Computational Algebraic Topology Topic B: Lecture II: The Topology of Paradox

Samson Abramsky

Department of Computer Science
The University of Oxford

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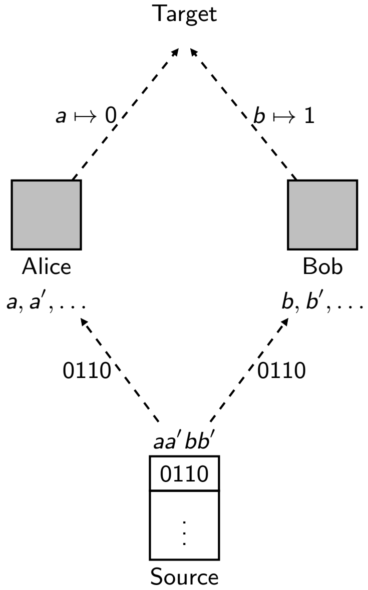
However, this view is **impossible to sustain** in the light of our **actual observations of (micro)-physical reality**.

Hidden Variables: The Mermin instruction set picture



The ‘Hardy Paradox’: Bell’s theorem without inequalities

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Hardy models: those whose **support** satisfies

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1			
(a_1, b_2)	0			
(a_2, b_1)	0			
(a_2, b_2)				0

Which ‘instruction set’ λ could the outcomes $(0,0)$ for measurements (a_1, b_1) could come? Clearly, we must have

$$\lambda : a_1 \mapsto 0, \quad b_1 \mapsto 0.$$

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Thus Hardy models are **contextual**. They cannot be explained by a classical source.

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
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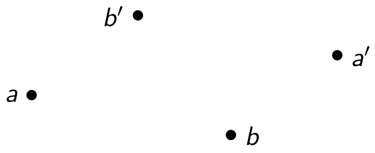

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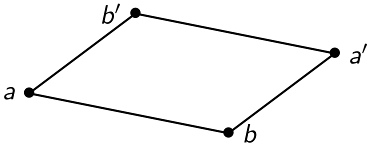

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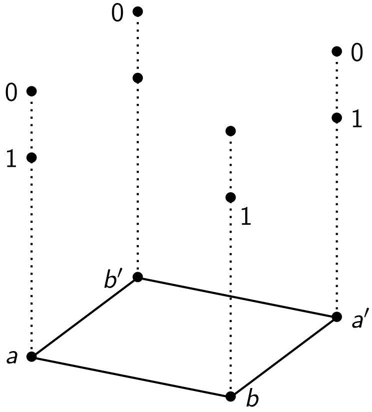

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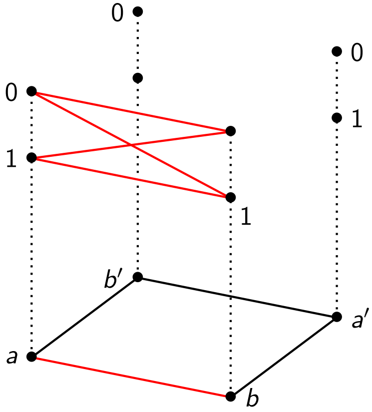

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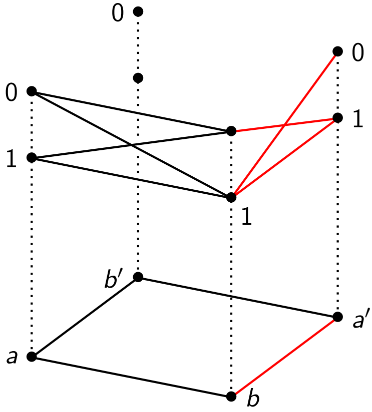

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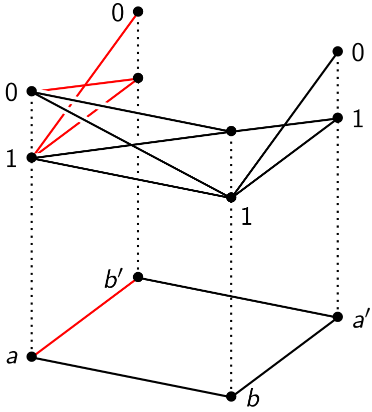

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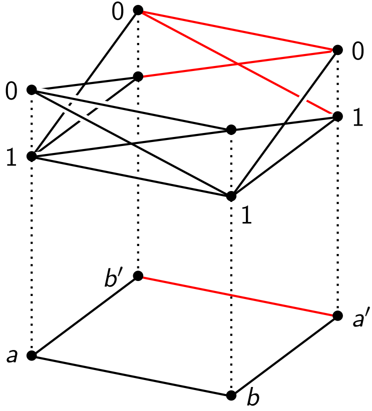

