Computational Algebraic Topology Topic B: Sheaf cohomology and applications to quantum non-locality and contextuality
Lecture 4

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The Support Presheaf

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This latter case, with which the logical and strong forms of contextuality are concerned, can equivalently be represented by a subpresheaf $\mathcal{S}$ of $\mathcal{E}$, where for each context $U \subseteq X$, $\mathcal{S}(U) \subseteq O^U$ is the set of all possible outcomes.

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Explicitly, $\mathcal{S}$ is defined as follows, where $\mathsf{supp}\left(p_C|_{U \cap C}\right)$ is the support of the marginal of p_C at $U \cap C$.

$$\mathcal{S}(U) := \left\{ s \in O^U \;\middle|\; \forall C \in \mathcal{M}.\; s|_{U \cap C} \in \mathsf{supp}\left(p_C|_{U \cap C}\right) \right\}$$

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Can be specified axiomatically. In particular, "flasque below the cover" corresponding to No-Signalling/compatibility.

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Definition

For any empirical model $\mathcal{S}$:

- For all $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\mathcal{S}$ is logically contextual at s, written $\mathsf{LC}(\mathcal{S}, s)$, if s is not a member of any compatible family. $\mathcal{S}$ is **logically contextual**, written $\mathsf{LC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for some s.
- $\mathcal{S}$ is **strongly contextual**, written $\mathsf{SC}(\mathcal{S})$, if $\mathsf{LC}(\mathcal{S}, s)$ for all s. Equivalently, it is strongly contextual if it has no global section, *i.e.* if $\mathcal{S}(X) = \varnothing$.

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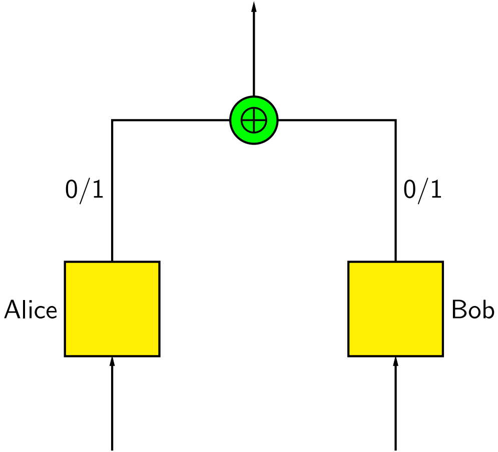
Motivation:

- Understand where AvN sits in the hierarchy of contextuality properties
- Characterise the quantum states which give rise to maximal degrees of non-locality/contextuality.

The XOR Game



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$$\{X_1, Y_1, X_2, Y_2, X_3, Y_3\} \longrightarrow \{+1, -1\}$$

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The **expected values** of these measurements give information about the **parity** of the support.

The Pauli Operators

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The 1-qubit operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Relations:

$$X^2 = Y^2 = Z^2 = I$$

$$XY = iZ, \quad YZ = iX, \quad ZX = iY,$$

$$YX = -iZ, \quad ZY = -iX, \quad XZ = -iY$$

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However, this can never be the case for any assignment

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Clearly, these are **inconsistent**.

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Proposition

If an empirical model e is AvN, *then it is strongly contextual.*

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A Galois correspondence between Pauli operators and states/vectors in the Hilbert space $\mathbb{C}^n$:

$$gRv \iff gv = v.$$

Closure operators on sets of group elements and of vectors:

$$S^\perp := \{v \mid \forall g \in S.\, gRv\}, \qquad V^\perp := \{g \mid \forall v \in V.\, gRv\}.$$

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The subgroups of $\mathcal{P}_n$ which stabilise non-trivial subspaces must be commutative, and only contain elements with global phases ± 1.

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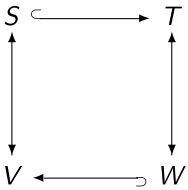

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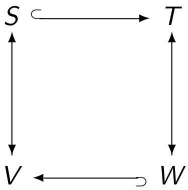

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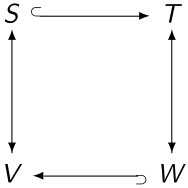

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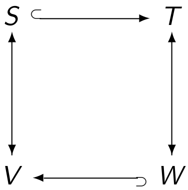

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Note that the correspondence is tight: a rank k subgroup determines a dimension 2^{n-k} subspace.

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Question:

How can we characterise when this happens?

AvN Triples

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Define an AvN **triple** in $\mathcal{P}_n$ to be (e, f, g) (order is important) with global phases $+1$, which pairwise commute, and additionally satisfy the following conditions:

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On the other hand, condition (A1) implies that under any global assignment/section on the variables, we can cancel the repeated items in each column, and deduce an even parity for h.

