Geometric interpretation:

β_0 'counts' connected components
β_1 'counts' cycles
β_2 'counts' voids
β_3 'counts' 3-dimensional cavities, etc.

Fact:
Homology = refinement of Euler characteristic

For a finite $\triangle$-complex

$$\chi(K) = \beta_0(K) - \beta_1(K) + \beta_2(K) - \dots$$

Functoriallity

Homology = functor from Spaces to Abelian Groups

$$\triangle\text{-Complexes} \xrightarrow{H_n} \text{Abelian Groups}$$

$$K \mapsto H_n(K)$$

$$(f : K \to L) \mapsto (f : H_n(K) \to H_n(L))$$

Continuous maps between spaces induce homomorphisms of homology groups!

Brouwer Fixed Point Theorem:
Every continuous map $f : D^2 \to D^2$ has a fixed point.

Proof:

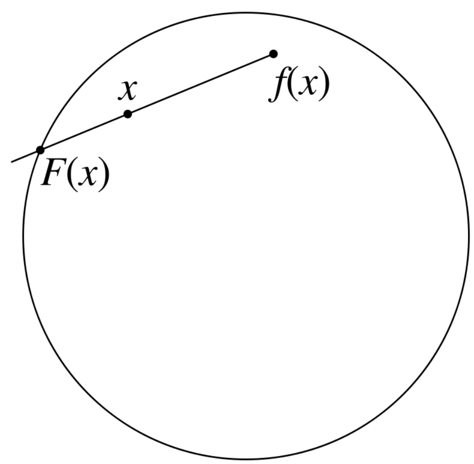
Assume there is no point $x \in D^2$ with $f(x) = x$.

Then $F : D^2 \to S^1$ is well defined and continuous.

Note $F \circ incl : S^1 \hookrightarrow D^2 \to S^1$ is the identity.

So $F \circ incl : H_1(S^1) \to H_1(D^2) \to H_1(S^1)$ is the identity as well.

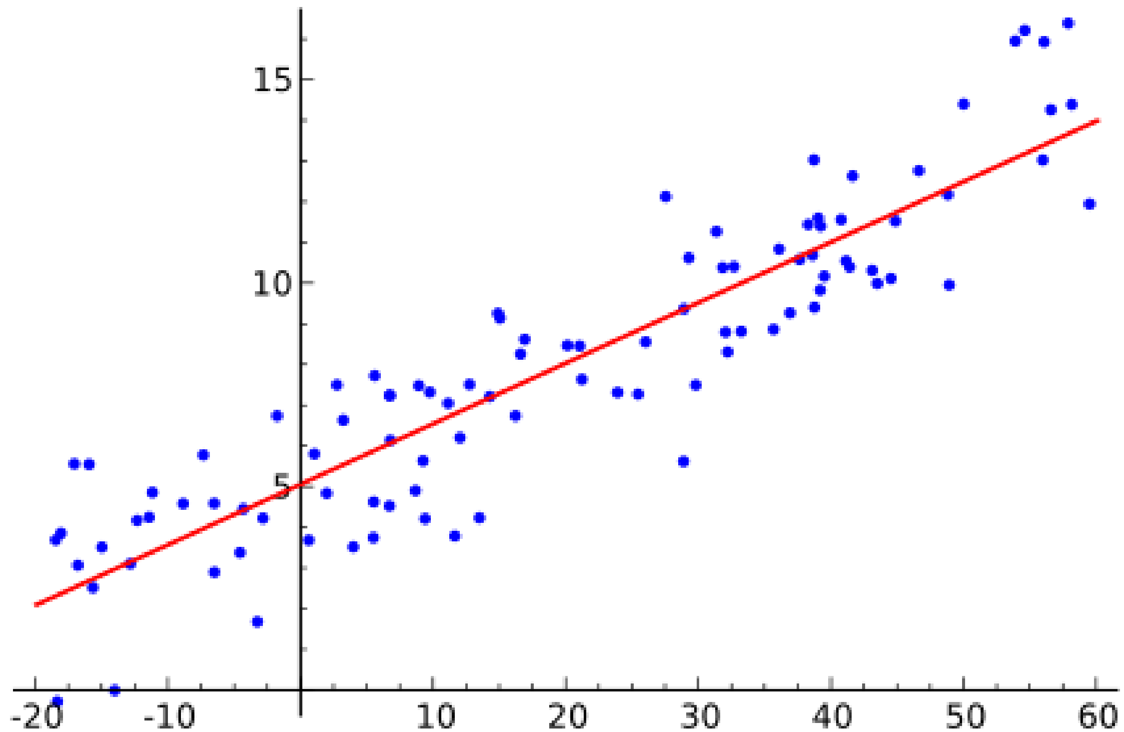
But $H_1(S^1) = \mathbb{Z}$ and $H_1(D^2) = 0$. Contradiction!



