

Computational Algebraic Topology Topic B

Lecture III: Quantum Realizability

Samson Abramsky

Department of Computer Science
The University of Oxford

Brief review of Hilbert spaces

Hilbert space is a complex inner product space. There is a norm defined from the inner product, and the space has to be complete in this norm.

The salient notion of basis is **orthonormal basis**: a basis consisting of pairwise orthogonal unit vectors.

Up to isomorphism, **there is only one Hilbert space in each dimension**.

So for ordinary QM, the possibilities are (in principle) just \mathbb{C}^n and $\ell_2(\omega)$.

C^* algebras are an elegant algebraic approach, but not really more general: by the Gelfand-Naimark theorem, every C^* algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.

Quantum information mostly restricts consideration to finite dimensions: \mathbb{C}^n .

Finite dimensional linear algebra: isn't that trivial?

No!

Tensor Product

Compound systems in QM are represented by tensor products $\mathcal{H} \otimes \mathcal{K}$ of the corresponding Hilbert spaces \mathcal{H} and \mathcal{K} .

This is where Alice and Bob live!

If \mathcal{H} has ONB $\{\psi_i\}$ and \mathcal{K} has ONB $\{\phi_j\}$ then $\mathcal{H} \otimes \mathcal{K}$ has ONB $\{\psi_i \otimes \phi_j\}$.

If we represent qubit space with a standard basis $\{|0\rangle, |1\rangle\}$, then n -qubit space has basis

$$\{|s\rangle : s \in \{0, 1\}^n\}$$

Quantum Realizability

Quantum Mechanics has been axiomatized with sufficient precision (by von Neumann, c. 1932) to allow a precise definition of the class QM of quantum realizable empirical models for a given observational scenario.

The main ingredients:

- States are given by unit vectors in complex Hilbert space
- Dynamics are given by the Schrödinger equation, whose solutions are given by unitary maps on the Hilbert space.
- Observables are given by self-adjoint operators on the Hilbert space.
- The possible outcomes of an observable $A = \sum_i \lambda_i P_i$ are given by the eigenvalues λ_i .
- The probability of getting the outcome λ_i when measuring A on the state $|\psi\rangle$ is given by the **Born rule**:

$$|\langle e_i | \psi \rangle|^2.$$

Caveats

Quantum information has to consider **noisy** environments, hence unsharp measurements and preparations.

Thus one studies mixed rather than pure states (density operators rather than vectors), unsharp measurements (POVM's) rather than sharp (projective) measurements, etc.

However, one can always resort to a larger-dimensional Hilbert space, and recover mixed from pure states, unsharp from sharp measurements by tracing out the additional degrees of freedom.

Formally, this is underwritten by results such as the Stinespring Dilation theorem.

Informally, appeal to “the Church of the larger Hilbert space”.

We shall stick to the simplest level of presentation . . .

Operational Interpretation of QM

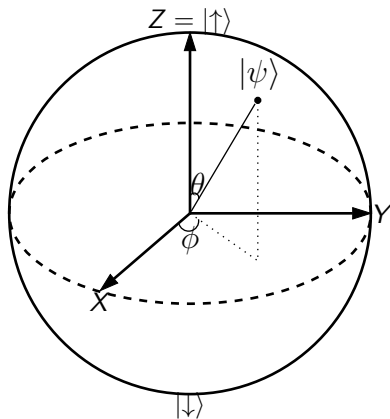
These mathematical structures are associated with **operational procedures** which can be performed in the lab (or observed in nature):

- Preparation procedures to produce quantum states
- Measurement devices: interferometers, photon detectors etc.
- Empirical probabilities of getting outcomes when measuring a state produced by preparation P with measurement device D .

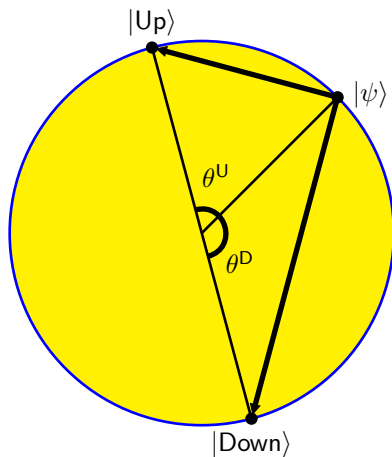
This leads to the study of **generalized probabilistic theories** as a means of studying the space of “possible physical theories” via their operational content.

Developments such as **device-independent QKD**.