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Note that the conditions imply that the product $e.f.g = -h$, which translates into a condition of odd parity on the support of any state stabilised by these operators for the measurement h.

On the other hand, condition (A1) implies that under any global assignment/section on the variables, we can cancel the repeated items in each column, and deduce an even parity for h.

This means that any state in V_S, where S is the subgroup generated by $\{e, f, g\}$, admits an AvN argument. Note that this is a 2^{n-3}-dimensional space, assuming e, f, g are independent.

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Example from Mermin, yielding a GHZ argument:

$$\begin{array}{ccc} X & Y & Y \\ Y & X & Y \\ Y & Y & X \end{array}$$

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\begin{array}{rr}
a_0 + 2b_0 \equiv 0 \bmod 3 & a_1 + 2c_0 \equiv 0 \bmod 3 \\
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- In fact, the ring structure is the essential ingredient.
- So, consider **any commutative ring** R.

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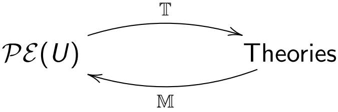
- A set of assignments $S \subseteq \mathcal{E}(C)$ determines an R-linear theory, $\mathbb{T}_R(S) := \{\phi \mid \forall s \in S.\; s \models \phi\}$.
- A system of equations Γ has a set of satisfying assignments, $\mathbb{M}(\Gamma) := \{s \in \mathcal{E}(C) \mid \forall \phi \in \Gamma.\; s \models \phi\}$.



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The model $\mathcal{S}$ is AvN_R if $\mathbb{T}_R(\mathcal{S})$ is **inconsistent**, meaning there is **no** global assignment $g \colon X \longrightarrow R$ consistent with the eqs:

$$\forall C.\ g|_C \models \mathbb{T}_R(\mathcal{S}(C)).$$

Affine Closure

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This means that

$$\mathsf{aff} \;\leq\; \mathbb{M} \circ \mathbb{T} \,, \tag{1}$$

where $\mathsf{aff}\, S$ stands for the **affine closure** of a set $S \subseteq \mathcal{E}(U)$:

$$\mathsf{aff}\, S \;:=\; \left\{ \sum_{i=1}^{t} c_i s_i \;\middle|\; s_i \in S, c_i \in R, \sum_{i=1}^{t} c_i = 1 \right\} \,.$$

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In the particular case of vector spaces (*i.e.* when R is a field), This is an **equality**.

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Let $\mathcal{S}$ be an empirical model on the scenario $\langle X, \mathcal{M}, R \rangle$. We define its **affine closure**, $\mathsf{Aff}\,\mathcal{S}$, as the empirical model given, at each $C \in \mathcal{M}$, by

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Since $\mathbb{T}_R(\mathcal{S})$ is given as the union of the theories at each maximal context, the Galois connection above lifts to the level of empirical models. We also have

$$\mathsf{Aff} \leq \mathbb{M} \circ \mathbb{T}$$

with equality when R is a field.

Proposition

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R\rangle$. Then,

$$\mathsf{AvN}(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}).$$

If R is a field, the converse also holds.

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We shall begin by revisiting our description of the cohomology invariant.

We give a higher-level description, in terms of the connecting homomorphism of the long exact sequence.

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We define two auxiliary presheaves related to $\mathcal{F}$. Firstly, $\mathcal{F}|_U$ is defined by

$$\mathcal{F}|_U(V) := \mathcal{F}(U \cap V) .$$

There is an evident presheaf map $p \colon \mathcal{F} \longrightarrow \mathcal{F}|_U$ given as

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Secondly, $\mathcal{F}_{\bar{U}}$ is defined by $\mathcal{F}_{\bar{U}}(V) := \ker(p_V)$. Thus, we have an exact sequence of presheaves

$$\mathbf{0} \longrightarrow \mathcal{F}_{\bar{U}} \longrightarrow \mathcal{F} \xrightarrow{p} \mathcal{F}|_U \; . \qquad (2)$$

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The **relative cohomology of $\mathcal{F}$ with respect to U** is defined to be the cohomology of the presheaf $\mathcal{F}_{\bar{U}}$.