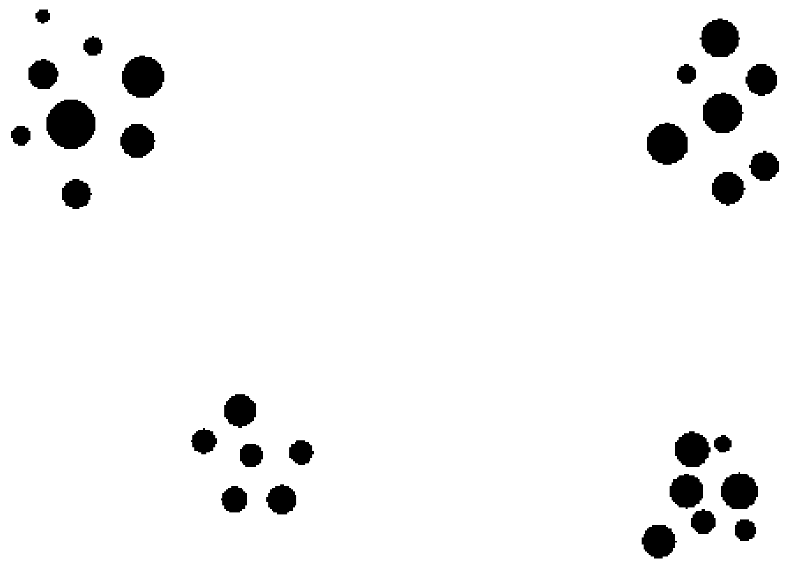
Topological Data Analysis

"Data has shape and shape matters!"

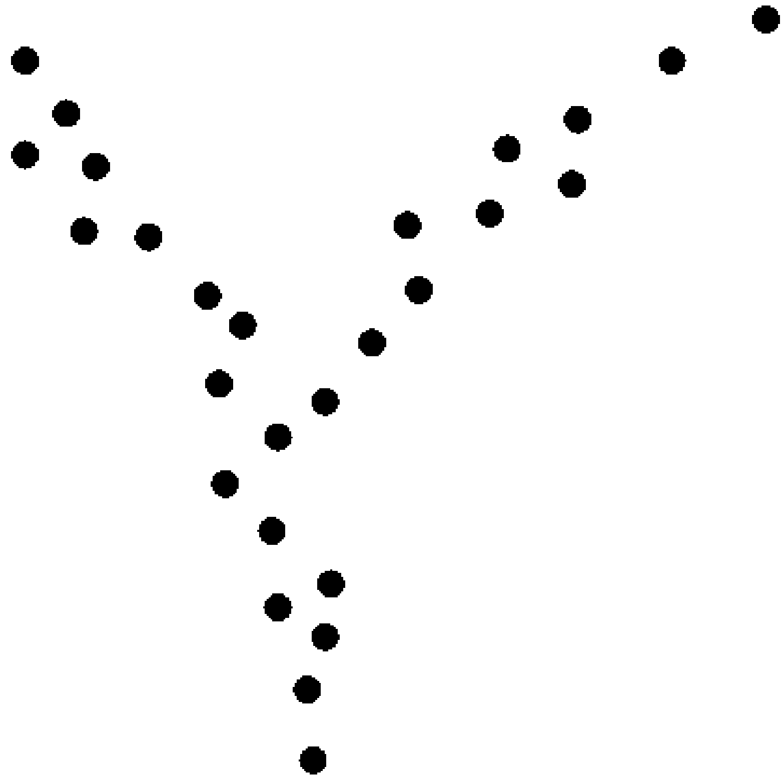
Most developed tool: **Persistent Homology**



