Computational Algebraic Topology Topic B: Sheaf cohomology and applications to quantum non-locality and contextuality
Lecture 4

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Presheaves

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$$\rho^V_U : P(V) \longrightarrow P(U)$$

subject to the functoriality requirements: if $U \subseteq V \subseteq W$, then

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Functoriality is easily verified: in this notation

$$(f|_V)_U = f|_U.$$

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- **Morphisms** of presheaves are just natural transformations.
- The category of all presheaves on a space X has a very rich structure — it is a **topos**. We shall not go into this aspect.
- However, there is an important conceptual aspect which should be understood. Presheaves allow us to formalise the concept of **variable set**. The variation is essentially over **contexts**. So presheaves provide the natural setting for talking about contextuality!

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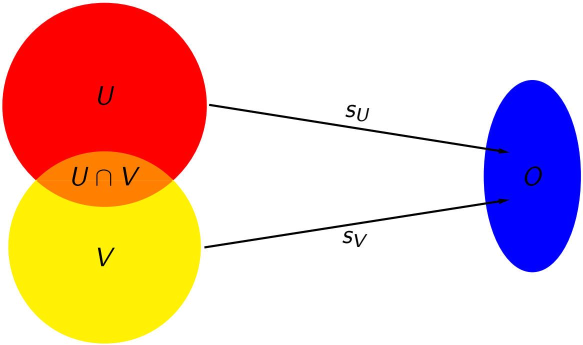
The presheaf P is a **sheaf** if for every open cover $\mathcal{U}$, it satisfies the sheaf condition for $\mathcal{U}$.

Gluing functional sections



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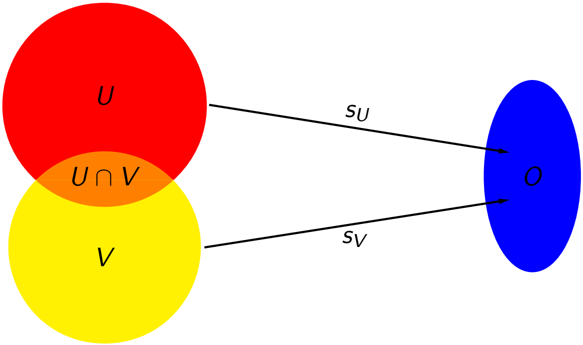

If $s_U|_{U \cap V} = s_V|_{U \cap V}$, they can be glued to form

$$s : U \cup V \longrightarrow O$$

such that $s|_U = s_U$ and $s|_V = s_V$.

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In particular, this is one of the main intuitions behind **sheaf cohomology**.

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Each row of the table specifies a **probability distribution** on events O^C for a given choice of measurements C.

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The fact that the behaviour of these observable outcomes cannot be accounted for by some context-independent global description of reality corresponds to the geometric fact that these local sections cannot be glued together into a **global section**.

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The quantum phenomena of **non-locality** and **contextuality** correspond exactly to the existence of obstructions to global sections in this sense.

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Features in **tropical geometry** (the **max-plus** semiring).

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We can compose this functor with $U \mapsto O^U$, to form a presheaf $\mathcal{F} : \mathcal{P}(X)^{\mathsf{op}} \longrightarrow \mathbf{Set}$.

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$$\sum_{x \in X} \phi(x) = 1.$$

We write $\mathcal{D}_R(X)$ for the set of R-distributions on X.
Examples: $\mathbb{R}_{\geq 0}$ (probability distributions), $\mathbb{B}$ (non-empty subsets), $\mathbb{R}$ (signed measures).

Functorial action: Given a function $f : X \to Y$, we define

$$\mathcal{D}_R(f) : \mathcal{D}_R(X) \to \mathcal{D}_R(Y) :: d \mapsto [y \mapsto \sum_{f(x)=y} d(x)].$$

This yields a functor $\mathcal{D}_R : \mathbf{Set} \longrightarrow \mathbf{Set}$.

We can compose this functor with $U \mapsto O^U$, to form a presheaf $\mathcal{F} : \mathcal{P}(X)^{\mathsf{op}} \longrightarrow \mathbf{Set}$.

Contextual Probability Theory!

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A setting for **contextual probability**.

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E.g. in the bipartite case, consider $C = \{m_a, m_b\}$, $C' = \{m_a, m'_b\}$.
Fix $s_0 \in \mathcal{E}(\{m_a\})$. Compatibility implies

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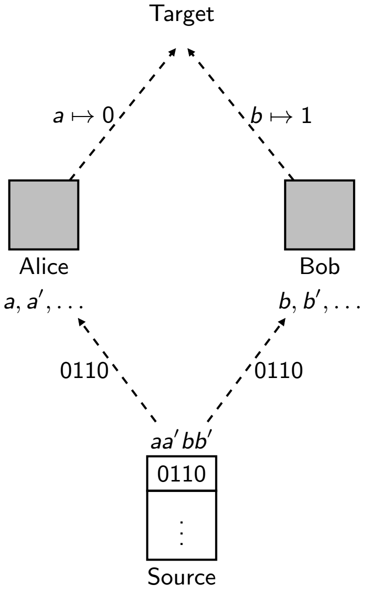
In other words, Bob's choice of measurement cannot influence Alice's outcome.