The Bloch sphere representation of qubits



Truth makes an angle with reality



Properties of the Qubit

Note the following key features:

- States of the qubit are represented as points on the surface of the sphere. Note that there are a continuum of possible states.
- Each pair (Up, Down) of antipodal points on the sphere define a possible measurement that we can perform on the qubit. Each such measurement has two possible outcomes, corresponding to Up and Down in the given direction. We can think of this physically e.g. as measuring Spin Up or Spin Down in a given direction in space.
- When we subject a qubit to a measurement (Up, Down), the state of the qubit determines a probability distribution on the two possible outcomes. The probabilities are determined by the **angles** between the qubit state $|\psi\rangle$ and the points ($|\text{Up}\rangle, |\text{Down}\rangle$) which specify the measurement. In algebraic terms, $|\psi\rangle, |\text{Up}\rangle$ and $|\text{Down}\rangle$ are unit vectors in the complex vector space \mathbb{C}^2 , and the probability of observing Up when in state $|\psi\rangle$ is given by the square modulus of the inner product:

$$|\langle\psi|\text{Up}\rangle|^2.$$

This is known as the **Born rule**. It gives the basic predictive content of quantum mechanics.

Qubits vs. Bits

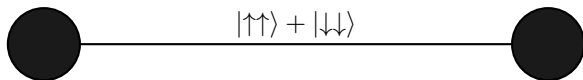
The sense in which the qubit generalises the classical bit is that, for each question we can ask — *i.e.* for each measurement — there are just two possible answers. We can view the states of the qubit as superpositions of the classical states 0 and 1, so that we have a probability of getting each of the answers for any given state.

But in addition, we have the important feature that there are a continuum of possible questions we can ask. However, note that on each run of the system, we can only ask **one** of these questions. We cannot simultaneously observe Up or Down in two different directions. Note that this corresponds to the feature of the scenario we discussed, that Alice and Bob could only look at one their local registers on each round.

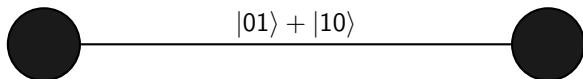
Note in addition that a measurement has an **effect** on the state, which will no longer be the original state $|\psi\rangle$, but rather one of the states Up or Down, in accordance with the measured value.

Quantum Entanglement

Bell state:



EPR state:



Compound systems are represented by **tensor product**: $\mathcal{H}_1 \otimes \mathcal{H}_2$. Typical element:

$$\sum_i \lambda_i \cdot \phi_i \otimes \psi_i$$

Superposition encodes **correlation**.

Einstein's 'spooky action at a distance'. Even if the particles are spatially separated, measuring one has an effect on the state of the other.

Bell's theorem: QM is **essentially non-local**.

A Probabilistic Model Of An Experiment

Example: The Bell Model

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
a_1	b_1	1/2	0	0	1/2
a_1	b_2	3/8	1/8	1/8	3/8
a_2	b_1	3/8	1/8	1/8	3/8
a_2	b_2	1/8	3/8	3/8	1/8

Important note: this is **physically realizable!**

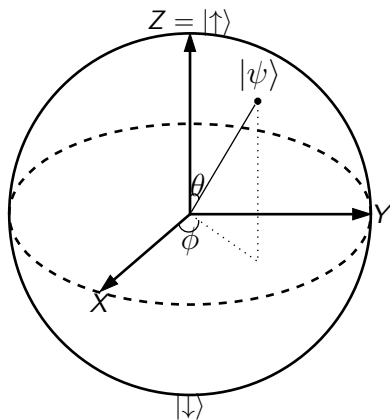
Generated by Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY -plane, at relative angle $\pi/3$.

Extensively tested experimentally.

Computing the Bell table



Spin measurements lying in the equatorial plane of the Bloch sphere

Spin Up: $(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)/\sqrt{2}$, Spin Down: $(|\uparrow\rangle + e^{i(\phi+\pi)}|\downarrow\rangle)/\sqrt{2}$

X itself, $\phi = 0$:

Spin Up $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ and Spin Down $(|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$.

Computing the Bell table

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
a	b	0	1/2	1/2	0
a'	b	3/8	1/8	1/8	3/8
a	b'	3/8	1/8	1/8	3/8
a'	b'	3/8	1/8	1/8	3/8

Alice: $a = X$, a' at $\phi = \pi/3$ (on **first** qubit)

Bob: $b = X$, b' at $\phi = \pi/3$ (on **second** qubit)

The event in yellow is represented by

$$\frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \otimes \frac{|\uparrow\rangle + e^{i4\pi/3}|\downarrow\rangle}{\sqrt{2}} = \frac{|\uparrow\uparrow\rangle + e^{i4\pi/3}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + e^{i4\pi/3}|\downarrow\downarrow\rangle}{2}.$$

Probability of this event M when measuring (a, b') on $B = (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)/\sqrt{2}$ is given by Born rule:

$$|\langle B|M\rangle|^2.$$

Computing Bell by Born

Since the vectors $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ are pairwise orthogonal, $|\langle B|M\rangle|^2$ simplifies to

$$\left| \frac{1 + e^{i4\pi/3}}{2\sqrt{2}} \right|^2 = \frac{|1 + e^{i4\pi/3}|^2}{8}.$$

Using the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$|1 + e^{i\theta}|^2 = 2 + 2\cos\theta.$$

Hence

$$\frac{|1 + e^{i4\pi/3}|^2}{8} = \frac{2 + 2\cos(4\pi/3)}{8} = \frac{1}{8}.$$

The other entries can be computed similarly.