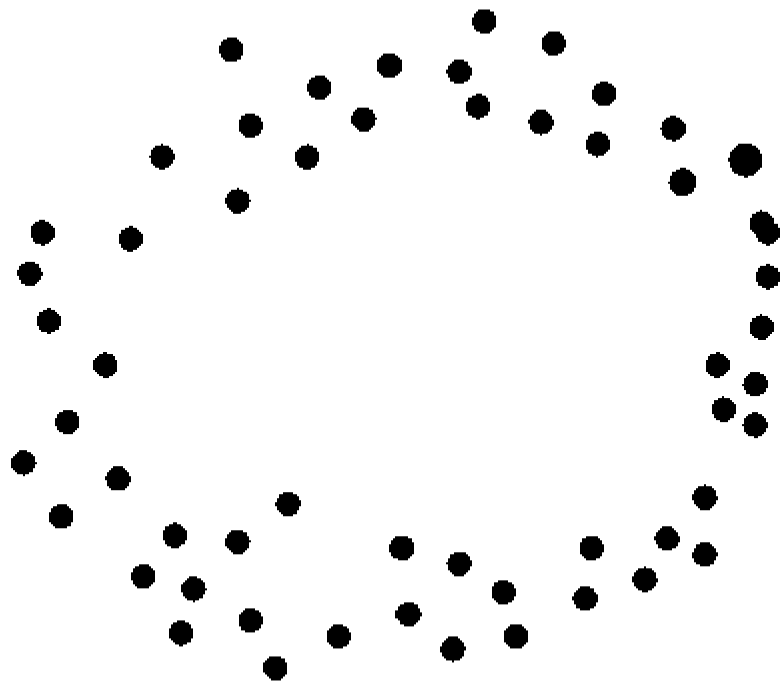
Linear regression



Clustering



Y-formation



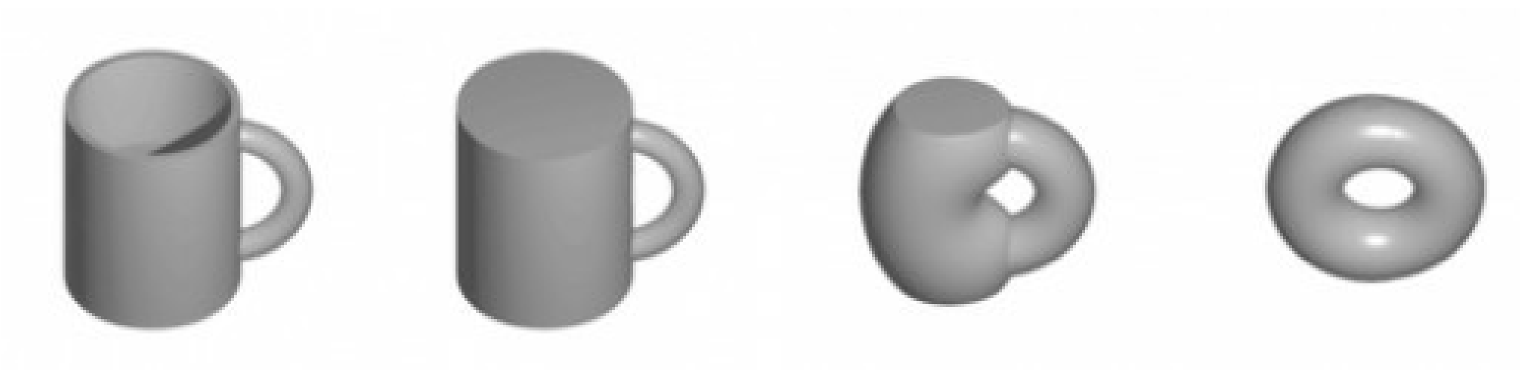
Circle

Why topology?

