Computational Algebraic Topology Topic B: Sheaf cohomology and applications to quantum non-locality and contextuality
Lecture 5

Samson Abramsky

Department of Computer Science, University of Oxford

Introduction

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We show that for a number of salient examples, including PR boxes, GHZ states, and the 18-vector configuration due to Cabello et al. giving a proof of the Kochen-Specker theorem in four dimensions, the obstruction does not vanish, thus yielding cohomological witnesses for contextuality.

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Based on:

- S. Abramsky and A. Brandenburger, The Sheaf-Theoretic Structure of Non-Locality and Contextuality. *New Journal of Physics*, 13(2011), 113036, 2011.
- S. Abramsky, S. Mansfield and R. Soares Barbosa, The Cohomology of Non-Locality and Contextuality, in *Proceedings of QPL 2011*, Electronic Proceedings in Theoretical Computer Science, 2011.

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We work over a finite discrete space X, which we think of as a set of **measurement labels**. We fix a finite cover $\mathcal{U}$, with $\bigcup \mathcal{U} = X$, which represents the set of **compatible families of measurements**, *i.e.* those which can be made jointly. Fixing a finite set O of **outcomes**, we have the presheaf of sets $\mathcal{E}$ on X, where $\mathcal{E}(U) := O^U$, and restriction is simply function restriction: given $U \subseteq U'$,

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An empirical model e is a compatible family $\{e_C\}_{C \in \mathcal{U}}$, where e_C is a probability distribution on $\mathcal{E}(C)$. The support of e determines a sub-presheaf S_e of $\mathcal{E}$:

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Here $e_U = e_C|U$ for any $C \in \mathcal{U}$ such that $U \subseteq C$. The compatibility of the family $\{e_C\}$ ensures that this is independent of the choice of C.

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The results from AB show that if a model is local or non-contextual in the usual sense, then it is possibilistically extendable. Thus possibilistic non-extendability is a sufficient condition for **non-locality** or **contextuality**. Strong contextuality is a much stronger condition. Thus these properties witness strong forms of the non-classical behaviour exhibited by quantum mechanics.

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The **nerve** $\mathrm{N}(\mathcal{U})$ of the cover $\mathcal{U}$ is defined to be the abstract simplicial complex comprising those finite subsets of $\mathcal{U}$ with non-empty intersection. Concretely, we take a q-simplex to be a list $\sigma = (C_0, \ldots, C_q)$ of elements of $\mathcal{U}$, with $|\sigma| := \cap_{j=0}^{q} C_j \neq \varnothing$. Thus a 0-simplex (C) is a single element of the cover $\mathcal{U}$. We write $\mathrm{N}(\mathcal{U})^q$ for the set of q-simplices.

Cochains and coboundaries

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Given a $q+1$-simplex $\sigma = (C_0, \ldots, C_{q+1})$, there are q-simplices

$$\partial_j(\sigma) := (C_0, \ldots, \widehat{C}_j, \ldots, C_{q+1}), \qquad 0 \leq j \leq q$$

obtained by omitting one of the elements of the $q+1$-simplex. Note that:

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We shall now define the **Čech cochain complex**. For each $q \geq 0$, we define the abelian group $C^q(\mathcal{U}, \mathcal{F})$:

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We also define the **coboundary maps**

$$\delta^q : C^q(\mathcal{U}, \mathcal{F}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{F}).$$

For $\omega = (\omega(\tau))_{\tau \in \mathsf{N}(\mathcal{U})^q} \in C^q(\mathcal{U}, \mathcal{F})$, and $\sigma \in \mathsf{N}(\mathcal{U})^{q+1}$, we define:

$$\delta^q(\omega)(\sigma) \ := \ \sum_{j=0}^{q} (-1)^j \rho^{|\partial_j(\sigma)|}_{|\sigma|} \omega(\partial_j \sigma).$$

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We define the q-**th Čech cohomology group** $\check{H}^q(\mathcal{U}, \mathcal{F})$ to be the quotient group $Z^q(\mathcal{U}, \mathcal{F})/B^q(\mathcal{U}, \mathcal{F})$.