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Therefore, the map δ^0 can be corestricted to a map

$$\tilde{\delta}^0 \colon C^0(\mathcal{M}, \mathcal{F}) \longrightarrow Z^1(\mathcal{M}, \mathcal{F})$$

whose kernel is

$$Z^0(\mathcal{M}, \mathcal{F}) \cong \check{H}^0(\mathcal{M}, \mathcal{F})$$

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In summary, we have:

$$\check{H}^0(\mathcal{M}, \mathcal{F}) \xrightarrow{\ker \tilde{\delta}^0} C^0(\mathcal{M}, \mathcal{F}) \xrightarrow{\tilde{\delta}^0} Z^1(\mathcal{M}, \mathcal{F}) \xrightarrow{\operatorname{coker} \tilde{\delta}^0} \check{H}^1(\mathcal{M}, \mathcal{F}) \quad .$$

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Putting this together with the previous observation, we obtain the diagram below:

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & C^0(\mathcal{M}, \mathcal{F}_{\tilde{U}}) & \longrightarrow & C^0(\mathcal{M}, \mathcal{F}) & \longrightarrow & C^0(\mathcal{M}, \mathcal{F}|_U) & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle \tilde{\delta}^0} & & \downarrow{\scriptstyle \tilde{\delta}^0} & & \downarrow{\scriptstyle \tilde{\delta}^0} & & \\
\mathbf{0} & \longrightarrow & Z^1(\mathcal{M}, \mathcal{F}_{\tilde{U}}) & \longrightarrow & Z^1(\mathcal{M}, \mathcal{F}) & \longrightarrow & Z^1(\mathcal{M}, \mathcal{F}|_U) & &
\end{array}$$

whose two rows are short exact sequences.

Enter the Snake

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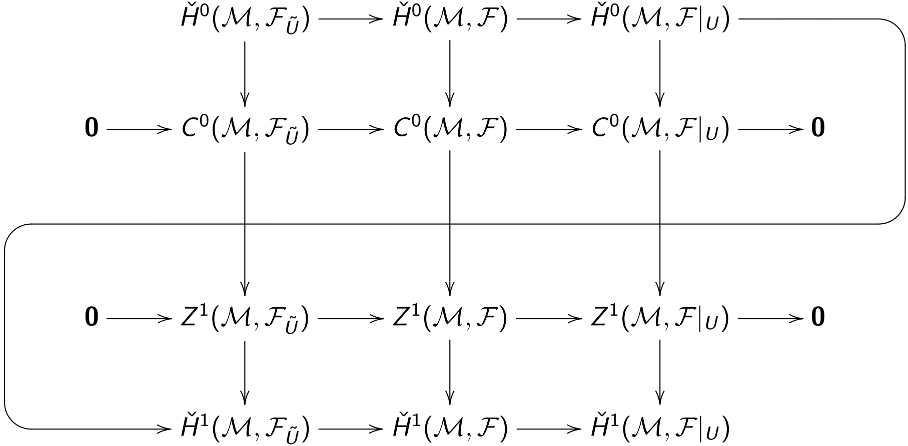
The snake lemma of homological algebra says that there exists a **connecting homomorphism** turning the kernels of the first row followed by the cokernels of the second into a long exact sequence, as shown in the following diagram.



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Definition

Let C_0 be an element of the cover $\mathcal{M}$ and $r_0 \in \mathcal{F}(C_0)$. Then, the **cohomological obstruction** of r_0 is the element $\gamma(r_0)$ of $\check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}_0})$, where γ is the connecting homomorphism.

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The following justifies regarding these as obstructions.

Proposition

Let the cover $\mathcal{M}$ be connected, $C_0 \in \mathcal{M}$, and $r_0 \in \mathcal{F}(C_0)$. Then, $\gamma(r_0) = 0$ if and only if there is a compatible family $\{r_C \in \mathcal{F}(C)\}_{C \in \mathcal{M}}$ such that $r_{C_0} = r_0$.

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We firstly consider how to build an abelian group from a set.

The free module functor

Definition

Given a ring R, we define a functor $F_R \colon \mathrm{Set} \longrightarrow R\text{-Mod}$ to the category of R-modules (and thus, in particular, to the category of abelian groups). For each set X, $F_R(X)$ is the set of functions $\phi \colon X \longrightarrow R$ of finite support. Given a function $f : X \longrightarrow Y$, we define:

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This assignment is easily seen to be functorial. We regard a function $\phi \in F_R(X)$ as a **formal** R**-linear combination** of elements of X:

$$\sum_{x \in X} \phi(x) \cdot x \ .$$

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Given a ring R, we define a functor $F_R\colon \mathrm{Set} \longrightarrow R\text{-Mod}$ to the category of R-modules (and thus, in particular, to the category of abelian groups). For each set X, $F_R(X)$ is the set of functions $\phi\colon X \longrightarrow R$ of finite support. Given a function $f\colon X \longrightarrow Y$, we define:

$$F_R f\colon F_R X \longrightarrow F_R Y :: \phi \longmapsto \lambda y. \sum_{f(x)=y} \phi(x) \; .$$

This assignment is easily seen to be functorial. We regard a function $\phi \in F_R(X)$ as a **formal R-linear combination** of elements of X:

$$\sum_{x \in X} \phi(x) \cdot x \; .$$

There is a natural embedding $x \mapsto 1 \cdot x$ of X into $F_R(X)$, which we shall use implicitly throughout.