Hidden Variables: The Mermin instruction set picture



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If d is a global section for the model $\{e_C\}$, we recover the predictions of the model by **averaging over the values of these hidden variables**:

$$e_C(s) \;=\; d|C(s) \;=\; \sum_{s' \in \mathcal{E}(X), s'|C = s} d(s') \;=\; \sum_{s' \in \mathcal{E}(X)} \delta_{s'|C}(s) \cdot d(s').$$

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Any factorizable (i.e. local) hidden-variable model defines a global section.

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So:

existence of a local hidden-variable model for a given empirical model
IFF
empirical model has a global section

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So a global section **is** a deterministic local hidden-variable model.

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Hence:

No such h.v. model exists (the empirical model is **non-local**/**contextual**)
IFF
there is an **obstruction to the existence of a global section**

Methods for showing obstructions to global sections

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(a_1, b_1)	1	1	1	1
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The PR Box

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

	00	01	10	11
ab	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
ab'	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
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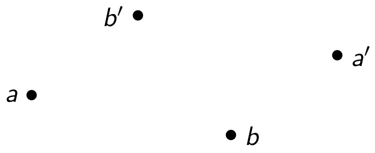

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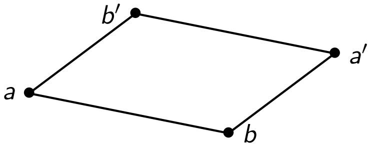

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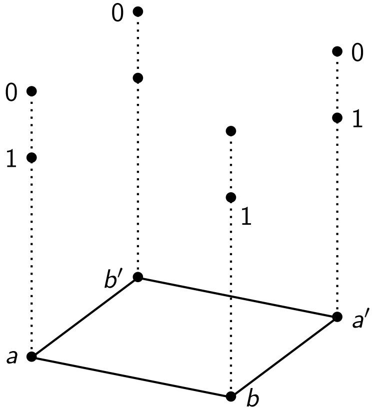

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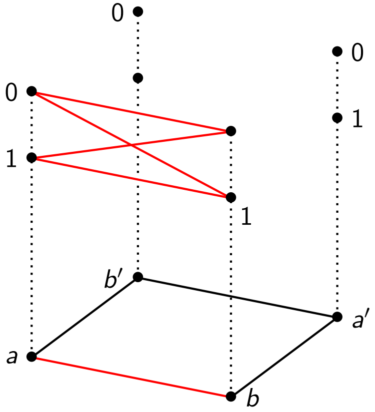

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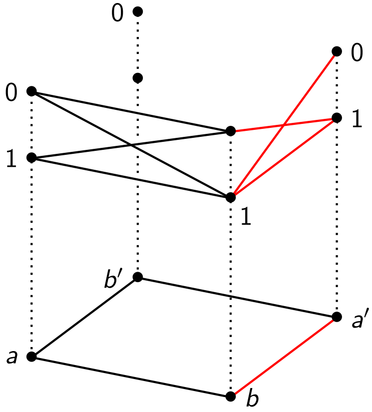

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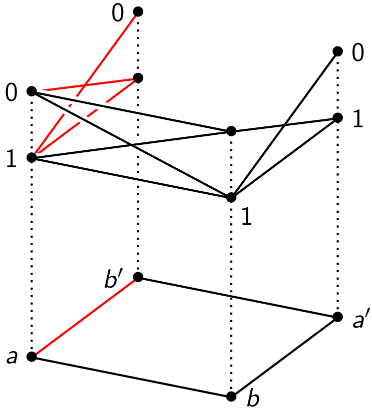

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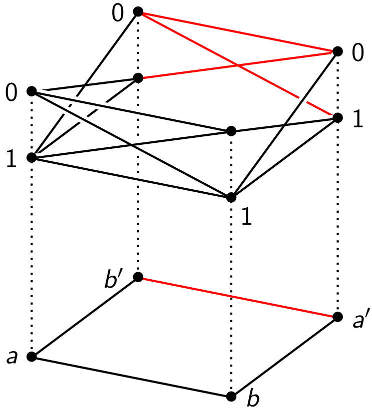