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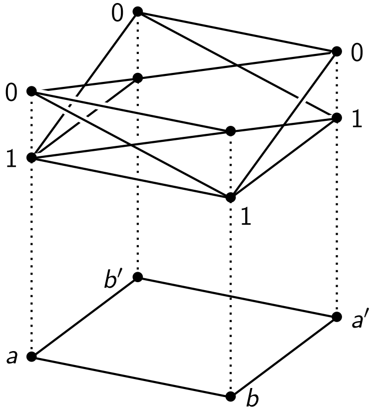

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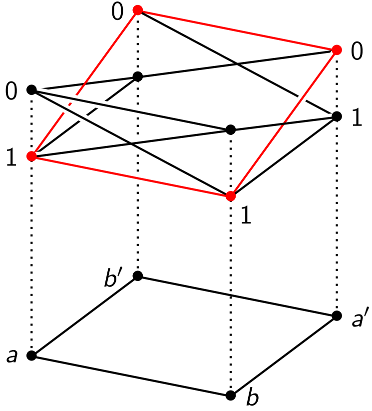

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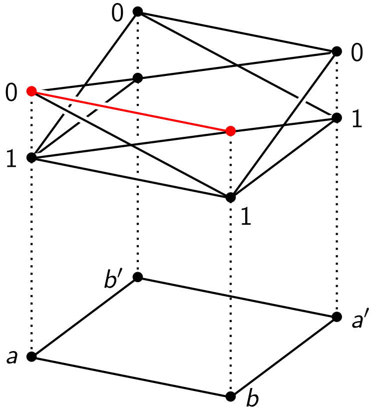

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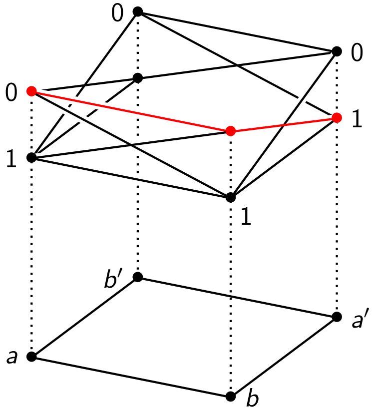

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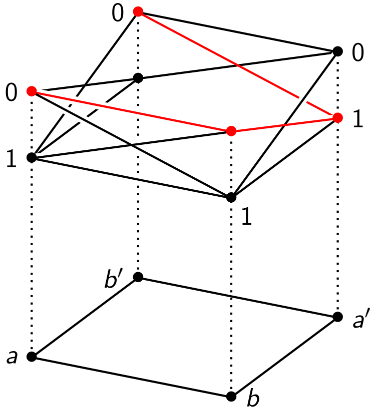

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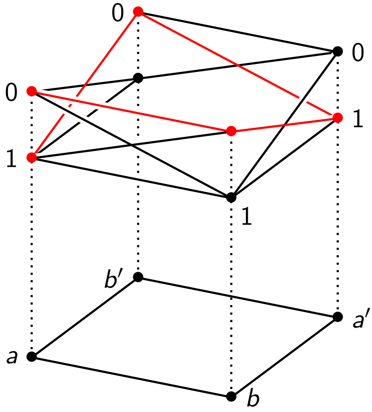

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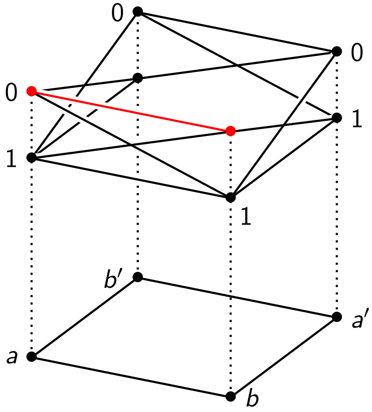

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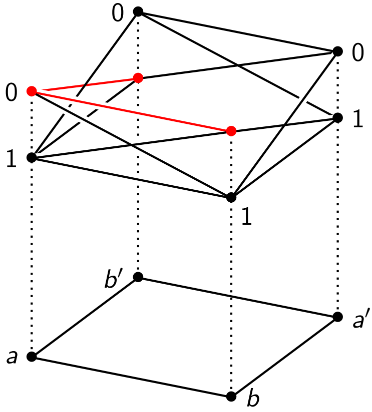

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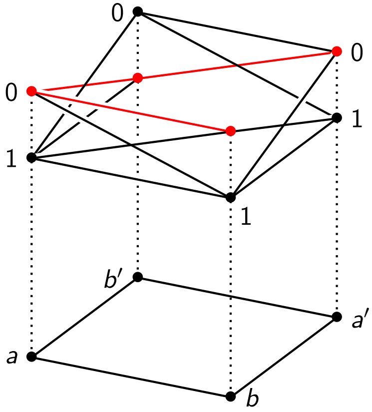