Mysteries of the Quantum Representation

Operationally, we see readings on measurement instruments, and observe probabilities of outcomes.

We never “see” a complex number!

And yet, QM uses this representation in complex Hilbert spaces to compute the positive real numbers corresponding to what we actually observe.

What convincing explanation can we give for this?

Attempts to find compelling axioms from which the QM representation in complex Hilbert space can be derived.

Lucien Hardy, “Quantum Mechanics from five reasonable axioms”

Other attempts by Masanes and Mueller, Brukner and Dakic, the Pavia group (D’Ariano, Chiribella and Perinotti), . . .

Empirical Models

Example: The Bell Model

A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1/2	0	0	1/2
a_1	b_2	3/8	1/8	1/8	3/8
a_2	b_1	3/8	1/8	1/8	3/8
a_2	b_2	1/8	3/8	3/8	1/8

Important note: this is **quantum realizable**.

Generated by Bell state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

subjected to measurements in the XY -plane, at relative angle $\pi/3$.

The PR Box

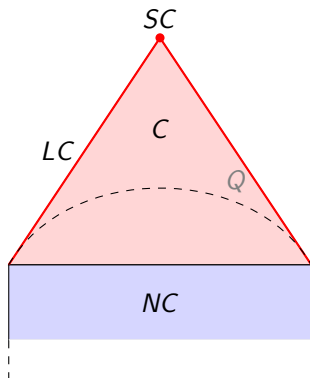
A	B	(0,0)	(1,0)	(0,1)	(1,1)
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

This satisfies No-Signalling, so is consistent with SR, but it is **not** quantum realisable.

The Quantum Set

A subtle convex set sandwiched between two polytopes.



Key question: find compelling principles to explain why Nature picks out the quantum set.

Quantum Realizations of Relational Models

A **quantum realization** of the n -partite Bell scenario $(M_1, \dots, M_n, O_1, \dots, O_n)$ is given by:

- Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$.
- For each $i = 1, \dots, n$, $m \in M_i$, and $o \in O_i$, a unit vector $\psi_{m,o}$ in \mathcal{H}_i , subject to the condition that the vectors $\{\psi_{m,o} : o \in O_i\}$ form an orthonormal basis of \mathcal{H}_i .
- A state ψ , i.e. a unit vector in $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$.

For each choice of measurement $\bar{m} \in M$, and outcome $\bar{o} \in O$, the usual 'statistical algorithm' of quantum mechanics defines a probability $p_{\bar{m}}(\bar{o})$ for obtaining outcome \bar{o} from performing the measurement \bar{m} on ρ :

$$p_{\bar{m}}(\bar{o}) = |\langle \psi | \psi_{\bar{m}, \bar{o}} \rangle|^2,$$

where $\psi_{\bar{m}, \bar{o}} = \psi_{\bar{m}_1, \bar{o}_1} \otimes \dots \otimes \psi_{\bar{m}_n, \bar{o}_n}$.

We take QM to be the class of empirical models which are realized by quantum systems in this fashion.

We take QM(d) to be the sub-class of models realisable in a Hilbert space of **finite** dimension d .

Quantum Realization of the Hardy Model

We consider the two-qubit system, with X_2 and Y_2 measurement in the computational basis. The eigenvectors for X_1 are taken to be

$$\sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle, \quad -\sqrt{\frac{2}{5}}|0\rangle + \sqrt{\frac{3}{5}}|1\rangle$$

and similarly for Y_1 . The state is taken to be

$$\sqrt{\frac{3}{8}}|10\rangle + \sqrt{\frac{3}{8}}|01\rangle - \frac{1}{2}|00\rangle.$$

One can then calculate the probabilities to be

$$p_{X_1 Y_2}(00) = p_{X_2 Y_1}(00) = p_{X_2 Y_2}(11) = 0,$$

and $p_{X_1 Y_1}(00) = 0.09$, which is very near the maximum attainable value.

The possibilistic collapse of this model is thus a Hardy model.

QM(d)

Proposition

The class QM(d) is in PSPACE. That is, there is a PSPACE algorithm to decide, given an empirical model, if it arises from a quantum system of dimension d .