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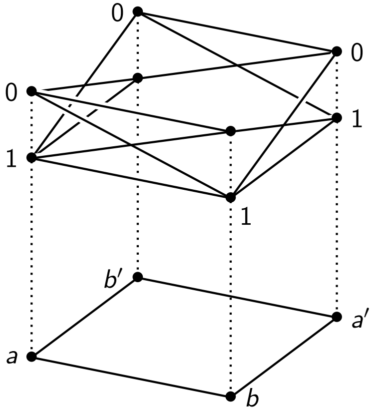

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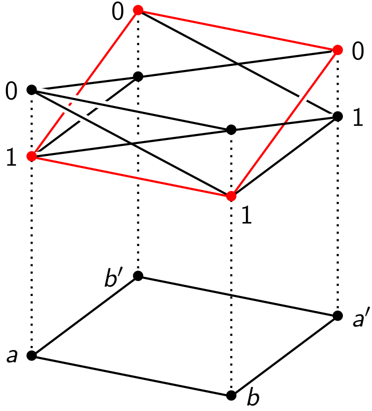

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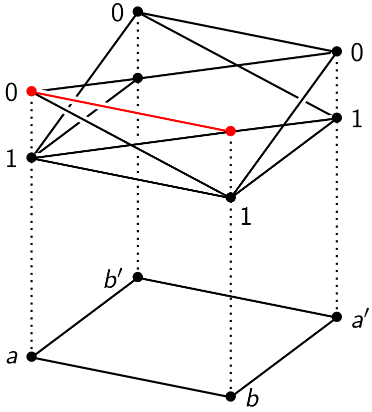

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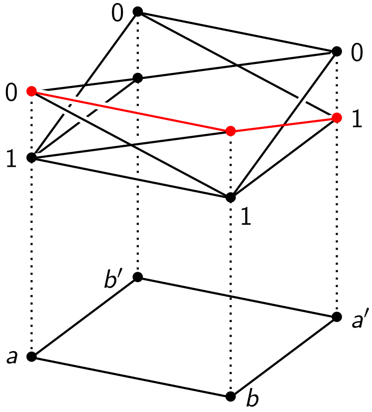

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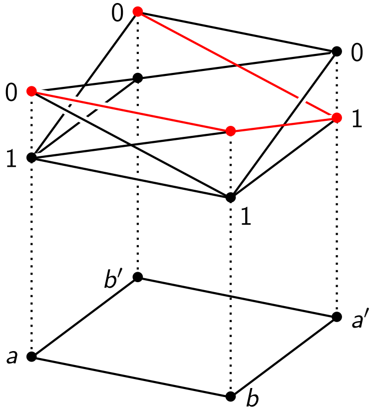

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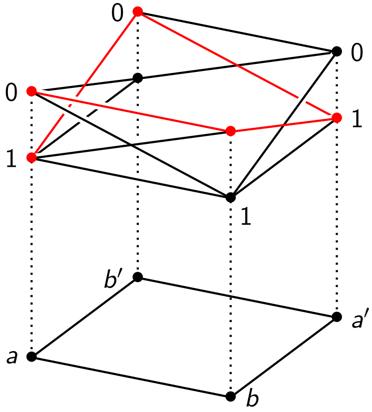

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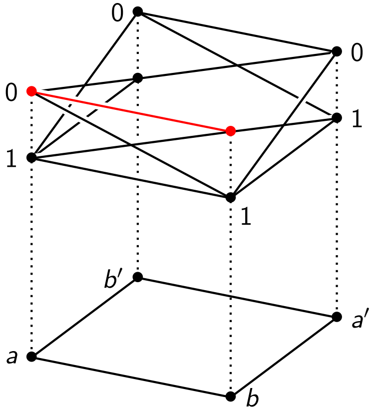

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

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ab'	×	✓	✓	✓
$a'b$	×	✓	✓	✓
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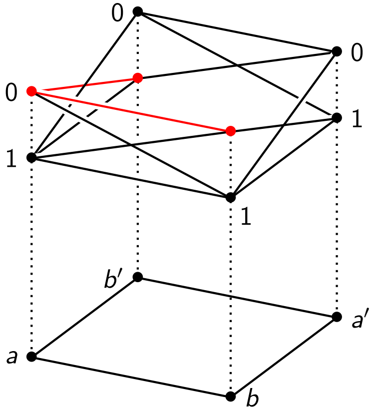

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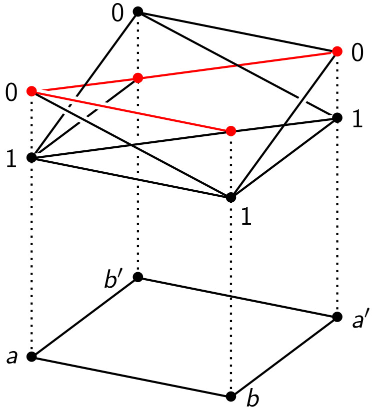