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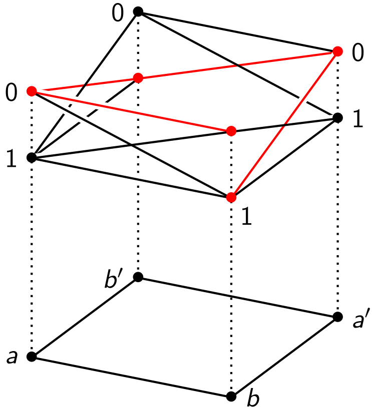

Strong Contextuality

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The PR Box

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Note this achieves the algebraic maximum of 4 for our logical Bell inequality.

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Note this achieves the algebraic maximum of 4 for our logical Bell inequality.

In terms of the XOR game, it is a **winning strategy**.

Bundle Pictures

Strong Contextuality

- E.g. the PR box:

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ab'	✓	×	×	✓
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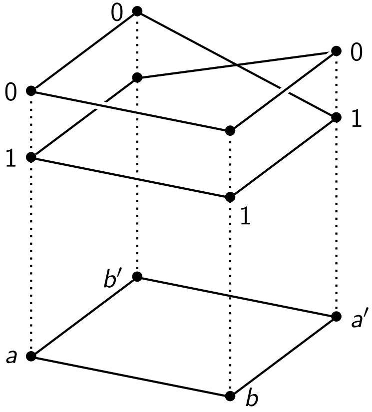
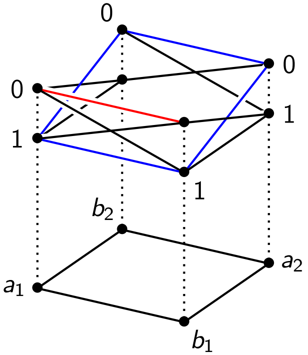
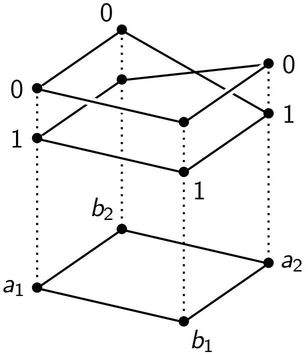


Visualizing Contextuality





The Hardy table and the PR box as bundles

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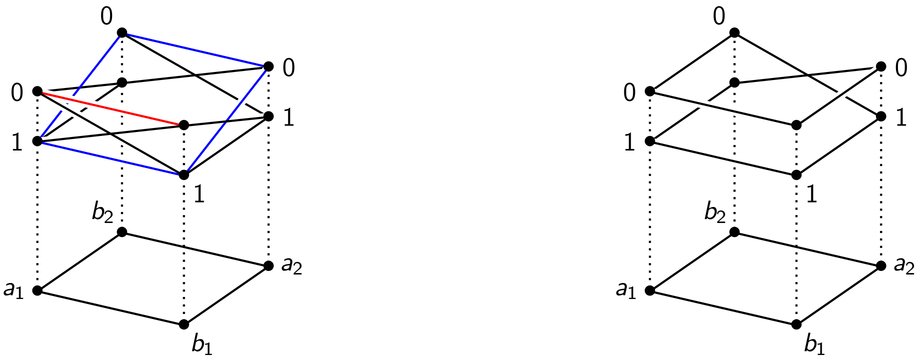

The Hardy table and the PR box as bundles

A hierarchy of degrees of contextuality:

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$

Contextuality, Logic and Paradoxes

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Liar cycles. A Liar cycle of length N is a sequence of statements

$$
\begin{aligned}
S_1 &: \; S_2 \text{ is true,} \\
S_2 &: \; S_3 \text{ is true,} \\
&\vdots \\
S_{N-1} &: \; S_N \text{ is true,} \\
S_N &: \; S_1 \text{ is false.}
\end{aligned}
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For $N = 1$, this is the classic Liar sentence

$$S : \; S \text{ is false.}$$

Contextuality, Logic and Paradoxes

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$$x_1 = x_2, \ \ \ldots, \ \ x_{n-1} = x_n, \ \ x_n = \neg x_1$$

The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

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We can regard each of these equations as fibered over the set of variables which occur in it:

$$\begin{array}{rrcl} \{x_1, x_2\}: & x_1 & = & x_2 \\ \{x_2, x_3\}: & x_2 & = & x_3 \\ & \vdots & & \\ \{x_{n-1}, x_n\}: & x_{n-1} & = & x_n \\ \{x_n, x_1\}: & x_n & = & \neg x_1 \end{array}$$