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Given a cocycle $z \in Z^q(\mathcal{U}, \mathcal{F})$, the **cohomology class** $[z]$ is the image of z under the canonical map

$$Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^q(\mathcal{U}, \mathcal{F}).$$

Compatible families

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A **compatible family** with respect to a cover $\mathcal{U} = \{C_1, \ldots, C_n\}$ is a family $\{r_i \in \mathcal{F}(C_i)\}_{i=1}^n$, such that, for all i, j:

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Proof Cochains $c = (r_i)_{C_i \in \mathcal{U}}$ in $C^0(\mathcal{U}, \mathcal{F})$ correspond to families $\{r_i \in \mathcal{F}(C_i)\}$. For each 1-simplex $\sigma = (C_i, C_j)$,

$$\delta^0(c)(\sigma) \;=\; r_i|C_i \cap C_j \;-\; r_j|C_i \cap C_j.$$

Hence $\delta^0(c) = 0$ if and only if the corresponding family is compatible. □

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Then $\mathcal{F}_{\bar{U}}$ is defined by $\mathcal{F}_{\bar{U}}(V) := \ker(p_V)$. Thus we have an exact sequence of presheaves

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The relative cohomology of $\mathcal{F}$ with respect to U is defined to be the cohomology of the presheaf $\mathcal{F}_{\bar{U}}$.

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We have the following refined version of Proposition 2.

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For any $C_i \in \mathcal{U}$, the elements of the relative cohomology group $\check{H}^0(\mathcal{U}, \mathcal{F}_{\bar{C}_i})$ correspond bijectively to compatible families $\{r_j\}$ such that $r_i = 0$.

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Proof By the previous Proposition, compatible families correspond to cocycles $r = (r_j)$ in $C^0(\mathcal{U}, \mathcal{F})$. By compatibility, $r_i | C_i \cap C_j = r_j | C_i \cap C_j$ for all j. Hence r is in $C^0(\mathcal{U}, \mathcal{F}_{\bar{U}_i})$ if and only if $r_i = p_{U_i}(r_i) = 0$. □

Application to our setting

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Given a commutative ring R, we define a functor $F_R : \mathbf{Set} \longrightarrow \mathbf{Set}$. For any set X, the **support** $\mathrm{supp}(\phi)$ of a function $\phi : X \to R$ is the set of $x \in X$ such that $\phi(x) \neq 0$. We define $F_R(X)$ to be the set of functions $\phi : X \to R$ of finite support. There is an embedding $x \mapsto 1 \cdot x$ of X in $F_R(X)$, which we shall use implicitly throughout.

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Given $f : X \to Y$, we define:

$$F_R f : F_R X \longrightarrow F_R Y :: \phi \mapsto [y \mapsto \sum_{f(x)=y} \phi(x)].$$

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In fact, $F_R(X)$ is the **free** R**-module generated by** X, and in particular, it is an abelian group; while $F_R(f)$ is a group homomorphism for any function f. In particular, taking $R = \mathbb{Z}$, $F_{\mathbb{Z}}(X)$ is the **free abelian group generated by** X.

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Given an empirical model e defined on the cover $\mathcal{U}$, we shall work with the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{F})$ for the abelian presheaf $\mathcal{F} := F_{\mathbb{Z}} S_e$. Note that, for any set of measurements U, $\mathcal{F}(U)$ is the set of **formal $\mathbb{Z}$-linear combinations of sections** in the support of e_U.

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To each $s \in S_e(C)$, we shall associate an element $\gamma(s)$ of a cohomology group, which can be regarded as an obstruction to s having an extension within the support of e to a global section. In particular, the existence of such an extension implies that the obstruction vanishes, yielding **cohomological witnesses** for contextuality and strong contextuality.

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For notational convenience, we shall fix an element $s = s_1 \in S_e(C_1)$. Because of the compatibility of the family $\{e_C\}$, there is a family $\{s_i \in S_e(C_i)\}$ with $s_1|C_1 \cap C_i = s_i|C_1 \cap C_i$, $i = 2, \ldots, n$.