Proof Outline

The condition for quantum realization of an empirical model can be written as the existence of a list of complex matrices satisfying some algebraic conditions. These can be written in terms of the entries of the matrices, and we can use the standard representation of complex numbers as pairs of reals.

The parameter d allows us to bound the dimensions of the matrices which need to be considered.

The whole condition can be written as an existential sentence $\exists v_1 \dots \exists v_k. \psi$, where ψ is a conjunction of atomic formulas in the signature $(+, 0, \times, 1, <)$, interpreted over the reals.

This fragment has PSPACE complexity (Canny). Moreover, the sentence can be constructed in polynomial time from the given empirical model. Hence membership of QM(d) is in PSPACE.

A Decision Problem

Can we bound the dimension d effectively, so that QM itself is decidable?

If we write

$$\text{QM}^{\text{fin}} := \bigcup_{d \in \mathbb{N}} \text{QM}(d),$$

then QM^{fin} is clearly **recursively enumerable** (r.e.).

Obviously $\text{QM}^{\text{fin}} \subseteq \text{QM}$.

The NPA Hierarchy

In “A convergent hierarchy of semidefinite programs characterising the set of quantum correlations” (2008), Navascues, Pironio and Acin gave a infinite hierarchy of conditions, expressed as semidefinite programs $\{P_n\}$, which could be used to test whether a given bipartite model was in QM.

This seems related to the **Lasserre hierarchy**, a very hot topic currently in optimisation and complexity.

Each successive semidefinite program P_n runs in polynomial time in its given program size (which grows ...).

Essentially, at level n we are looking at conditions on n -fold products of the projectors which would witness the quantum realizability of the model.

They prove that this sequence of tests is **complete**, in the sense that:

- if a model passes all the tests, it is in QM.
- If it fails any of the tests, it is **not** in QM.

This shows that QM is **co-r.e.** — the complement of an r.e. set.

In particular, the proof that there is a quantum realisation in the limit uses an **infinite-dimensional** Hilbert space.

Questions

Note firstly that if $\text{QM}^{\text{fin}} = \text{QM}$, then this set is **decidable** by the foregoing results.

Indeed, NPA show that the model admits a finite-dimensional quantum realisation if and only if a certain condition (a “rank loop”) holds at some finite level of the hierarchy.

Conjecture: the converse also holds.

Note that $\text{QM}^{\text{fin}} \neq \text{QM}$ has a clear **physical** significance: it is saying that to realise **finite** quantum correlations, in general we need **infinitely many degrees of freedom** in the physical system.

Thus we appear to have an equivalence between:

- a purely mathematical decision problem, and
- a question with a clear physical and operational content.

Tobias Fritz has pointed out interesting connections with the Kirschberg QWEP conjecture and the Connes Embedding Problem.

Other questions: e.g. generalise the NPA hierarchy to arbitrary measurement scenarios.

LHV is in NP

In order to simplify notation, we shall consider possibilistic empirical models of the form (U, e) , where $e \subseteq U^n \times U^n$. Thus we use the same underlying set U for both measurements and outcomes at each site.

We shall write $HV(n)$ for the class of models of this form which has a local hidden variable realisation (*i.e.* a boolean global section). We are interested in the algorithmic problem of determining if a structure (U, e) of arity n is in $HV(n)$.

Proposition

For each n , $HV(n)$ is in NP.

Proof

From the previous Proposition, it is clear that $HV(n)$ is defined by the following second-order formula interpreted over finite structures (U, e) :

$$\forall \vec{x}. \exists \vec{y}. R(\vec{x}, \vec{y}) \wedge \forall \vec{x}, \vec{y}. R(\vec{x}, \vec{y}) \rightarrow \exists f_1, \dots, f_n. \bigwedge_i f_i(x_i) = y_i \wedge \forall \vec{v}. R(\vec{v}, f(\vec{v})).$$

By standard quantifier manipulations, this can be brought into an equivalent Σ_1^1 form, and hence $HV(n)$ is in NP. □

Robust CSP

Samson Abramsky, Georg Gottlob and Phokion Kolaitis, 'Robust Constraint Satisfaction and Local Hidden Variables in Quantum Mechanics', in Proceedings of IJCAI 2013.

- Robust CSP: can every **consistent partial assignment** of a certain length be extended to a solution?
- Special cases studied previously by Beacham and Gottlob.
- Main results: ROBUST 3-COLOURABILITY and ROBUST 2-SAT are NP-complete.
- These are used to show that HV(n), $n > 2$, is NP-complete; smaller instances are in PTIME.
- The robust paradigm is an interesting and non-trivial extension of current theory, and worthy of further study.