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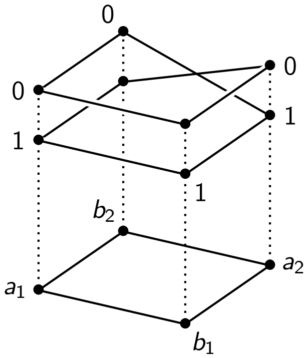
The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



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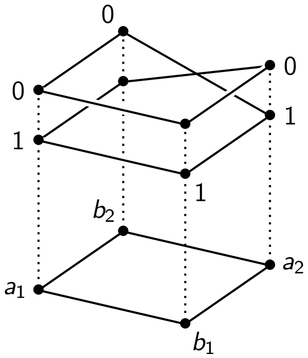

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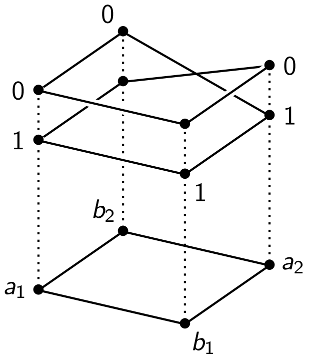

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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

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A joint outcome or **event** in a context C is $s \in O^C$, e.g. $s = \{a \mapsto 0, b \mapsto 1\}$.

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B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
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The original K-S construction used 117 variables!

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These distributions are the rows of our probability tables.

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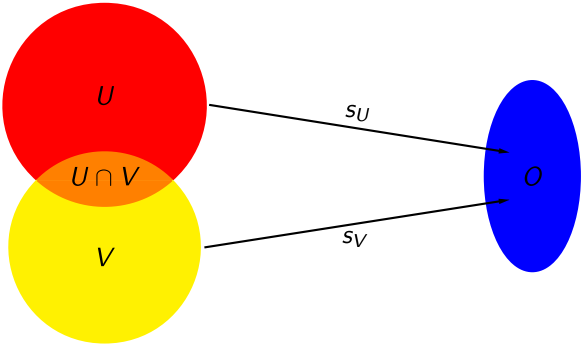
Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

Gluing functional sections



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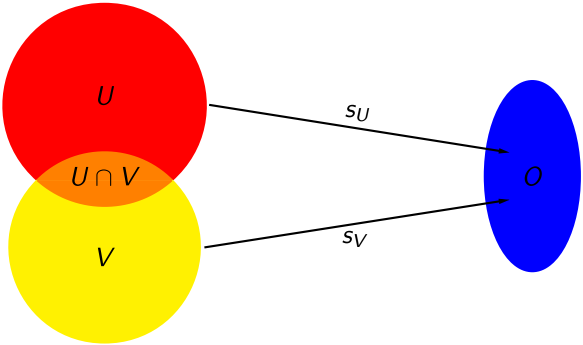

If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

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So compatibility says that the distributions on different contexts have consistent marginals.

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- This condition is generalized by compatibility – and this general form is satisfied by quantum systems.

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Moreover, the PR box has a **unique family of distributions** which satisfy these conditions.

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The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

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We thus get a strict hierarchy of empirical models:

$$\mathsf{NC} \subset \mathsf{QM} \subset \mathsf{NS}$$

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This satisfies No-Signalling, so is consistent with SR, but it is **not** quantum realisable.

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Moreover, convex combinations of compatible models are compatible.

The Quantum Set

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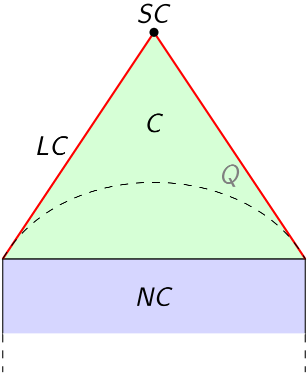

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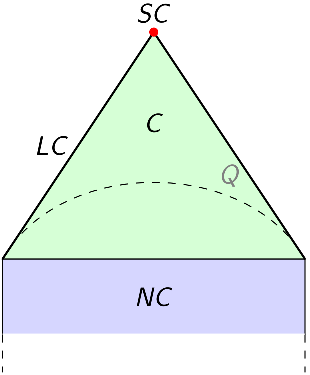

Key question: find compelling principles to explain why Nature picks out the quantum set.

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This is equivalent to saying that, for all $C \subseteq C'$, the restriction map

$$\rho^{C'}_{C} : \mathcal{S}(C') \longrightarrow \mathcal{S}(C) \quad :: s \mapsto s|_C$$

is surjective.

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Thus in terms of well-known quantum examples, we have

$$\text{Bell} < \text{Hardy} < \text{GHZ}$$