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To each $s \in S_e(C)$, we shall associate an element $\gamma(s)$ of a cohomology group, which can be regarded as an obstruction to s having an extension within the support of e to a global section. In particular, the existence of such an extension implies that the obstruction vanishes, yielding **cohomological witnesses** for contextuality and strong contextuality.

For notational convenience, we shall fix an element $s = s_1 \in S_e(C_1)$. Because of the compatibility of the family $\{e_C\}$, there is a family $\{s_i \in S_e(C_i)\}$ with $s_1|C_1 \cap C_i = s_i|C_1 \cap C_i$, $i = 2, \ldots, n$.

We define the cochain $c := (s_1, \ldots, s_n) \in C^0(\mathcal{U}, \mathcal{F})$. The coboundary of this cochain is $z := \delta^0(c)$.

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Proposition

The coboundary z of c vanishes under restriction to C_1, and hence is a cocycle in the relative cohomology with respect to C_1.

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Note that $\delta^1 : C^1(\mathcal{U}, \mathcal{F}_{\bar{C}_1}) \to C^2(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$ is the restriction of the coboundary map on $C^1(\mathcal{U}, \mathcal{F})$. Hence $z = \delta^0(c)$ is a cocycle. □

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We define $\gamma(s_1)$ as the cohomology class $[z] \in \check{H}^1(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$.

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Thus in general, we need not have $[z] = 0$.

Key Property of the Obstruction

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Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a family* $\{r_i \in \mathcal{F}(C_i)\}$ *with* $s_1 = r_1$*, and for all* i, j*:*

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Proof The obstruction vanishes if and only if there is a cochain $c' = (c'_1, \ldots, c'_n) \in C^0(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$ with $\delta^0(c') = \delta^0(c)$, or equivalently $\delta^0(c - c') = 0$, *i.e.* such that $c - c'$ is a cocycle.

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By Proposition 2, this is equivalent to $\{r_i := s_i - c'_i\}$ forming a compatible family. Moreover, $c' \in C^0(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$ implies $c'_1 = p_{C_1}(c'_1) = 0$, so $r_1 = s_1$.

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For the converse, suppose we have a family $\{r_i \in \mathcal{F}(C_i)\}$ as in (2).
We define $c' := (c'_1, \ldots, c'_n)$, where $c'_i := s_i - r_i$.
Since $r_1 = s_1$, $p_{C_i}(c'_i) = s_1|C_{1,i} - r_1|C_{1,i} = 0$ for all i, and $c' \in C^0(\mathcal{U}, \mathcal{F}_{\bar{C}_1})$.
We must show that $\delta^0(c') = z$, *i.e.* that $z_{i,j} = c'_i|C_{i,j} - c'_j|C_{i,j}$. This holds since $r_i|C_{i,j} = r_j|C_{i,j}$. □

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If the model e is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If e is not strongly contextual, then the obstruction vanishes for some section in the support.

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Thus we have a *sufficient condition* for contextuality in the non-vanishing of the obstruction.

False Positives

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We shall begin, however, with a false positive.

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	1	0	0	0
(A,B')	0	1	0	0
(A',B)	0	1	1	1
(A',B')	1	1	1	0

- Possibilistically non-local
- Not strongly contextual
- The section $(A,B) \to (0,0)$ witnesses non-locality
- All other sections belong to compatible families of sections

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	s_1	s_2	s_3	s_4
(A,B')	0	s_6	s_7	s_8
(A',B)	0	s_{10}	s_{11}	s_{12}
(A',B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{aligned} r_2|A &= 1\cdot(A \mapsto 0) + 1\cdot(A \mapsto 1) - 1\cdot(A \mapsto 1) &= r_1|A, \\ r_2|B' &= 1\cdot(B' \mapsto 1) + 1\cdot(B' \mapsto 0) - 1\cdot(B' \mapsto 1) &= r_4|B' \end{aligned}$$

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The Hardy Model

- $\gamma(s_1)$ vanishes!
- This example illustrates that false positives do arise
- The cohomological obstruction does not show the non-locality of the Hardy model