- is robust against noise
- can extract essential information
- can suppress information
- can reveal higher dimensional structure/dependencies

- has computable signatures

Robustness against noise...

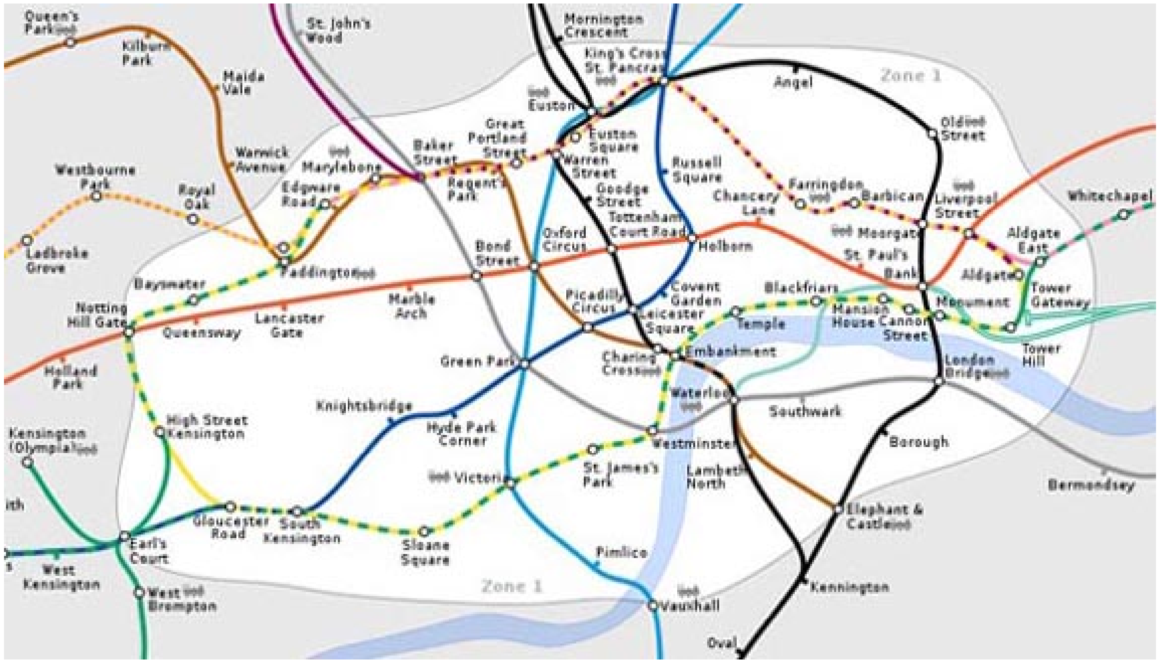


A mug is topologically equivalent to a doughnut

Extracting essential information...