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There is a natural embedding $x \mapsto 1 \cdot x$ of X into $F_R(X)$, which we shall use implicitly throughout.

In fact, $F_R(X)$ is the **free R-module generated by** X; and in particular, $F_{\mathbb{Z}}(X)$ is the **free abelian group generated by** X.

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With each local section, $s \in \mathcal{S}(C)$, in the support of an empirical model, we associate the **cohomological obstruction** $\gamma_{F_R\mathcal{S}}(s)$.

- If there exists some local section $s_0 \in \mathcal{S}(C_0)$ such that $\gamma_{F_R\mathcal{S}}(s_0) \neq 0$, we say that $\mathcal{S}$ is **cohomologically logically contextual**, or $\mathsf{CLC}_R(\mathcal{S})$.
- If $\gamma_{F_R\mathcal{S}}(s) \neq 0$ for all local sections, we say that e is **cohomologically strongly contextual**, or CSC_R.

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The following proposition justifies considering cohomological obstructions as witnessing contextuality.

Proposition

- CLC_R *implies* LC.
- CSC_R *implies* SC.

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Unfortunately, this condition is not, in general, necessary. It is possible that "false positives" arise in the form of families $\{r_C \in F_R\mathcal{S}(C)\}_{C\in\mathcal{M}}$ which are not *bona fide* global sections in $\mathcal{S}(X)$ in which genuine global sections do not exist.

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However, cohomological obstructions over $\mathbb{Z}$ (in fact, $\mathbb{Z}_2$ is enough) provide witnesses of strong contextuality for a number of well-studied models, including: the GHZ model, the Peres–Mermin "magic" square, and the 18-vector Kochen–Specker model, the PR box, and the Specker triangle.

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The chain of implications

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Here is the main result:

Theorem

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R \rangle$. Then:

$$\mathsf{AvN}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_{\mathbb{Z}}(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathcal{S}) \;.$$

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We have already seen the first and fourth implications. The third implication is an instance of the following general result:

Proposition

Let $h \colon R' \longrightarrow R$ be a ring homomorphism. Then, for any $C \in \mathcal{M}$ and $s \in \mathcal{S}(C)$, $\gamma_{F_{R'}\mathcal{S}}(s) = 0$ implies $\gamma_{F_R\mathcal{S}}(s) = 0$, and so $\mathsf{CSC}_R \Rightarrow \mathsf{CSC}_{R'}$ and $\mathsf{CLC}_R \Rightarrow \mathsf{CLC}_{R'}$.

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The counit is the natural transformation $\epsilon \colon F_R \circ U \Rightarrow Id_{R\text{-Mod}}$ given, for each R-module M, by the evaluation map

$$\epsilon_M \colon F_R U(M) \longrightarrow M :: r \longmapsto \sum_{x \in M} r(x) x \ .$$

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Then the map ϵ_M, by virtue of being an R-module homomorphism, maps the formal linear combinations of elements of S, $F_R(S)$, which coincide with the linear span in $F_R U(M)$ of $\eta[S] = \{1 \cdot s \mid s \in S\}$, to the linear span of S in M, $\mathrm{span}_M S$.

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Thus the counit can be horizontally composed to yield a natural transformation, or map of presheaves,

$$\mathrm{id}_{\mathcal{E}} * \epsilon \colon F_R \circ U \circ \mathcal{E} \longrightarrow \mathcal{E},$$

given at each context $U \subseteq X$ by

$$\epsilon_{\mathcal{E}(U)} \colon F_R U \mathcal{E}(U) \longrightarrow \mathcal{E}(U).$$

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we conclude that the presheaf map restricts as follows:

$$\begin{array}{ccccc} F_R^{\mathsf{aff}}\,U\mathcal{S} & \rightarrowtail & F_R\,U\mathcal{S} & \rightarrowtail & F_R\,U\mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \epsilon \\ \mathsf{Aff}\,\mathcal{S} & \rightarrowtail & \mathsf{Span}\,\mathcal{S} & \rightarrowtail & \mathcal{E} \end{array}$$

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\end{array}$$

We can use this to transfer compatible families of formal affine combinations of sections to compatible families of $\mathrm{Aff}\,\mathcal{S}$, and hence to prove the second implication by contraposition.