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A, B)$	a	0	0	b
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$$r_1|C_{1,2} = r_2|C_{1,2} \quad \longrightarrow \quad a = c \qquad b = d$$

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$$r_2|C_{2,4} = r_4|C_{2,4} \quad \longrightarrow \quad c = h \qquad d = g$$

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Other Examples

The cohomology approach witnesses strong contextuality in a number of other well-known examples:

- GHZ model
- Peres-Mermin Square
- 18-vector Kochen-Specker model
- Other KS-type models

GHZ

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The support for (the relevant part of) GHZ is as follows:

	000	001	010	011	100	101	110	111
ABC	1	0	0	1	0	1	1	0
$AB'C'$	0	1	1	0	1	0	0	1
$A'BC'$	0	1	1	0	1	0	0	1
$A'B'C$	0	1	1	0	1	0	0	1

Equational form

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We display the coefficients for a candidate family as follows:

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$A'B'C$	0	m	n	0	o	0	0	p

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The constraints arising from the requirements that $r_i|C_{i,j} = r_j|C_{i,j}$ are:

$$
\begin{array}{rclcrcl}
a+b & = & e+f & \quad & c+d & = & g+h \\
a+c & = & i+k & & b+d & = & j+l \\
a+d & = & n+o & & b+c & = & m+p \\
f+g & = & j+k & & e+h & = & i+l \\
e+g & = & m+o & & f+h & = & n+p \\
i+j & = & m+n & & k+l & = & o+p
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All cases for GHZ have been machine-checked in mod 2 arithmetic, and it has been confirmed that the cohomology obstruction witnesses the impossibility of extending any section in the support to all measurements; thus *cohomology witnesses the strong contextuality of GHZ*.

Kochen-Specker-type Models

- In a Kochen-Specker problem, we wish to assign the outcome 1 to a single measurement in each context
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	1000	0100	0010	0001
$ABCD$	a	b	c	d
$AEFG$	a	e	f	g
$HICJ$	h	i	c	j
$HKGL$	h	k	g	l
$BEMN$	b	e	m	n
$IKNO$	i	k	n	o
$PQDJ$	p	q	d	j
$PRFL$	p	r	f	l
$QRMO$	q	r	m	o

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$$
\begin{aligned}
b + c + d &= e + f + g \\
a + b + d &= h + i + j \\
a + c + d &= e + m + n \\
a + b + c &= p + q + j \\
a + f + g &= b + m + n \\
a + e + f &= h + k + l \\
a + e + g &= p + r + l \\
i + c + j &= k + g + l \\
h + c + j &= k + n + o \\
h + i + c &= p + q + d \\
h + g + l &= i + n + o \\
h + k + g &= p + r + f \\
b + e + n &= q + r + o \\
b + e + m &= i + k + o \\
i + k + n &= q + r + m \\
q + d + j &= r + f + l \\
p + d + j &= r + m + o \\
p + f + l &= q + m + o
\end{aligned}
$$

A Class of KS-type Models

Proposition (Abramsky-Brandenburger)

A necessary condition for Kochen-Specker-type models to have a global section is:

$$\gcd\{d_m \mid m \in X\} \quad \mid \quad |\mathcal{U}|,$$

where $d_m := |\{C \in \mathcal{U} \mid m \in C\}|$

Corollary

All models that do not satisfy the above condition are therefore strongly contextual

A Class of KS-type Models

Proposition (AMB)

If $\gamma(s)$ vanishes for some section s in the support of a connected Kochen-Specker-type model, then the GCD condition holds for that model

Corollary

The vanishing of the cohomological obstruction is a complete invariant for the non-locality/contextuality of any connected KS-type model that violates the GCD condition

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  We would like to use the machinery of homological algebra and exact sequences to obtain more conceptual and general results.

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  We would like to use the machinery of homological algebra and exact sequences to obtain more conceptual and general results.
- Use additional structure of cohomology: products, Steenrod squares etc. to create refined invariants of quantum mechanical behavior.
- See if cohomology can be applied to entanglement classes to study the structure of multipartite quantum entanglement, and to develop new invariants of quantum entanglement.