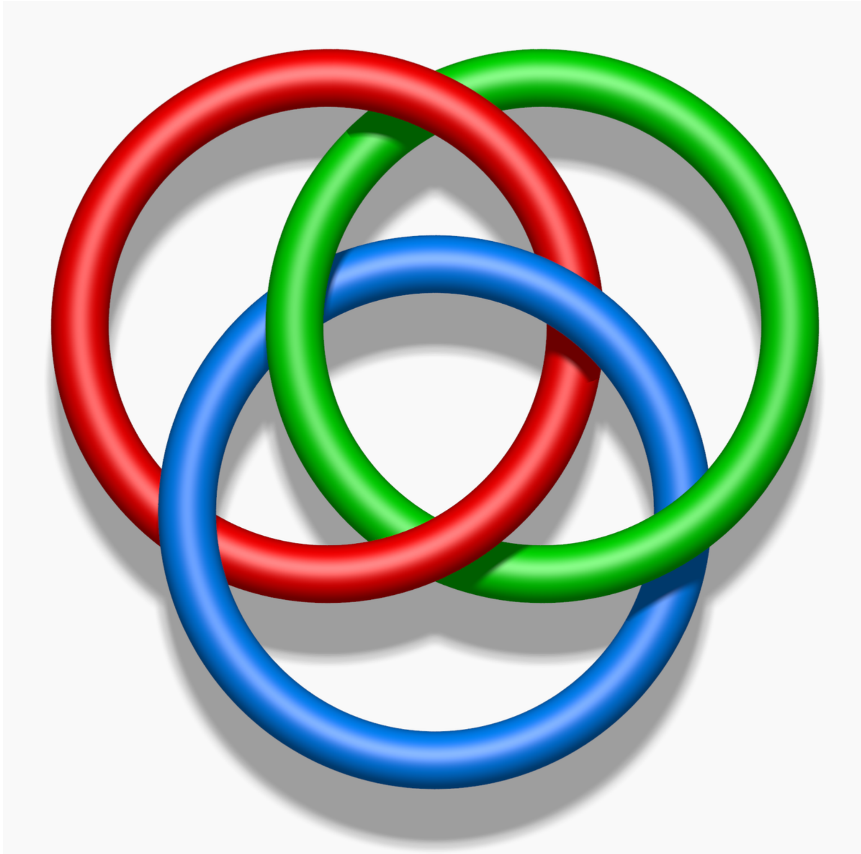
A topological map of the London Underground

Suppressing information...



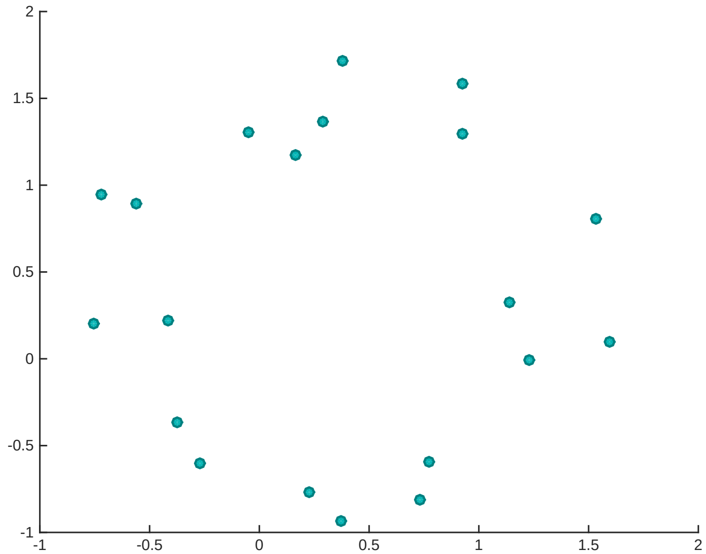
A geometric map

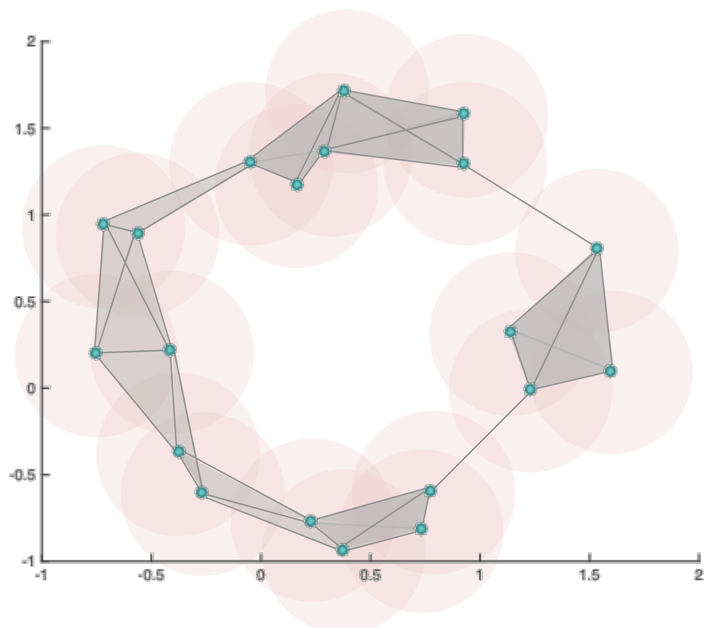
Revealing higher dimensional dependencies...