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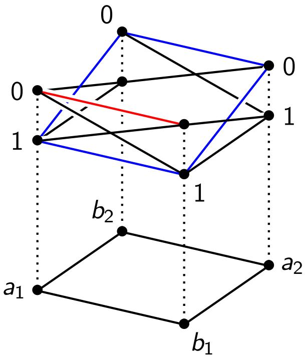
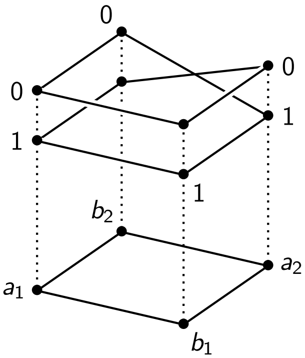
Then we have:

$$\mathsf{AvN}_R(e, s_0) \;\Rightarrow\; \mathsf{LC}(\mathsf{Aff}\,\mathcal{S}, s_0) \;\Rightarrow\; \mathsf{CLC}_R(\mathcal{S}, s_0) \;\Rightarrow\; \mathsf{CLC}_{\mathbb{Z}}(\mathcal{S}, s_0) \;\Rightarrow\; \mathsf{LC}(\mathcal{S}, s_0)\ .$$

Visualizing Contextuality





The Hardy table and the PR box as bundles

Contextuality, Logic and Paradoxes

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Liar cycles. A Liar cycle of length N is a sequence of statements

$$\begin{aligned} S_1 &: \; S_2 \text{ is true}, \\ S_2 &: \; S_3 \text{ is true}, \\ &\vdots \\ S_{N-1} &: \; S_N \text{ is true}, \\ S_N &: \; S_1 \text{ is false}. \end{aligned}$$

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The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

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We can regard each of these equations as fibered over the set of variables which occur in it:

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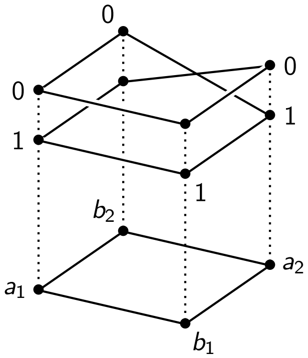
The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



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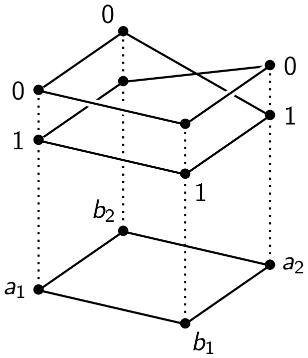

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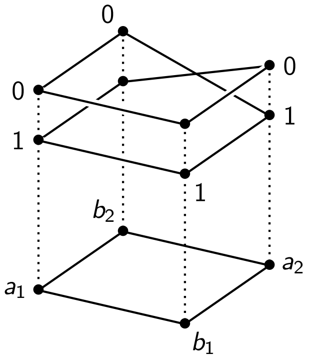

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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

The Robinson Consistency Theorem

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A classic result:

Theorem (Robinson Joint Consistency Theorem)

Let T_i be a theory over the language L_i, $i = 1, 2$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

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This example is well-known in the quantum contextuality literature as the **Specker triangle**.

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...	...	...	...

From possibility models to databases

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Consider again the Hardy model:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
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Change of perspective:

a_1, a_2, b_1, b_2	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
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a_1	b_2
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There is no **universal relation**: no table

a_1	a_2	b_1	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

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Relational databases	measurement scenarios
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set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
relation/set of tuples	boolean distribution on joint outcomes
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We can also consider probabilistic databases and other generalisations;
cf. provenance semirings.

Applications to Natural Language Semantics

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$$s_1 = \{John(x), Man(x)\}, \quad s_2 = \{donkey(y), \neg Man(y)\}, \quad s_3 = \{grey(z)\}.$$

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However, using the cover

$$f_1 : x \mapsto a, \quad f_2 : y \mapsto b, \quad f_3 : z \mapsto b$$

we do have a gluing:

$$s = \{John(a), Man(a), donkey(b), \neg Man(b), grey(b)\}.$$

Contextual Semantics

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For an accessible overview of Contextual Semantics, see the article in the *Logic in Computer Science* Column, Bulletin of EATCS No. 113, June 2014 (and `arXiv`).

People

Comrades in Arms in Contextual Semantics:

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Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida

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- General characterisation of **All-versus-Nothing** arguments. The cohomology invariant captures contextuality for all such models. Large classes of quantum examples using stabiliser groups.

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