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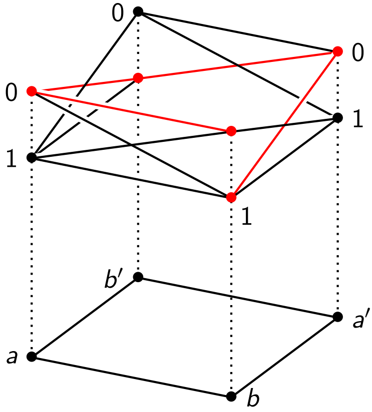

Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Bundle Pictures

Strong Contextuality

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	00	01	10	11
ab	✓	×	×	✓
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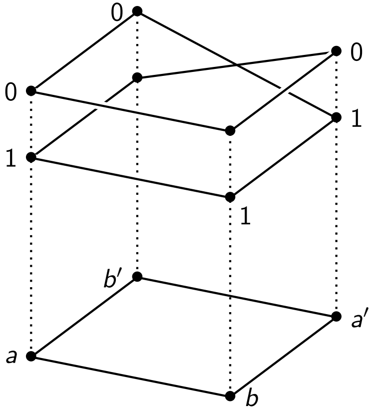
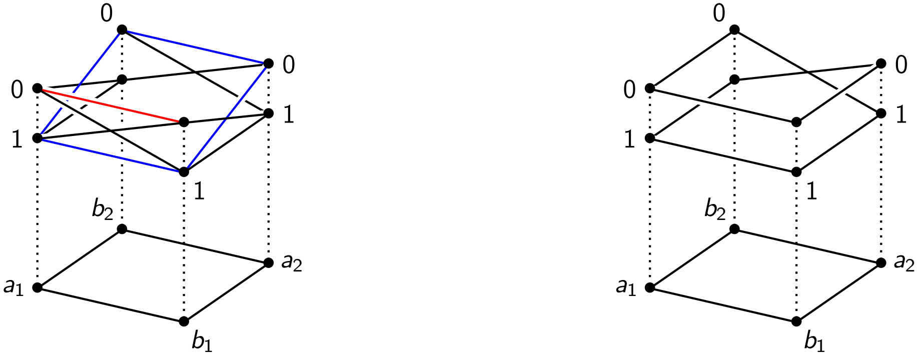


Visualizing Contextuality



The Hardy table and the PR box as bundles

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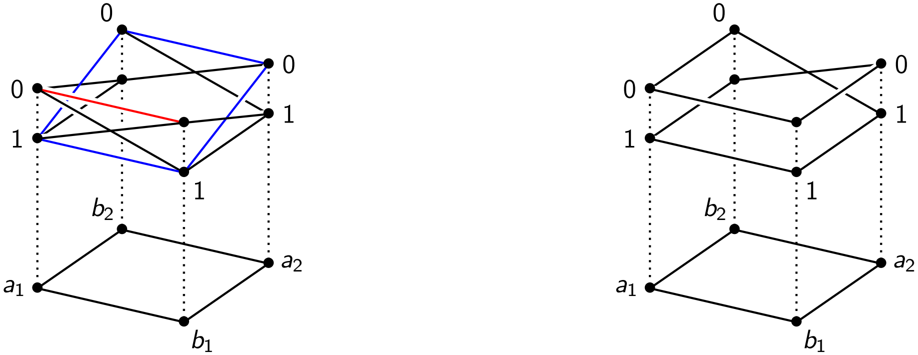

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A hierarchy of degrees of contextuality:

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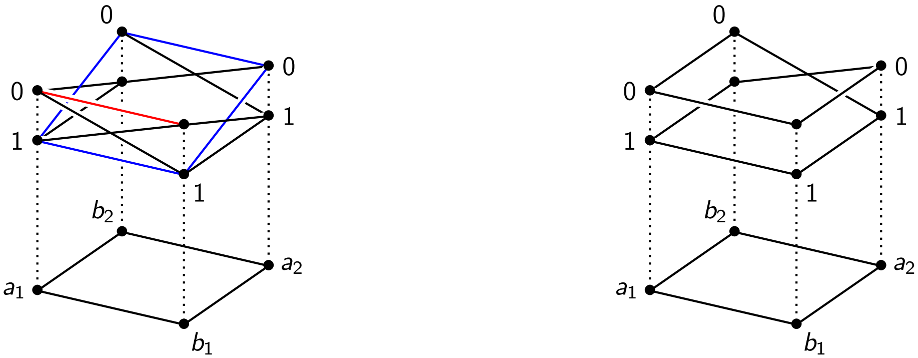

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Indexed family of sets $\{X_i\}_{i \in I}$.

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Sheaves on X are equivalently formulated as continuous maps $p : Y \longrightarrow X$ which are **local homeomorphisms** (*espaces étalé*).

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Thus in terms of well-known examples, we have

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