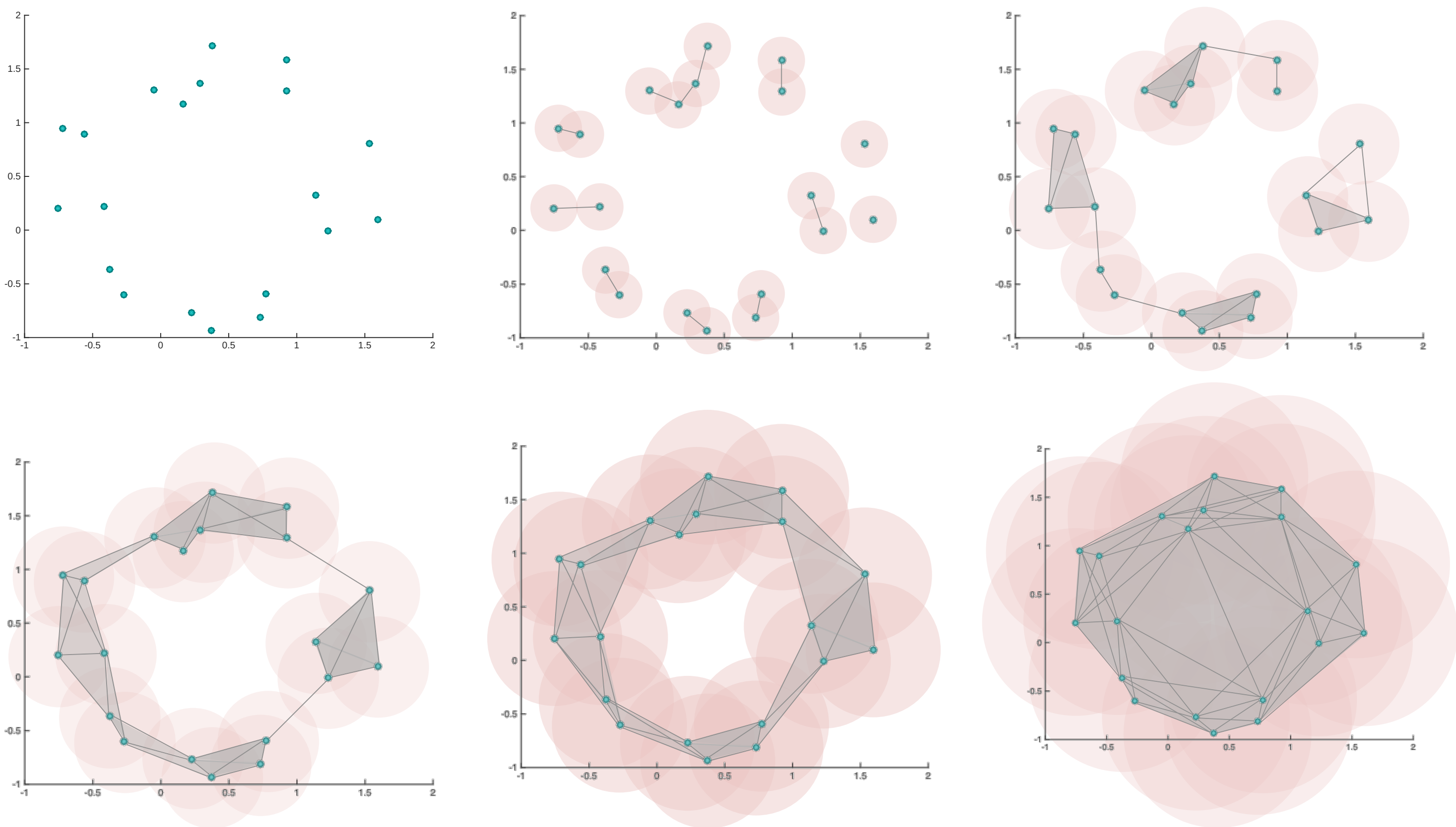


Borromean rings

Topology of point clouds

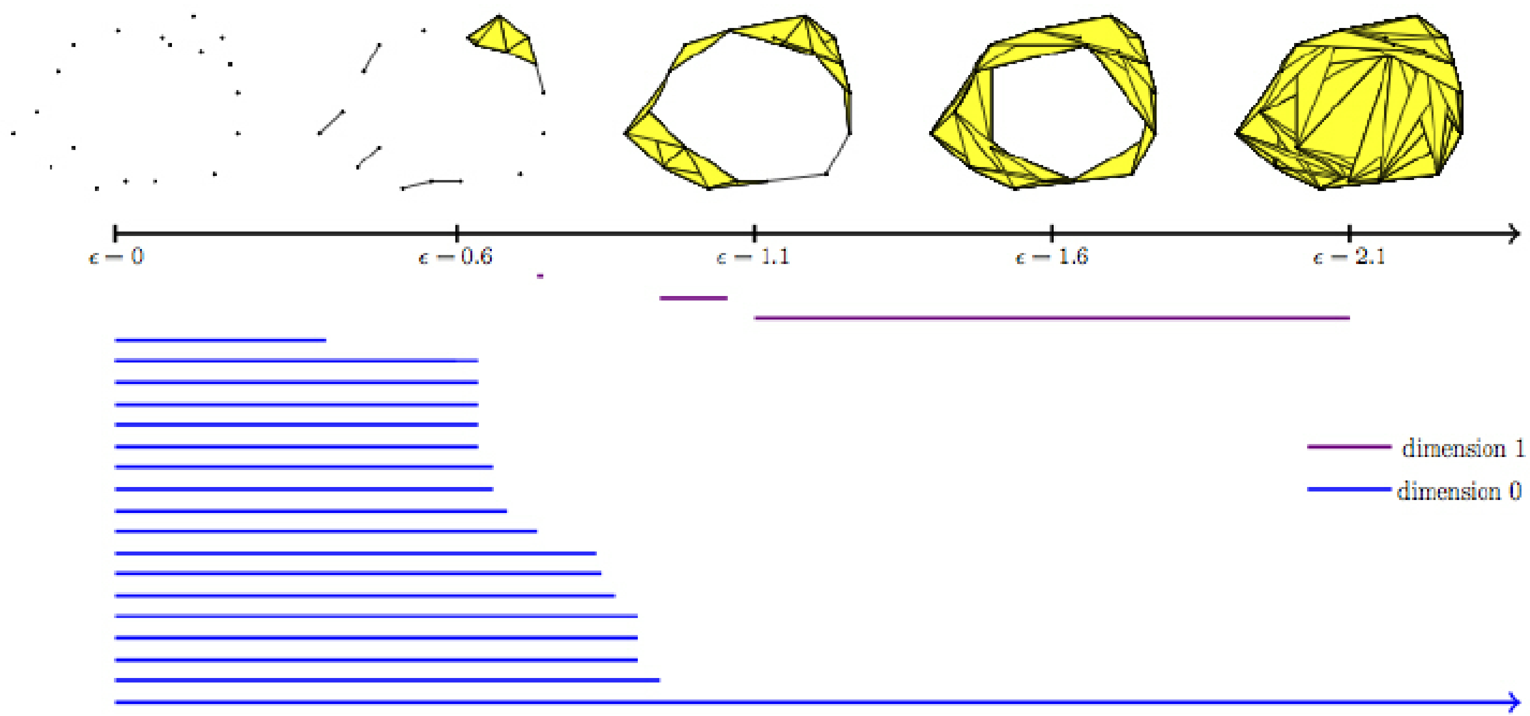






Source: Ghrist

Barcodes



Working over a field.

Source: Otter et al.

Functoriallity is of essence!

Applications:

medicine, network analysis, material science, sensor networks etc.

Emmy Noether